PHYSICAL REVIEW A VOLUME 44, NUMBER 8 15 OCTOBER 1991

Amplitude equation for modulated Rayleigh-Bénard convection

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(Received 8 May 1991)

Using a systematic perturbation expansion for slightly supercritical driving, the equation for the slow spatiotemporal variation of the amplitude of convective rolls is derived when the temperatures of the horizontal fluid boundaries are modulated harmonically in time. The modulation frequencies are considered to be large compared with the amplitude growth rate, but small enough to avoid the generation of thermal Stokes layers. The marginally stable, time-dependent linear functions determining the fast spatiotemporal periodic variations of the convective fields as well as all coefficients entering into the amplitude equation depend on the modulation and the Prandtl number. Analytic expressions for all these quantities are given for small modulation amplitudes and idealized horizontal boundary conditions. The time-dependent convective heat current is compared with previous results from few-mode Galerkin approximations.

PACS number(s): 47.25.Qv, 47.20.Ky, 47.20.Bp, 03.40.Gc

I. INTRODUCTION

The response of the fluid in the Rayleigh-Bénard problem to a time-periodic modulation of the externally imposed temperature difference has recently been investigated theoretically and experimentally (see Refs. [1–15] and works cited therein). Typical findings are the stabilization of the conductive state by low-frequency modulation with small modulation amplitude, the pattern competition between roll and hexagonal convective patterns, and the modulation induced decrease of the initial slope of the convective heat current for roll convection.

Theoretical work so far has used Galerkin decompositions of the convective fields into spatial normal modes, thereby transforming the underlying hydrodynamic partial differential equations into ordinary differential systems for the mode amplitudes [1,2,4–8,10, 15]. Here we present a natural extension of the nonlinear perturbation method of Segel [16] and Newell and Whitehead [17] to roll convection under time modulated Rayleigh numbers to derive the amplitude equation for the slow spatiotemporal variation of the envelope of the convective roll fields.

According to theoretical results [5,10,15] obtained for laterally infinite layers a subcritical bifurcation to hexagonal patterns exists for time-dependent driving in addition to the supercritical bifurcation to roll patterns. The pattern and stability competition between rolls and hexagons was observed experimentally [14] in a cylindrical cell. Additional experiments [11,18] showed that in cylindrical cells with a side wall with thermal conductivity different from the fluid the thermal mismatch between the wall and the fluid generates for time-dependent heating horizontal currents [19,20] normal to the wall that induce convective rolls oriented along the wall.

The experimental setup we have in mind is an annular fluid layer heated from below that is confined between two concentric cylinders of large radii with a radius ratio close to 1 and a radial gap with a width of the order of one to two times the height of the layer. Furthermore the thermal conductivity of the cylindrical walls should be close to that of the fluid to reduce the thermal mismatch between fluid and walls and with it the modulation induced radial heat currents normal to the walls. Then (for modulation periods not much smaller than the characteristic vertical thermal diffusion time and not too large modulation amplitudes) we expect the convective pattern to consist of radially oriented rolls, i.e., normal to the walls. Due to the large radii they are practically parallel to each other with a vertical cross section in the middle of the gap that is similar to that in a pattern of straight parallel rolls. Thus in such a setup one would suppress hexagonal convection [5,10,14,15] as well as vertical vorticity [21].

In this work we consider a vertical cross-section perpendicular to the roll axes through the fluid layer in the idealized situation of straight, truly parallel rolls without any field variation along the roll axes, i.e., a two-dimensional system. Furthermore we impose for mathematical convenience idealized free slip horizontal boundary conditions at the top and bottom of the fluid layer. The difference in the modulation induced effects on roll convection between rigid and free slip horizontal boundaries is well understood, at least within few-mode Galerkin models [22].

We present a systematic nonlinear perturbation theory of the hydrodynamic field equations using a Poincaré-Lindstedt technique combined with a multiple scale analysis. That amounts to an expansion in powers of $(\epsilon-\epsilon_c)^{1/2}$, where $\epsilon-\epsilon_c$ is the appropriately reduced distance of the mean Rayleigh number from the critical one for the onset of convection in the presence of temperature modulation. We consider modulation frequencies that are large compared with amplitude growth rates, $(\epsilon-\epsilon_c)/\tau$,

$$\epsilon-\epsilon_c \ll \omega\tau \,, \tag{1.1}$$

where τ is the characteristic time scale entering into the amplitude equation. On the other hand the modulation period should not be very much smaller than the vertical thermal diffusion time to avoid the "skin effect" of the generation of a narrow thermal Stokes boundary layer [4,9,12]. Note that this is not really a limitation on experiments since the interesting modulation effects for small amplitudes occur when the modulation period is of the order of the thermal diffusion time. Also the condition (1.1) does not pose a problem for experiments—in fact for the experimentally realized modulation frequencies [6,8,11,13,14] it is impossible not to fulfill (1.1) when measuring close to the onset of convection. We should like to stress again that we consider here the two time scales $\tau/(\epsilon-\epsilon_c)$ and $2\pi/\omega$ to be separated and *not* the situation $\omega\tau \simeq \epsilon-\epsilon_c$. For the latter case Hall [23,24] has derived an amplitude equation for modulated Taylor vortex flow, the form of which has also been used [7] in the context of modulated convection.

Here we find to lowest order in the perturbation expansion that, e.g., the vertical velocity field has the form

$$w(x,z;t)=A(x,t)f_c(x,z;t)+\text{c.c.} \tag{1.2}$$

Here

$$f_c(x,z;t)\simeq e^{ik_c x}\sqrt{2}\sin(\pi z)f_c(t) \tag{1.3}$$

is the marginal solution of the linearized equations at threshold $\epsilon=\epsilon_c$ and $f_c(t)$ varies on the time scale of the modulation. The complex amplitude A is the solution of the Ginzburg-Landau amplitude equation

$$\tau\partial_t A(x,t)=[\xi^2\partial_x^2+\epsilon-\epsilon_c-g|A(x,t)|^2]A(x,t) . \tag{1.4}$$

In contrast to $f_c(x,z;t)$ the amplitude A varies slowly, i.e., on the scales $\tau/(\epsilon-\epsilon_c)$ and $\xi/(\epsilon-\epsilon_c)^{1/2}$. All parameters $k_c,\epsilon_c,\tau,\xi^2,g$ are functions of the modulation, in particular of the frequency and the modulation amplitude, but otherwise the modulation operating on a time scale that is fast relative to $\tau/(\epsilon-\epsilon_c)$ does not enter into Eq. (1.4) for the slowly varying amplitude.

As an aside we mention that the amplitude equation for gravity modulation [25] has the same form with similar coefficients. However, gravity modulation is rather difficult to be realized experimentally with sizable modulation amplitudes. Hence, we do not address this case here.

In Sec. II we define the problem on the basis of the Oberbeck-Boussinesq equations. Since linear properties play a key role in the investigation of weakly nonlinear systems, we compile known linear results and complete them by new ones. In Sec. III we derive the amplitude equation and present the dependence of the coefficients on frequency and modulation amplitudes. The application of the amplitude equation to the determination of the vertical convective heat current is presented in Sec. IV. Appendixes A, B, and C contain details of the perturbation expansion.

II. TEMPERATURE MODULATION

A. System

We investigate convection in the form of straight parallel rolls in a laterally infinite horizontal fluid layer of height d that is heated from below. We shall scale lengths, times, temperatures, and pressures by d, d^2/κ, $\nu\kappa/(\alpha g d^3)$, and $\rho\kappa^2/d^2$, respectively. Here g is the gravitational acceleration, ρ is the mean fluid density, α the thermal expansion coefficient, ν the kinematic viscosity, and κ the thermal diffusivity.

The external driving force is modulated harmonically in time by varying the temperatures of the lower and/or the upper horizontal boundary

$$T_l(t)=T_u+R_0[1+\Delta_l\cos(\omega t)] , \tag{2.1a}$$

$$T_u(t)=T_u+R_0\Delta_u\cos(\omega t-\phi) . \tag{2.1b}$$

We consider here the relative amplitudes Δ_l and Δ_u to be related by

$$\Delta_l-\Delta_u e^{i\phi}=\Delta . \tag{2.1c}$$

Then the Rayleigh number

$$R(t)=T_l(t)-T_u(t)=R_0[1+\Delta\cos(\omega t)] \tag{2.2}$$

is modulated with relative amplitude Δ around its mean R_0. Special cases are in-phase modulation ($\phi=0$), including bottom plate modulation ($\Delta_l=\Delta$, $\Delta_u=0$); out-of-phase modulation ($\phi=\pi$), including top plate modulation ($\Delta_l=0$, $\Delta_u=\Delta$) and modulation of both boundaries with half amplitude ($\Delta_l=\Delta_u=\Delta/2$).

The system with Prandtl number $\sigma=\nu/\kappa$ is described by the nonlinear Oberbeck-Boussinesq equations

$$(\partial_t+\mathbf{u}\cdot\boldsymbol{\partial})\mathbf{u}=-\boldsymbol{\partial}P+\sigma(\theta\mathbf{e}_z+\partial^2\mathbf{u}) , \tag{2.3a}$$

$$(\partial_t+\mathbf{u}\cdot\boldsymbol{\partial})\theta=-\mathbf{u}\cdot\boldsymbol{\partial}T_{\text{cond}}+\partial^2\theta , \tag{2.3b}$$

$$\boldsymbol{\partial}\cdot\mathbf{u}=0 . \tag{2.3c}$$

Here the Cartesian components of the velocity field $\mathbf{u}(\mathbf{x},t)$ are u, v, and w and the temperature $T(\mathbf{x},t)=T_{\text{cond}}(z,t)+\theta(\mathbf{x},t)$ is decomposed in terms of the conduction field T_{cond} and the contribution θ due to convection. The pressure P also refers to convection. The position vector $\mathbf{x}$ has components x,y,z. The conduction profile $T_{\text{cond}}(z,t)$ obeys the heat diffusion equation and fulfills the boundary conditions (2.1). It contains a linear term in z and a contribution describing damped thermal waves which propagate into the fluid layer. Note that the modulation induced z and t dependence of the vertical temperature profile implies a (z,t)-dependent heat current (A4) entering into (2.3b).

For low frequencies T_{cond} deviates only slightly from a linear profile. But for high-frequency modulation the heat waves enter only into a narrow thermal Stokes boundary layer [4,9,12], thus causing additional exponential spatial behavior. This thermal skin effect restricts the applicability of an approximation used by us (cf. below) to low-frequency modulation.

We consider free slip, perfectly heat conducting horizontal boundaries [22]:

$$w=\partial_z^2 w=\theta=0 \quad \text{at } z=0,1 . \tag{2.4}$$

Furthermore, we restrict ourselves to convective solutions in the form of straight parallel rolls with wave vector $\mathbf{k}=(k,0,0)$ by setting $v=0$ and discarding any y dependence of the fields. In this way we suppress the vertical vorticity that was shown [21] to arise close to the onset in laterally infinite layers between free-slip boundaries. Convective rolls in the annular experimental set-up described in the introduction seem to come closest to our idealized system. Their cross section at midgap in an annulus with large cylinder radii is roughly like a cross section of straight parallel rolls without y variation.

In the following we use the reduced distance

$$\epsilon=\frac{R_0}{R_c^{\mathrm{stat}}}-1 \tag{2.5}$$

of the mean Rayleigh number R_0 from the threshold $R_c^{\mathrm{stat}}=27\pi^4/4$ for the onset of convection in the absence of modulation as a control parameter.

B. Linear convective properties

Venezian [1] found the threshold for the onset of convection for small modulation amplitudes,

$$\frac{R_c(\Delta,\omega)}{R_c^{\mathrm{stat}}}-1=\epsilon_c(\Delta,\omega)=\Delta^2\epsilon_c^{(2)}(\omega)+\mathcal{O}(\Delta^4)\ , \tag{2.6}$$

by looking for a marginal, i.e., periodic solution of the linearized differential equations. Here and also in all other temporal mean values of the present paper solely even orders in the modulation amplitude Δ appear because the transformation $\Delta\to-\Delta$ in the drive (2.2) adds only a phase shift of π.

Modulation typically causes a stabilization of the conductive state. The effect is strongest for Prandtl numbers $\sigma\simeq 1$ and for low frequencies ω. However, for high frequencies a weak destabilization is possible for appropriate Prandtl numbers. Because of the vertical dependence of the conductive heat current each mode $\sin(n\pi z)$ of the vertical expansion of temperature and velocity yields an additive contribution [1] $\epsilon_{cn}^{(2)}(\omega)$ to the threshold shift in order Δ^2:

$$\epsilon_c^{(2)}(\omega)=\sum_{n=1}^{\infty}\epsilon_{cn}^{(2)}(\omega)\ . \tag{2.7}$$

We consider only low frequencies and experimentally relevant Prandtl numbers

$$0<\omega\lesssim 20 \quad \text{for } 0.1\lesssim\sigma\lesssim 1\ , \tag{2.8a}$$

$$0<\omega\lesssim 2\pi \quad \text{for } \sigma=\mathcal{O}(10)\ , \tag{2.8b}$$

where the problems associated with the Stokes layer are negligible. Then Venezian [1] finds that the threshold shift $\epsilon_c^{(2)}(\omega)$ in order Δ^2 is well approximated by the contribution $\epsilon_{c1}^{(2)}(\omega)$ of the mode $\sin(\pi z)$:

$$\epsilon_c^{(2)}(\omega)\simeq\epsilon_{c1}^{(2)}(\omega) =\frac{1}{2}\frac{\sigma}{(\sigma+1)^2}\times\left[\left[1+\frac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}\right]\left[1+\frac{\omega^2}{(2\pi)^4}\right]\right]^{-1}, \tag{2.9a}$$

$$q_c^{(0)2}=\frac{3\pi^2}{2}\ . \tag{2.9b}$$

Also the Lorenz model of Ahlers, Hohenberg, and Lücke [7] yields the result (2.9). We derive all following results for low frequencies. Then the mathematical problem is crucially simplified by retaining the first linear unstable mode, $\sin(\pi z)$, only.

Temperature modulation affects also the critical wave number

$$k_c(\Delta,\omega)=k_c^{(0)}[1+\Delta^2k_c^{(2)}(\omega)+\mathcal{O}(\Delta^4)] \tag{2.10}$$

of the convection pattern. In order Δ^2 it is slightly shifted down from the value $k_c^{(0)}=\pi/\sqrt{2}$ for stationary drive by

$$k_c^{(2)}(\omega)=-\frac{1}{4}\frac{\omega^2}{q_c^{(0)4}}\frac{\sigma}{(\sigma+1)^4}\times\left[\left[1+\frac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}\right]^2\left[1+\frac{\omega^2}{(2\pi)^4}\right]\right]^{-1} \tag{2.11}$$

(see Fig. 1). Although the shift of the critical wave number is about one order smaller than the threshold shift in order Δ^2, a consistent treatment has to include it. Venezian [1] first mentioned the expansion of the critical wave number, but he did not give its dependence on ω and σ because $k_c^{(2)}(\omega)$ does not enter into the threshold shift. Later papers [5–8,10,15] on modulated convection used $k_c^{(0)}$ as the critical wave number instead of (2.10) thus neglecting a small correction. We note that the present derivation [25] of the amplitude equation in Sec. III needs the correct form [(2.10) and (2.11)].

Furthermore, we get from linear analysis the curvature

$$\xi^2(\Delta,\omega)=\tfrac{1}{2}\partial_k^2\epsilon(k,\Delta,\omega)\Big|_{k_c(\Delta,\omega)} \tag{2.12}$$

of the stability curve $\epsilon(k,\Delta,\omega)$ at the critical point (see

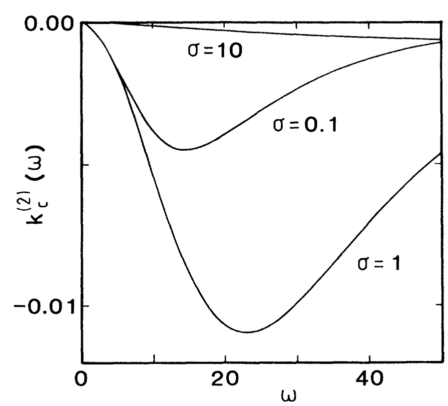

FIG. 1. The shift (2.11) of the critical wave number $k_c(\Delta,\omega)=k_c^{(0)}[1+\Delta^2k_c^{(2)}(\omega)]$ in order Δ^2 caused by temperature modulation with amplitude Δ is shown as a function of the frequency ω for different Prandtl numbers σ.

Fig. 2). We discuss this quantity in Sec. III in the framework of the amplitude equation. We summarize the above results in Fig. 2. Temperature modulation typically induces an upward shift of the marginal stability curve. Convection sets in with a slightly enlarged spatial periodicity due to the lowering of the critical wave number. The curvature ξ^2 of the neutral curve is increased.

Moreover one finds [1,25] that the time dependence of the critical velocity field at threshold is dominated in the low-frequency range by the contribution of the $\sin(\pi z)$ mode,

$$w(\mathbf{x},t)=Af_c(t)e^{ik_c x}\sqrt{2}\sin(\pi z)+\text{c.c.}+\text{h.m.} \qquad (2.13)$$

Contributions from higher modes (h.m.) $\sin(n\pi z)(n\geq 2)$ are negligible. The temporal variation $f_c(t;\Delta,\omega)$ of the first growing $n=1$ mode obeys the differential equation

$$\left[\frac{1}{\sigma q_c^4}\partial_t^2+\frac{\sigma+1}{\sigma}\frac{1}{q_c^2}\partial_t +1-\frac{R_c^{\text{stat}}}{q_c^6/k_c^2}(1+\epsilon_c)C(t)\right]f_c(t)=0 . \qquad (2.14)$$

The external modulation enters via the projection of $T_{\text{cond}}(z,t)$ onto the $n=1$ mode,

$$C(t;\Delta,\omega)=1+\Delta c(t;\omega) , \qquad (2.15a)$$

$$c(t;\omega)=\tfrac{1}{2}\Gamma(\omega)e^{-i\omega t}+\text{c.c.} , \qquad (2.15b)$$

$$\Gamma(\omega)=\frac{1}{1-i\omega/(2\pi)^2} . \qquad (2.15c)$$

Here

$$\mathbf{q}_c(\Delta,\omega)=(k_c(\Delta,\omega),0,\pi) , \qquad (2.16a)$$

$$q_c^2(\Delta,\omega)=k_c^2(\Delta,\omega)+\pi^2 =q_c^{(0)2}[1+\Delta^2\tfrac{2}{3}k_c^{(2)}(\omega)+\mathcal{O}(\Delta^4)] . \qquad (2.16b)$$

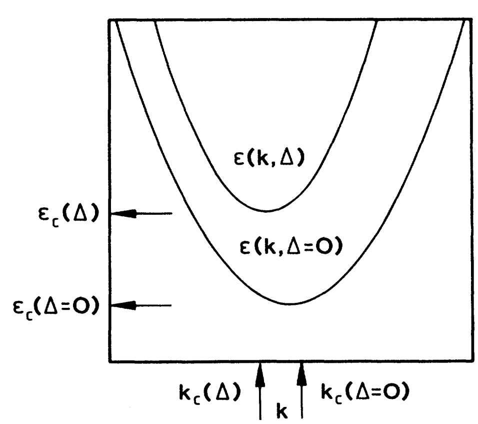

FIG. 2. Schematic stability curve $\epsilon(k,\Delta)$ for temperature modulation (the argument ω is omitted) and for the stationary case $\epsilon(k,\Delta=0)$ as a function of wave number k. Modulated driving typically stabilizes the conductive state, decreases the wave number of the convective pattern at onset, and increases the curvature of the neutral curve at the critical point.

The frequency dependence of the matrix element [(2.15c) and (A24)] and its appearance in the form $|\Gamma(\omega)|^2$ in the shifts, (2.9a) and (2.11), is due to the nonlinearity of the conductive temperature profile. Further dependences on ω and σ arise from the eigenvalues of the unmodulated differential operator in Eq. (2.14). Those contributions within linear theory which vanish for infinite Prandtl number, e.g., the convective threshold shift $\epsilon_{c1}^{(2)}(\omega)$, can be traced back to the second-order time derivative in (2.14). The presence of this term in (2.14) for finite Prandtl numbers complicates the analysis. We mention that the damped Mathieu equation (2.14) can be interpreted in terms of a parametrically modulated damped oscillator. Then the stabilization of the basic state by modulation is caused by inertia in the language of the oscillator [7,26].

It is convenient to normalize the time-periodic solution $f_c(t;\Delta,\omega)$ of the linear equation (2.14) such that its temporal average is 1. We have evaluated f_c with a straightforward but lengthy expansion in terms of the modulation amplitude Δ inclusively up to order Δ^3 (see Appendix C). For briefness, we only give the first two terms,

$$f_c(t;\Delta,\omega)\simeq 1+\frac{\Delta}{\omega\tau^{(0)}}\sin(\omega t) -\frac{1}{4}\left[\frac{\Delta}{\omega\tau^{(0)}}\right]^2\cos(2\omega t)+\mathcal{O}(\Delta^3) , \qquad (2.17a)$$

$$1/\tau^{(0)}=q_c^{(0)2}\sigma/(\sigma+1) . \qquad (2.17b)$$

Here a small quantity in the Δ and Δ^2 terms as well as slight phase shifts are neglected.

Figure 3 shows the full time dependence $f_c(t)$ obtained by numerical integration of Eq. (2.14) for three combina-

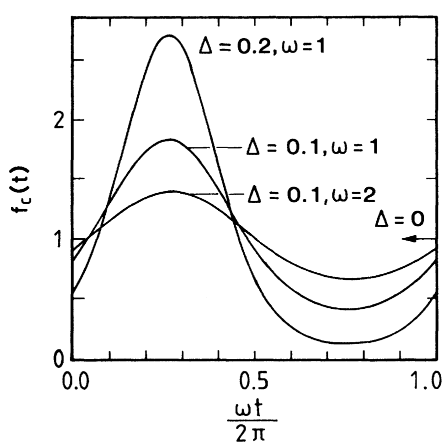

FIG. 3. One period of the critical time dependence $f_c(t;\Delta,\omega)$ of the vertical velocity field w due to the low-frequency mode $\sin(\pi z)$ obtained by numerical integration of (2.14) for various amplitudes Δ and frequencies ω. The Rayleigh number is $R(t)=R_c(\Delta,\omega)[1+\Delta\cos(\omega t)]$ and the Prandtl number is $\sigma=1$. With increasing $\Delta/(\omega\tau^{(0)})$ the marginal response of the system becomes more and more anharmonic. We use a normalization such that the temporal average is identical to the unmodulated case, $\langle f_c(t;\Delta,\omega)\rangle=f_c(\Delta=0)=1$ (arrow).

tions of Δ and ω at a Prandtl number $\sigma=1$. The response of the critical unstable mode is phase shifted by about $\pi/2$ relative to the external cosine modulation (2.2). In general, the anharmonicity of the time dependence increases with growing Δ/ω: with increasing Δ the drive reaches larger supercritical and subcritical values while with decreasing ω the system is a longer time supercritical and subcritical. Both effects increase the anharmonicity. Only for

$$\frac{\Delta}{\omega\tau^{(0)}} \ll 1 \tag{2.18}$$

can the higher harmonics be ignored. For fixed Prandtl number σ, this is realized either for weak modulation, i.e., small Δ, or by modulation with a moderately high frequency in the range (2.8). Furthermore, the analytical form (2.17a) for $f_c(t)$ shows that decreasing the modulation frequency at fixed Prandtl number σ has the same effect on the anharmonicity of the marginal field response $f_c(t)$ as increasing the diffusive relaxation rate $1/\tau^{(0)}$ (2.17b) at fixed ω via increasing σ. We checked that analytical and numerical results agree in the validity range of the expansion for small Δ.

III. DERIVATION OF THE AMPLITUDE EQUATION

To evaluate the convective fields and the convective heat current a nonlinear analysis [16,17] is required to determine the amplitude A in (1.2). This will be supplied by the amplitude equation.

For slightly supercritical drive, i.e., $\epsilon-\epsilon_c \ll 1$, the convective roll fields are small and grow continuously with the distance $\epsilon-\epsilon_c$ from the critical point. This is exploited within the framework of a Poincaré-Lindstedt expansion where the distance from threshold as well as the fields are expanded in powers of a small parameter η. The linear stability curve (Fig. 2) shows that above threshold modes with wave numbers from a band around the critical wave number k_c can grow. The superposition of the plane waves from this band produces a slow spatial variation of an envelope. These linear unstable modes grow initially with a rate $\sim(\epsilon-\epsilon_c)/\tau$. If the latter is small compared with the frequency of the modulation, $\epsilon-\epsilon_c \ll \omega\tau$, then the slow growth becomes independent of the response to the periodic modulation. We take this into account with the method of multiple scale analysis by introducing slow length scales $X_n=\eta^n x$ and slow time scales $T_n=\eta^n t$ in addition to the fast variations in x and t.

We combine the Poincaré-Lindstedt expansion and the multiple scale analysis,

$$\epsilon-\epsilon_c=\eta\epsilon_1+\eta^2\epsilon_2+\cdots \; , \tag{3.1a}$$

$$\mathbf{u}(\mathbf{x},t)=\eta[\mathbf{u}_0(\mathbf{x},X,t,T)+\eta\mathbf{u}_1(\mathbf{x},X,t,T)+\cdots] \; , \tag{3.1b}$$

$$\theta(\mathbf{x},t)=\eta[\theta_0(\mathbf{x},X,t,T)+\eta\theta_1(\mathbf{x},X,t,T)+\cdots] \; , \tag{3.1c}$$

denoting with X and T all slow length and time scales, respectively,

$$X=\{X_1,X_2,\ldots\} \; , \quad T=\{T_1,T_2,\ldots\} \; . \tag{3.2}$$

Having evaluated the "coefficients" $\epsilon_n,\mathbf{u}_n,\theta_n$ the expansion parameter η is then eliminated in (3.1). The logic is to combine the functional dependence of (3.1a) and, say, (3.1c) on η into the final desired form $\theta=\theta(\epsilon-\epsilon_c)$, for example. In the present case of a roll pattern it turns out (cf. further below) that the fields grow like $(\epsilon-\epsilon_c)^{1/2}$. We consider the complex amplitude A of the convection rolls as a function of the slow variables,

$$A(x,t)=\eta[A_0(X,T)+\eta A_1(X,T)+\cdots] \; . \tag{3.3}$$

By inserting the Poincaré-Lindstedt expansion and the multiple scale analysis into the Oberbeck-Boussinesq equations the nonlinear problem is decomposed in a sequence of linear equations with inhomogeneities depending in general nonlinearly on previous "coefficients." We thereby factorize the convection fields (3.1b) and (3.1c) into a part reflecting the fast critical spatiotemporal response and into a slowly varying amplitude. Integrals over the rapidly varying functions enter into the solubility conditions, i.e., the amplitude equation. In Appendix A we perform the calculation of the convective fields in the first two η orders. Starting with the critical linear solution, the same spatial normal modes are excited with temperature modulation with moderately low frequency as in the case of stationary heating, viz., one linear mode of the velocity field and two temperature modes, i.e., one linear and one nonlinear mode. The temporally periodic behavior of the slightly supercritical fields in order η^2 follows from the periodic time dependence of the critical mode of order η. These convective fields enter into the Fredholm alternatives in order η^2 and in order η^3. Integrating over the fast variables $\mathbf{x}$ and t one guarantees via solvability conditions the absence of secular growth. That leads to partial differential equations in the slow scales X and T. We need the variables X_1,X_2 and T_1,T_2 as well as the constants ϵ_1,ϵ_2.

For rolls the solvability condition in order η^2,

$$\tau\partial_{T_1}A_0(X,T)=(\zeta\nabla_{X_1}+\epsilon_1)A_0(X,T) \; , \tag{3.4}$$

is linear in the amplitude $A_0(X,T)$. Newell [27] and Brand, Lomdahl, and Newell [28,29] determined for time-independent drive relations between derivatives of the linear growth exponent $s(\epsilon,k)$ of a wave train at the stability curve $\epsilon(k)$ and the linear coefficients of the amplitude equation. We generalize in Appendix B this method of identification to the case of modulated driving.

The linear growth exponent $s(\epsilon,k,\Delta,\omega)$ (B12) determines the slow temporal variation of the amplitude of the critical mode. In our case s vanishes at the convective onset, i.e., it displays no imaginary part at threshold. Also the group velocity vanishes at the critical point,

$$\zeta=-i\tau\partial_k s(\epsilon,k,\Delta,\omega)\Big|_{\epsilon_c,k_c}=0 \; . \tag{3.5}$$

Then marginality entails via (3.4), as in the case of temporally constant heating, that a finite amplitude $A_0(X,T)$ is independent of the slow scale $T_1=\eta t$:

$$\partial_{T_1} A_0(X,T)=0 \; , \tag{3.6}$$

$$\epsilon_1=0 \; . \tag{3.7}$$

Note that the above Fredholm alternative in order η^2, Eq. (3.4), leads to two physical consequences. First, the vanishing group velocity prevents variation of A_0 on the time scale T_1. Secondly, the absence of a quadratic nonlinearity in order η^2, Eq. (3.4), enforces $\epsilon_1=0$ so that the amplitude of a roll pattern does not bifurcate linearly in the distance $\epsilon-\epsilon_c$ from the critical point. This agrees with previous results of Roppo, Davis, and Rosenblat [5] and Ahlers, Hohenberg, and Lücke [7]. For hexagonal (H) convection, on the other hand, Roppo, Davis, and Rosenblat [5] and Hohenberg and Swift [10] found a backwards bifurcation with a linear variation $\sim|\epsilon-\epsilon_c|$. In view of their result we expect for hexagonal modes in the η^2 solvability condition the appearance of a quadratic nonlinearity enforcing in Δ^2 a finite $\epsilon_{1H}\neq 0$.

For roll convection, the first nonlinear amplitude combination arises in the η^3 solvability condition

$$\tau\partial_{T_2} A_0(X,T)=[\xi^2\nabla^2_{X_1}+\epsilon_2-g|A_0(X,T)|^2]A_0(X,T) \; . \tag{3.8}$$

To eliminate the slow auxiliary variables in favor of the original x,t we multiply (3.8) with η^3 and use (3.1a), (3.3), and (A10) and we obtain the amplitude equation

$$\tau\partial_t A(x,t)=[\xi^2\partial_x^2+\epsilon-\epsilon_c-g|A(x,t)|^2]A(x,t) + \mathcal{O}(\epsilon-\epsilon_c)^2 \tag{3.9}$$

for modulated convection. It is the generalization of the Newell-Whitehead-Segel equation [16,17] in one spatial dimension for externally modulated drive with small amplitude and low frequency. The amplitude equation of Ginzburg-Landau–type describes the spatiotemporal dynamics of the slowly varying amplitude $A(x,t)$ of the convection rolls close to the critical point. It is exact at threshold. The relaxation time τ and the curvature ξ^2 of the marginal curve in the critical point are results of linear stability theory and enter into the amplitude equation via multiple scale analysis. Whereas the response of the system to the externally modulated drive occurs on the time scale $2\pi/\omega$, the amplitude $A(x,t)$ of the convection rolls with wave number $k=k_c+Q$ varies on the longer time scale $\tau/|\epsilon-\epsilon_c-\xi^2Q^2|$ [(A41) and (A42)]. Hence, the amplitude equation (3.9) is valid for

$$|\epsilon-\epsilon_c-\xi^2Q^2| \ll \omega\tau \; , \quad |Q| \ll k_c \tag{3.10}$$

with regard to the temporal and spatial variations, respectively.

The coefficient g of the cubic nonlinearity measures the nonlinear coupling between the $\sin(2\pi z)$ temperature modes and the vertical velocity field [(A19a), (A20a), (A30b), (A34), and (C4)]. The nontrivial constant solution

$$A=\left[\frac{\epsilon-\epsilon_c}{g}\right]^{1/2} e^{i\chi}+\mathcal{O}(\epsilon-\epsilon_c) \tag{3.11}$$

of the amplitude equation, where χ is any real phase, shows that the convection fields at threshold grow proportional to $(\epsilon-\epsilon_c)^{1/2}$. Since g is positive, the amplitude equation exhibits the supercritical bifurcation that is characteristic for convection rolls.

The form of the amplitude equation for weak temperature modulation is the same as for unmodulated drive. But in the modulated case, the coefficients τ, ξ^2, and g become functions of modulation amplitude and frequency:

$$\tau(\Delta,\omega)=\tau^{(0)}[1+\Delta^2\tau^{(2)}(\omega)+\mathcal{O}(\Delta^4)] \; , \tag{3.12a}$$

$$\xi^2(\Delta,\omega)=\xi^{2(0)}[1+\Delta^2\xi^{2(2)}(\omega)+\mathcal{O}(\Delta^4)] \; , \tag{3.12b}$$

$$g(\Delta,\omega)=g^{(0)}[1+\Delta^2 g^{(2)}(\omega)+\mathcal{O}(\Delta^4)] \; . \tag{3.12c}$$

The nonlinear coefficient depends on the definition of the amplitude A, and all three coefficients reflect the normalization used in the amplitude equation [(1.4) and (3.9)]. For stationary heating, $\Delta=0$, one has [16,17]

$$\tau^{(0)}=\frac{2}{3\pi^2}\frac{\sigma+1}{\sigma} \; , \quad \xi^{2(0)}=\frac{8}{3\pi^2} \; , \quad g^{(0)}=\frac{2}{3\pi^2} \; . \tag{3.13}$$

The order Δ^2 shifts are (cf. Appendix C)

$$\tau^{(2)}(\omega)=\frac{\sigma^3}{(\sigma+1)^4}\frac{1+\dfrac{1}{\sigma^2}\left[1+\dfrac{7}{6}\dfrac{\omega^2}{q_c^{(0)4}}\right]}{\left[1+\dfrac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}\right]^2\left[1+\dfrac{\omega^2}{(2\pi)^4}\right]} \; , \tag{3.14a}$$

$$\xi^{2(2)}(\omega)=\frac{1}{2}\frac{\sigma}{(\sigma+1)^2}\frac{1+\dfrac{8}{3}\dfrac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}+3\dfrac{\omega^4}{(\sigma+1)^4q_c^{(0)8}}}{\left[1+\dfrac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}\right]^3\left[1+\dfrac{\omega^2}{(2\pi)^4}\right]} \; , \tag{3.14b}$$

$$g^{(2)}(\omega)=\vartheta_0^{(2)}(\omega)+S^{(2)}(\omega) \; , \tag{3.14c}$$

where

$$\vartheta_0^{(2)}(\omega)=\frac{1}{2}\frac{\sigma^2}{(\sigma+1)^2}\frac{\dfrac{q_c^{(0)4}}{\omega^2}+\dfrac{1}{(\sigma+1)^2}\left[1+\dfrac{1}{3\sigma}\dfrac{\omega^2}{q_c^{(0)4}}\right]}{\left[1+\dfrac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}\right]^2\left[1+\dfrac{\omega^2}{(2\pi)^4}\right]}, \tag{3.15a}$$

$$S^{(2)}(\omega)=s^{(2)}(\omega)+2\epsilon_{c1}^{(2)}(\omega), \tag{3.15b}$$

$$s^{(2)}(\omega)=\frac{\sigma^3}{(\sigma+1)^4}\frac{\dfrac{3}{4}+\dfrac{7}{4\sigma}-\dfrac{1}{\sigma^2}\left[1-\dfrac{3}{8}\dfrac{\omega^2}{q_c^{(0)4}}\right]}{\left[1+\dfrac{\omega^2}{(\sigma+1)^2q_c^{(0)4}}\right]^2\left[1+\dfrac{\omega^2}{(2\pi)^4}\right]^2}. \tag{3.15c}$$

In the above coefficients we recognize the contribution $|\Gamma(\omega)|^2$ of the matrix element $\Gamma(\omega)$ (2.15c) with the frequency dependence of the nonlinear vertical conductive temperature profile. Furthermore the linear operator in (2.14) generates more complicated ω,σ dependences in (3.14) and (3.15). In addition into $g^{(2)}(\omega)$ there enters the time dependence $\vartheta(t)$ of the $\sin(2\pi z)$ temperature mode, given by (A35).

We show $\tau^{(2)}$, $\xi^{2(2)}$, and $g^{(2)}$ as a function of frequency for some Prandtl numbers in Figs. 4 and 5. The validity is restricted by (2.8) to low frequencies. Temperature modulation increases the relaxation time τ as well as the curvature ξ^2 similarly to the low-frequency convective threshold shift. We find $\xi^{2(2)}(\omega)\simeq\epsilon_{c1}^{(2)}(\omega)$ and, for Prandtl numbers $\sigma\simeq1$, $\tau^{(2)}(\omega)\simeq\epsilon_{c1}^{(2)}(\omega)$. Thus, the effect of modulation is biggest for Prandtl numbers $\sigma\simeq1$. Figure 5 shows that temperature modulation increases g and thereby decreases the amplitude A (3.11) of convection rolls. In the frequency range of Fig. 5 we can approximate

$$g^{(2)}(\omega)\simeq\vartheta_0^{(2)}(\omega)\simeq\frac{1}{2}\frac{1}{(\omega\tau^{(0)})^2}. \tag{3.16}$$

Note that one has to ensure $\Delta^2g^{(2)}(\omega)\ll1$ to guarantee the validity of the small Δ expansion.

IV. MODULATED CONVECTION

The amplitude equation yields the exact solution of the convective fields at threshold. For detailed formulas we refer to Appendix A. Here we first consider the vertical velocity field for the nonlinear saturated periodic state

$$w(\mathbf{x},t)=\left[\frac{\epsilon-\epsilon_c}{g}\right]^{1/2}f_c(t)e^{i(k_cx+\chi)}\sqrt{2}\sin(\pi z)+\text{c.c.}+\mathcal{O}(\epsilon-\epsilon_c). \tag{4.1}$$

In Fig. 6 we show the time dependence of the mode amplitude $[(\epsilon-\epsilon_c)/g]^{1/2}f_c(t)$ for a modulation that is slightly supercritical and allows us to neglect higher-order terms $\sim(\epsilon-\epsilon_c)$. The function $f_c(t)$ is numerically obtained, whereas ϵ_c and g are taken from our analytical results. The linear time dependence $f_c(t)$ was discussed already in Fig. 3. Concerning the nonlinear result, Fig. 6 reveals that modulation decreases the temporal average

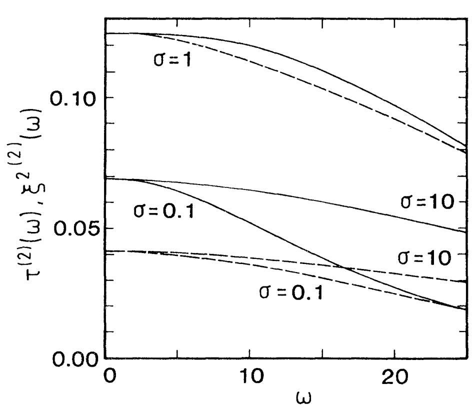

FIG. 4. The shifts of the relaxation time $\tau(\Delta,\omega)=\tau^{(0)}[1+\Delta^2\tau^{(2)}(\omega)]$ (solid lines) and of $\xi^2(\Delta,\omega)=\xi^{2(0)}[1+\Delta^2\xi^{2(2)}(\omega)]$ (dashed lines) of the amplitude equation in order Δ^2 by temperature modulation with frequency ω for various Prandtl numbers.

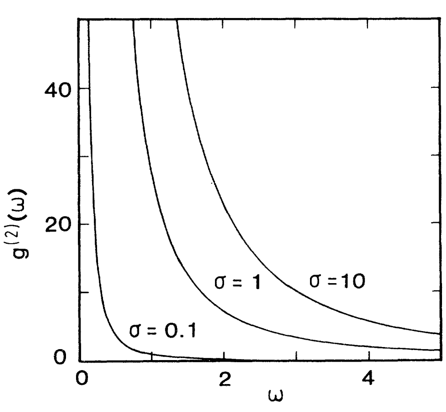

FIG. 5. Frequency dependence of the nonlinear coefficient $g(\Delta,\omega)=g^{(0)}[1+\Delta^2g^{(2)}(\omega)]$ of the amplitude equation in order Δ^2 for various Prandtl numbers. With increasing frequency g decreases and the convection amplitude increases.

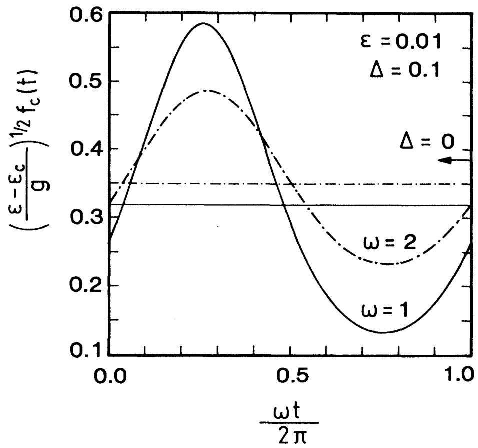

FIG. 6. One period of the amplitude of the vertical velocity field (4.1) for a Rayleigh number $R(t)/R_c^{\text{stat}}=(1+\epsilon)[1+\Delta\cos(\omega t)]$ such that ϵ is slightly above $\epsilon_c(\Delta,\omega)$. The Prandtl number used is $\sigma=1$. Horizontal lines denote the corresponding temporal averages. Note the modulation induced decrease of the mean intensity in comparison with the unmodulated intensity (arrow).

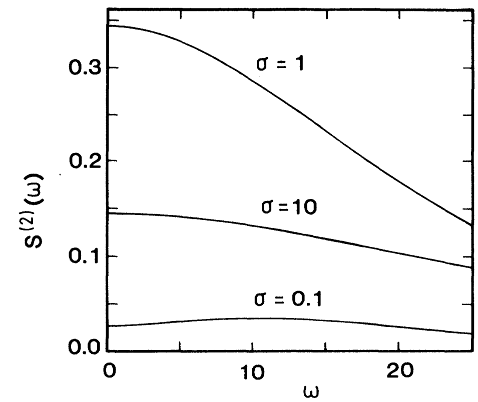

FIG. 7. The second-order coefficient of the modulation induced shift of the slope $S(\Delta,\omega)=1-\Delta^2 S^{(2)}(\omega)$ of the temporally averaged convective heat current $\langle j_{\text{conv}}(t)\rangle$ [(3.15b), (3.15c), and (2.9)] is shown vs modulation frequency for several Prandtl numbers σ.

of the vertical velocity field w; low-frequency modulation yields the biggest effect.

In the following we evaluate the laterally averaged reduced vertical convective heat current (cf. Appendix A):

$$j_{\text{conv}}(z,t)=\frac{1}{R_c^{\text{stat}}}\langle w(\mathbf{x},t)\theta(\mathbf{x},t)-\partial_z\theta(\mathbf{x},t)\rangle_\perp \ . \tag{4.2}$$

This quantity, into which the amplitude A enters quadratically (A45), is experimentally accessible [6,8,11, 13,14]. With the constant solution (3.11) of the amplitude equation the convective heat current at the lower plate (A44)–(A48) reads

$$\begin{aligned} j_{\text{conv}}(t)&=j_{\text{conv}}(z=0,t)\\ &=2(\epsilon-\epsilon_c)SZ(t)+\mathcal{O}(\epsilon-\epsilon_c)^{3/2} \ , \end{aligned} \tag{4.3}$$

where S and the function $Z(t)$ are explained further below. The time-averaged convective current is independent of z:

$$\langle j_{\text{conv}}(z,t)\rangle=\langle j_{\text{conv}}(t)\rangle=2(\epsilon-\epsilon_c)S+\mathcal{O}(\epsilon-\epsilon_c)^{3/2} \ . \tag{4.4}$$

The initial slope S (C6) follows from the nonlinear coefficient g of the amplitude equation. In Appendix C we evaluate the slope for small modulation amplitude Δ in the low-frequency regime,

$$S(\Delta,\omega)=1-\Delta^2 S^{(2)}(\omega)+\mathcal{O}(\Delta^4) \ . \tag{4.5}$$

The second-order coefficient $S^{(2)}$ enters into the shift $g^{(2)}$ [(3.12c) and (3.14c)] and is given by (3.15b), (3.15c), and (2.9). Since $S^{(2)}$ is positive for finite Prandtl number σ temperature modulation decreases the slope (see Fig. 7). Hence, modulation suppresses the growth of the temporally averaged convective heat current above threshold. For a modulation amplitude $\Delta=0.5$ we calculate the shift $\Delta^2 S^{(2)}(\omega)\simeq 0.08$ for low frequencies ω and for Prandtl numbers $\sigma\simeq 1$. Our result (4.5) for the slope agrees with the Lorenz model of Ahlers, Hohenberg, and Lücke [7] for free slip boundary conditions. The modulation induced reduction of the slope found in the experiments of Niemela and Donnelly [13], i.e., a system with rigid boundaries, is considerably larger.

The temporal variation of the convective heat current is given by the function $Z(t)$, which solves the equation

$$\left[\frac{1}{(2\pi)^2}\partial_t+1\right]Z(t)=\left[\frac{1}{2\sigma q_c^2}\partial_t+1\right]f_c^2(t)/\langle f_c^2(t)\rangle \ , \tag{4.6}$$

where the time dependence $f_c(t)$ of the vertical velocity field w follows from Eq. (2.14). We have evaluated $Z(t)$ both analytically and numerically. The results agree in the validity range $\Delta \lesssim \frac{1}{2}\omega\tau^{(0)}$ of the analytical expansion for small modulation amplitude Δ. For low frequencies one finds approximately

$$Z(t)\simeq f_c^2(t)/\langle f_c^2(t)\rangle \ , \tag{4.7}$$

which yields together with (2.17) directly the small Δ behavior.

The full time-dependent convective heat current $j_{\text{conv}}(t)$ is shown in Figs. 8 and 9 over one period for some values of Δ, ω, and ϵ at a fixed Prandtl number $\sigma=1$. The results are based on a numerical evaluation of Eqs. (2.14), (A35), and (A36) as well as the analytical formulas (2.6), (2.9), (4.5), (3.15b), and (3.15c). We note that Eqs. (A35) and (A36) together with (A47) are equivalent to Eq. (4.6). The plots present the equilibrated state after the transients have died out. The temporal behavior of j_{conv} being roughly governed by $f_c^2(t)$ has some features in common with the velocity field: the phase shift to the externally modulated drive slightly above threshold is around $\pi/2$, and the temporal mean is lowered by in-

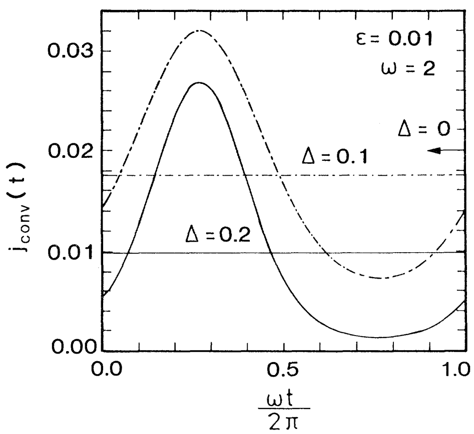

FIG. 8. The vertical convective heat current (4.3) is presented over one period of the temperature modulation $R(t)/R_c^{\rm stat}=(1+\epsilon)[1+\Delta\cos(\omega t)]$ for $\sigma=1$. Increasing the modulation amplitude—more precisely the ratio $\Delta/(\omega\tau^{(0)})$—increases the "anharmonicity" of the current and decreases its temporal average (4.4) (horizontal lines) relative to the unmodulated case (arrow).

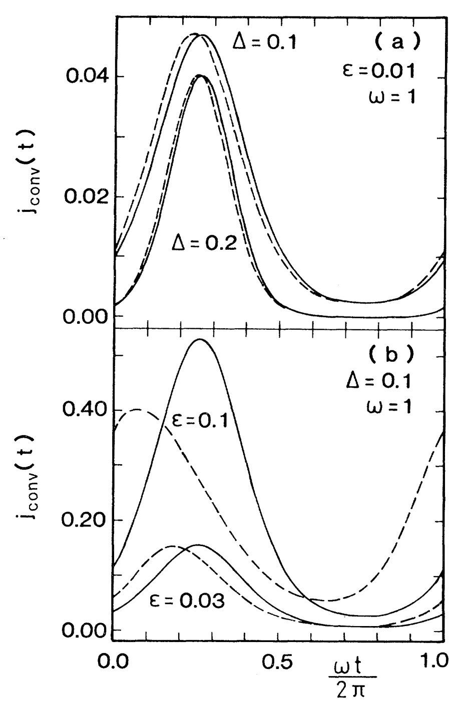

FIG. 9. Comparison of the convective current (4.3) resulting from our amplitude equation approach (solid lines) with the result of the Lorenz model [6,7] (dashed lines) for different parameters ϵ and Δ of the temperature modulation $R(t)/R_c^{\rm stat}=(1+\epsilon)[1+\Delta\cos(\omega t)]$. In each case $\omega=1$ and $\sigma=1$. In (a) for $\epsilon=0.01$ with the two different modulation amplitudes Δ shown we expect the convective heat current to be quantitatively correct. Increasing ϵ in (b) for fixed $\Delta=0.1$ higher orders in the distance from threshold are no longer negligible. Note the different ordinate scales in (a) and (b).

creasing Δ. Furthermore $j_{\rm conv}\sim f_c^2(t)$ becomes more "anharmonic" when the quotient $\Delta/(\omega\tau^{(0)})$ is increased (cf. Sec. II B), which can be done by changing Δ, ω, or σ.

Finally, we compare in Fig. 9 the result of the time-dependent convective heat current given by the amplitude equation with the prediction of the Lorenz model of Ahlers, Hohenberg, and Lücke [6,7]. Close to the threshold ϵ_c quantitative agreement of both methods is found by analytical as well as numerical evaluation. The present amplitude equation yields the exact result *in the limit* $\epsilon\rightarrow\epsilon_c$ for small Δ and ω, and the Lorenz model reproduces it with the exception of the slight Δ, ω, and σ dependence of the critical wave number of the convection rolls which has not been included [6,7]. While the correction to the critical wave number $k_c^{(0)}$ induced by modulation is needed to derive the amplitude equation, the difference that results from discarding it is not recognizable on the plots of the convective heat current in Figs. 8 and 9. The amplitude equation as well as the Lorenz model give the convective heat current correctly only to first order in $\epsilon-\epsilon_c$. The ϵ range in which higher orders in $\epsilon-\epsilon_c$ resulting from higher modes are negligible is not obvious. We found that for $\epsilon-\epsilon_c=\mathcal{O}(10^{-3})$ the agreement between Lorenz model [6,7] and amplitude equation is perfect [25] but already for $\epsilon-\epsilon_c=\mathcal{O}(10^{-2})$ there are deviations [Fig. 9(a)] that grow with increasing supercritical driving as shown in Fig. 9(b) for $\epsilon-\epsilon_c=\mathcal{O}(0.1)$.

V. CONCLUSION

With a systematic perturbation theory of the hydrodynamic field equations we have derived an amplitude equation to study the influence of harmonic temperature modulation on the thermal instability in the Rayleigh-Bénard problem for stress-free boundary conditions. This method determines the convection fields of a straight roll pattern at threshold exactly.

Our results are restricted to modulation with moderately low frequencies. This case is interesting because the effects of modulation are strongest if the period of the drive is of the order of the vertical diffusion time. On the other hand, the modulation period is short compared to the amplitude growth time $\tau/(\epsilon-\epsilon_c)$ so that these two time scales are well separated.

Temperature modulation shifts the convective threshold, the critical wave number, the curvature of the stability curve in the critical point, the relaxation time of the amplitude, and, via nonlinear interaction, the slope of the convective heat current. The temporally periodic response of the vertical velocity field and of the convective heat current is discussed.

The purpose of the present paper is motivated theoretically rather than experimentally. The main idea was to develop an appropriate amplitude equation description for the parametrically modulated Rayleigh-Bénard problem when the pattern formation involves fast as well as slow time scales being well separated. To elucidate the method, we have used some idealizations and approximations to simplify the system under investigation. However, we should like to note that the results of the amplitude equation agree with the Lorenz model of Ahlers, Hohenberg, and Lücke [6–8] for free boundaries, and also qualitatively with measurements [6,8,13].

The following extensions, which are aimed at incorporating the experimental setup more quantitatively, are partly applied in other theories and are able to improve agreement with experiments. Rigid boundary conditions [2,7,8,10,15] complicate the vertical dependence of the velocity field and influence both linear and nonlinear results in a quantitative manner. Side walls of the cell lead to an imperfect bifurcation and force convection [6–8,11,13,18]. Deterministic and stochastic effects [11,19] emerge during the pattern formation. Since experiments [6,8,11,13,14] often realize stronger supercritical drives and larger modulation amplitudes, it is ultimately desirable to include higher orders in the perturbation theory.

APPENDIX A: NONLINEAR PERTURBATION THEORY

Here we give formulas to clarify the perturbation procedure leading to the amplitude equation for the convective fields of a roll pattern.

1. System of equations

Eliminating the pressure from the momentum balance (2.3a) and combining it with the heat balance (2.3b) we transform the Oberbeck-Boussinesq equations into the system

$$\mathcal{L}w=N\ , \tag{A1a}$$

$$(\partial_t-\partial^2)\theta=R_0C(z,t)w-(\mathbf{u}\cdot\boldsymbol{\partial})\theta\ , \tag{A1b}$$

$$\boldsymbol{\partial}\cdot\mathbf{u}=0\ . \tag{A1c}$$

The linear differential operator is defined by

$$\mathcal{L}=(\partial_t-\partial^2)\left[\frac{1}{\sigma}\partial_t-\partial^2\right]\partial^2-R_0C(z,t)\partial_x^2\ , \tag{A2}$$

and the convective nonlinearities read

$$N=-\partial_x^2[(\mathbf{u}\cdot\boldsymbol{\partial})\theta]+\frac{1}{\sigma}(\partial_t-\partial^2)\mathbf{e}_z\cdot\boldsymbol{\partial}\times\{\boldsymbol{\partial}\times[(\mathbf{u}\cdot\boldsymbol{\partial})\mathbf{u}]\}\ . \tag{A3}$$

Temperature modulation affects the heat balance [(2.3b) and (A1b)], thus entering (A1a) also, via the z- and t-dependent vertical conductive heat current

$$J_{\text{cond}}(z,t)=-\partial_z T_{\text{cond}}(z,t)=R_0C(z,t) \tag{A4a}$$

with

$$C(z,t)=1+\Delta c(z,t)\ , \tag{A4b}$$

$$\Delta c(z,t)=\tfrac{1}{2}[\Delta_l\Xi(z)-\Delta_u\Xi(1-z)e^{i\phi}]e^{-i\omega t}+\text{c.c.}\ , \tag{A4c}$$

$$\Xi(z)=\sqrt{i\omega}\,\frac{\cos[\sqrt{i\omega}(1-z)]}{\sin(\sqrt{i\omega})}\ . \tag{A4d}$$

It describes a wavelike profile and follows the external drive with the same frequency ω. To solve the system (A1)–(A4), we expand the convection fields $\mathbf{u}=(u,0,w)$ and θ into normal modes. In bra and ket notation, which we find advantageous for evaluating scalar products, we use plane waves with wave number k

$$|k\rangle=e^{ikx} \tag{A5}$$

for the lateral expansion and trigonometric functions

$$|\mathcal{S}(n\pi)\rangle=\sqrt{2}\sin(n\pi z) \tag{A6}$$

for the vertical expansion. The above spatial variations are combined in the form

$$|\mathcal{S}(\mathbf{q})\rangle=|k\rangle|\mathcal{S}(\pi)\rangle=e^{ikx}\sqrt{2}\sin(\pi z)\ . \tag{A7}$$

Furthermore we express the real temporally periodic functions of period $2\pi/\omega$ in terms of the Fourier series:

$$f(t)=\tfrac{1}{2}\sum_{\mu=0}^{\infty}f_\mu e^{-i\mu\omega t}+\text{c.c.} \tag{A8}$$

To formulate mode projections and solvability conditions and to calculate averages, we use the standard scalar product for finite periodic functions V and W:

$$\langle V|W\rangle=\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}dx\int_0^1dz\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\ V^*(\mathbf{x},t)W(\mathbf{x},t)\ . \tag{A9}$$

2. Perturbation expansion

The introduction of slow length and time scales in the framework of the multiple scale analysis generates the expansion

$$\partial_x\to\partial_x+\eta\nabla_{X_1}+\eta^2\nabla_{X_2}+\cdots\ , \tag{A10a}$$

$$\partial_t\to\partial_t+\eta\partial_{T_1}+\eta^2\partial_{T_2}+\cdots\ , \tag{A10b}$$

with

$$\nabla_{X_n}=\frac{\partial}{\partial X_n}\ ,\quad \partial_{T_n}=\frac{\partial}{\partial T_n}\ . \tag{A11}$$

Inserting the Poincaré-Lindstedt expansion and the multiple scale analysis into (A1a) the successive orders in η yield the equations

$$\mathcal{L}_0w_0=0\ , \tag{A12a}$$

$$\mathcal{L}_0w_1+\mathcal{L}_1w_0=N_0\ , \tag{A12b}$$

$$\mathcal{L}_0w_2+\mathcal{L}_1w_1+\mathcal{L}_2w_0=N_1\ . \tag{A12c}$$

Here the linear operator $\mathcal{L}$ is decomposed into

$$\mathcal{L}=\mathcal{L}_0+\eta\mathcal{L}_1+\eta^2\mathcal{L}_2+\cdots\ , \tag{A13}$$

$$\mathcal{L}_0=(\partial_t-\partial^2)\left[\frac{1}{\sigma}\partial_t-\partial^2\right]\partial^2-R_c^{\mathrm{stat}}(1+\epsilon_c)C(z,t)\partial_x^2\ , \tag{A14a}$$

$$\mathcal{L}_1=\left[\frac{2}{\sigma}\partial_t-\frac{\sigma+1}{\sigma}\partial^2\right]\partial^2\partial_{T_1}+\left[(\partial_t-\partial^2)\left[\frac{1}{\sigma}\partial_t-\partial^2\right]-\left[\frac{\sigma+1}{\sigma}\partial_t-2\partial^2\right]\partial^2\right]2\partial_x\nabla_{X_1}$$
$$-R_c^{\mathrm{stat}}C(z,t)[(1+\epsilon_c)2\partial_x\nabla_{X_1}+\epsilon_1\partial_x^2]\ , \tag{A14b}$$

$$\mathcal{L}_2=\left[\frac{2}{\sigma}\partial_t-\frac{\sigma+1}{\sigma}\partial^2\right]\left[\partial^2\partial_{T_2}+2\partial_x\nabla_{X_1}\partial_{T_1}\right]+\partial^2\left[\frac{1}{\sigma}\partial_{T_1}^2-\frac{\sigma+1}{\sigma}2\partial_x\nabla_{X_1}\partial_{T_1}\right]$$
$$+\left[(\partial_t-\partial^2)\left[\frac{1}{\sigma}\partial_t-\partial^2\right]-\left[\frac{\sigma+1}{\sigma}\partial_t-2\partial^2\right]\partial^2\right](\nabla_{X_1}^2+2\partial_x\nabla_{X_2})-\left[\frac{\sigma+1}{\sigma}\partial_t-3\partial^2\right]4\partial_x^2\nabla_{X_1}^2$$
$$-R_c^{\mathrm{stat}}C(z,t)[(1+\epsilon_c)(\nabla_{X_1}^2+2\partial_x\nabla_{X_2})+\epsilon_1 2\partial_x\nabla_{X_1}+\epsilon_2\partial_x^2]\ , \tag{A14c}$$

and the related nonlinear inhomogeneities

$$N=\eta^2(N_0+\eta N_1\cdots) \tag{A15}$$

read

$$N_0=-\partial_x^2[(\mathbf{u}_0\cdot\boldsymbol{\partial})\theta_0]+\frac{1}{\sigma}(\partial_t-\partial^2)\mathbf{e}_z\cdot\boldsymbol{\partial}\times\{\boldsymbol{\partial}\times[(\mathbf{u}_0\cdot\boldsymbol{\partial})\mathbf{u}_0]\}\ , \tag{A16a}$$

$$N_1=-\partial_x^2[(\mathbf{u}_0\cdot\boldsymbol{\partial})\theta_1+u_0\nabla_{X_1}\theta_0+(\mathbf{u}_1\cdot\boldsymbol{\partial})\theta_0]-2\partial_x\nabla_{X_1}[(\mathbf{u}_0\cdot\boldsymbol{\partial})\theta_0]$$
$$+\frac{1}{\sigma}(\partial_t-\partial^2)\mathbf{e}_z\cdot(\boldsymbol{\partial}\times\{\boldsymbol{\partial}\times[(\mathbf{u}_0\cdot\boldsymbol{\partial})\mathbf{u}_1+u_0\nabla_{X_1}\mathbf{u}_0+(\mathbf{u}_1\cdot\boldsymbol{\partial})\mathbf{u}_0]+\mathbf{e}_x\times\nabla_{X_1}[(\mathbf{u}_0\cdot\boldsymbol{\partial})\mathbf{u}_0]\}$$
$$+\mathbf{e}_x\times\nabla_{X_1}\{\boldsymbol{\partial}\times[(\mathbf{u}_0\cdot\boldsymbol{\partial})\mathbf{u}_0]\})+\frac{1}{\sigma}(\partial_{T_1}-2\partial_x\nabla_{X_1})\mathbf{e}_z\cdot\boldsymbol{\partial}\times\{\boldsymbol{\partial}\times[(\mathbf{u}_0\cdot\boldsymbol{\partial})\mathbf{u}_0]\}\ . \tag{A16b}$$

$\mathbf{e}_x$ and $\mathbf{e}_z$ are unit vectors. We note that the arrangement of the derivatives with respect to fast and slow variables in the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, together with the ansatz of convection rolls [(A19a), (A20a), (A30a), and (A31a)], allows an identification of the linear coefficients τ, ζ, and ξ^2 of the amplitude equation according to Eqs. (C1)–(C3), (C5), and (C7).

For a roll pattern under low-frequency modulation the convective momentum transport drops out, and only the contribution of the convective heat transfer $(\mathbf{u}_0\cdot\boldsymbol{\partial})\theta_1$ in order η^3 survives in the nonlinear decomposition [(A15) and (A16)], as in the case of stationary drive. The first two η orders of Eq. (A1b) for the temperature field read

$$(\partial_t-\partial^2)\theta_0=R_c^{\mathrm{stat}}(1+\epsilon_c)C(z,t)w_0\ , \tag{A17a}$$

$$(\partial_t-\partial^2)\theta_1+(\partial_{T_1}-2\partial_x\nabla_{X_1})\theta_0$$
$$=R_c^{\mathrm{stat}}C(z,t)[(1+\epsilon_c)w_1+\epsilon_1 w_0]-(\mathbf{u}_0\cdot\boldsymbol{\partial})\theta_0\ . \tag{A17b}$$

The nonlinear behavior discussed in this work arises from the term $(\mathbf{u}_0\cdot\boldsymbol{\partial})\theta_0$ in the heat balance in order η^2 (A17b). The equation of continuity (A1c) is decomposed into

$$\boldsymbol{\partial}\cdot\mathbf{u}_0=0\ , \tag{A18a}$$

$$\boldsymbol{\partial}\cdot\mathbf{u}_1+\nabla_{X_1}u_0=0\ . \tag{A18b}$$

In order η the Oberbeck-Boussinesq equations are solved by the critical fields for roll convection,

$$w_0(\mathbf{x},X,t,T)=\hat{w}_0(X,t,T)\mathcal{S}(\mathbf{q}_c)+\mathrm{c.c.}\ , \tag{A19a}$$

$$\theta_0(\mathbf{x},X,t,T)=\hat{\theta}_0(X,t,T)\mathcal{S}(\mathbf{q}_c)+\mathrm{c.c.}\ , \tag{A19b}$$

with u_0 following via incompressibility (A18a). The fast time dependence of the critical functions and the slow spatiotemporal variations of the amplitude are factorized:

$$\hat{w}_0(X,t,T)=A_0(X,T)f_c(t)\ , \tag{A20a}$$

$$\hat{\theta}_0(X,t,T)=A_0(X,T)\varphi_c(t)\ . \tag{A20b}$$

The fast critical response $f_c(t)$ of the velocity field follows from Eq. (2.14), rewritten in the form

$$\hat{\mathcal{L}}_0(\partial_t,t)f_c(t)=0 \tag{A21}$$

with the linear differential operator at threshold

$$\hat{\mathcal{L}}_0(\partial_t,t)=-q_c^2(\partial_t+q_c^2)\left[\frac{1}{\sigma}\partial_t+q_c^2\right]$$
$$+k_c^2R_c^{\mathrm{stat}}(1+\epsilon_c)C(t)\ , \tag{A22}$$

and

$$\varphi_c(t)=\frac{q_c^2}{k_c^2}\left[\frac{1}{\sigma}\partial_t+q_c^2\right]f_c(t) \tag{A23}$$

is the marginal temporal variation of the temperature field. Here $C(t)$ (2.15) is the relevant part of the time-dependence of the conductive heat current (A4) as given by the matrix element of $C(z,t)$ between the first vertical free slip mode

$$C(t)=\langle \mathcal{S}(\pi)|C(z,t)|\mathcal{S}(\pi)\rangle_z \ . \tag{A24}$$

We also have to solve the adjoint equation

$$\mathcal{L}_0^\dagger w_0^\dagger=0 \ , \tag{A25}$$

with the adjoint linear operator

$$\mathcal{L}_0^\dagger=(\partial_t+\partial^2)\left[\frac{1}{\sigma}\partial_t+\partial^2\right]\partial^2-R_c^{\text{stat}}(1+\epsilon_c)C(z,t)\partial_x^2 \ , \tag{A26}$$

by

$$w_0^\dagger(\mathbf{x},t)=f_c^\dagger(t)\mathcal{S}(\mathbf{q}_c) \ . \tag{A27}$$

Here $\mathcal{L}_0^\dagger$ is defined by the scalar product (A9) with respect to finite periodic functions whose z dependence is expressed into normal modes $\mathcal{S}(n\pi)$ (A6).

3. Results in order η^2 and η^3

Since $N_0=0$, the Fredholm alternative in order η^2 is linear in the fields:

$$\langle w_0^\dagger|\mathcal{L}_1|w_0\rangle=0 \ . \tag{A28}$$

It is transformed into

$$(-\tau\partial_{T_1}+\xi\nabla_{X_1}+\epsilon_1)A_0(X,T)=0 \ , \tag{A29}$$

with the relaxation time τ and the quantity ξ defined in the Eqs. (C1) and (C2), respectively, in terms of scalar products of $f_c(t)$ and the adjoint $f_c^\dagger(t)$ together with the linear operators and functions [(C7a), (C7b), and (C7d)] induced by multiple scale analysis. Our treatment in Appendix B then shows the simplification of Eqs. (3.4) and (A29) to (3.5)–(3.7).

Equations (A12b) and (A17b) in order η^2 are solved by

$$w_1(\mathbf{x},X,t,T)=\hat{w}_1(X,t,T)\mathcal{S}(\mathbf{q}_c)+\text{c.c.} \ , \tag{A30a}$$

$$\theta_1(\mathbf{x},X,t,T)=[\hat{\theta}_{1,1}(X,t,T)\mathcal{S}(\mathbf{q}_c)+\text{c.c.}] -\hat{\theta}_{1,2}(X,t,T)\frac{1}{\sqrt{2}}\mathcal{S}(2\pi) \tag{A30b}$$

and u_1 follows from u_0 and w_1 via incompressibility (A18b). The part varying with $\mathcal{S}(\mathbf{q}_c)$ arises from the linear terms in the differential equations. The convective nonlinearity $(\mathbf{u}_0\cdot\partial)\theta_0$ of the heat balance equation (A17b) contributes with the vertical dependence $\sin(2\pi z)$ to the temperature field. In the linear portion the rapid temporal variations and the slow dependences of space and time,

$$\hat{w}_1(X,t,T)=A_1(X,T)f_c(t)-2ik_c\nabla_{X_1}A_0(X,T)h(t) \ , \tag{A31a}$$

$$\hat{\theta}_{1,1}(X,t,T)=A_1(X,T)\varphi_c(t)+2ik_c\nabla_{X_1}A_0(X,T)\psi(t) \ , \tag{A31b}$$

consist of a homogeneous contribution $A_1(X,T)$ and an inhomogeneity connected with $\nabla_{X_1}A_0(X,T)$ due to multiple scale analysis. Apart from the response of the system to the modulated drive with the fast time scale t, the field structure of (A30) and (A31) corresponds to that of the stationary problem. From the coupling of the external modulation to the inhomogeneity only the solution $h(t)$ of the equation

$$\hat{\mathcal{L}}_0(\partial_t,t)h(t)=q_c^{(0)6}\mathcal{L}_X(\partial_t,t)f_c(t) \tag{A32}$$

has to be determined because it turns up in the curvature ξ^2 (C3), while

$$\psi(t)=\left[\frac{\pi^2}{k_c^4}\left[\frac{1}{\sigma}\partial_t+q_c^2\right]-\frac{q_c^2}{k_c^2}\right]f_c(t) -\frac{q_c^2}{k_c^2}\left[\frac{1}{\sigma}\partial_t+q_c^2\right]h(t) \tag{A33}$$

is not needed in detail for results discussed here. The operator $\mathcal{L}_X(\partial_t,t)$ is introduced by Eq. (C7b).

The convective nonlinearity produces the cubic nonlinearity in the amplitude equation [(1.4) and (3.9)] and enters into the convective heat current [(A45) and (A46)]. Its fast time dependence $\vartheta(t)$ is defined so that

$$\hat{\theta}_{1,2}(X,t,T)=|A_0(X,T)|^2\vartheta(t) \tag{A34}$$

and obeys the equation

$$\left[\frac{1}{(2\pi)^2}\partial_t+1\right]\vartheta(t)=\mathcal{N}(t) \ , \tag{A35}$$

with an inhomogeneity

$$\mathcal{N}(t)=\frac{1}{\pi}f_c(t)\varphi_c(t)=\frac{1}{2\pi}\frac{q_c^2}{k_c^2}\left[\frac{1}{\sigma}\partial_t+2q_c^2\right]f_c^2(t) \ . \tag{A36}$$

To evaluate the nonlinear solvability condition in order η^3,

$$\langle w_0^\dagger|\mathcal{L}_1w_1+\mathcal{L}_2w_0-N_1\rangle=0 \ , \tag{A37}$$

we collect the fields in order η and in order η^2. The nonlinearity arises from the contribution of the $\mathcal{S}(2\pi)$ mode of the temperature field to the expression $(\mathbf{u}_0\cdot\partial)\theta_1$ in (A16b). One finds

$$-\tau\partial_{T_1}A_1(X,T)+[-\tau\partial_{T_2}+\xi^2\nabla_{X_l}^2 +\epsilon_2-g|A_0(X,T)|^2]A_0(X,T)=0 \ . \tag{A38}$$

Since A_0 is independent of T_1 according to (3.6), the amplitude $A_1(X,T)$ does not show secular growth if

$$\partial_{T_1}A_1(X,T)=0 \tag{A39}$$

so that Eq. (A38) reduces to (3.8). The relaxation time τ, the curvature ξ^2 of the stability curve, and the nonlinear coefficient g are defined in (C1), (C3), and (C4), respectively. Their evaluation for weak modulation is provided by Appendix C with a perturbation expansion in the modulation amplitude Δ.

4. Convection fields

Eliminating the technical auxiliary variables from (A19), (A20), (A30), (A31), and (A34), the convection fields read

$$w(\mathbf{x},t)=\mathcal{S}(\mathbf{q}_c)[f_c(t)-h(t)2ik_c\partial_x]A(x,t)+\text{c.c.}+\mathcal{O}(\epsilon-\epsilon_c)^{3/2}\ , \qquad \text{(A40a)}$$

$$\theta(\mathbf{x},t)=\{\mathcal{S}(\mathbf{q}_c)[\varphi_c(t)+\psi(t)2ik_c\partial_x]A(x,t)+\text{c.c.}\}-|A(x,t)|^2\vartheta(t)\frac{1}{\sqrt{2}}\mathcal{S}(2\pi)+\mathcal{O}(\epsilon-\epsilon_c)^{3/2}\ . \qquad \text{(A40b)}$$

Here $A(x,t)$, being the solution of (3.9) with boundary and initial conditions that are still to be specified, is of order $(\epsilon-\epsilon_c)^{1/2}$ with corrections $\mathcal{O}(\epsilon-\epsilon_c)$: Consider for the sake of simplicity, e.g., the special case of a spatially periodic solution,

$$A(x,t)=\left[\left[\frac{\epsilon-\epsilon_c-\xi^2Q^2}{g}\right]^{1/2}\frac{|A(t=0)|\Lambda(t)}{\{(\epsilon-\epsilon_c-\xi^2Q^2)/g+|A(t=0)|^2[\Lambda^2(t)-1]\}^{1/2}}+\mathcal{O}(\epsilon-\epsilon_c)\right]e^{i(Qx+\chi)}\ , \qquad \text{(A41)}$$

that relaxes globally with exponential time dependence

$$\Lambda(t)=\exp\left[(\epsilon-\epsilon_c-\xi^2Q^2)\frac{t}{\tau}\right] \qquad \text{(A42)}$$

toward $[(\epsilon-\epsilon_c-\xi^2Q^2)/g]^{1/2}e^{i(Qx+\chi)}$. Here $Q=k-k_c$ is the distance of the wave number k from the critical value k_c. For this solution the contributions $\partial_x A(x,t)$ and $|A(x,t)|^2$ in (A40) are of order $\epsilon-\epsilon_c$. Thus, the fast temporal response of w,θ (A40) is dominated by $f_c(t)$ and $\varphi_c(t)$. In the middle of the band, i.e., for $Q=0$, the amplitude (A41) is spatially constant. For this case the long-time behavior of the fields is given by

$$\begin{bmatrix} w(\mathbf{x},t)\\ \theta(\mathbf{x},t)\end{bmatrix}=\left[\frac{\epsilon-\epsilon_c}{g}\right]^{1/2}e^{i\chi}\begin{bmatrix} f_c(t)\\ \varphi_c(t)\end{bmatrix}\mathcal{S}(\mathbf{q}_c)+\text{c.c.}+\mathcal{O}(\epsilon-\epsilon_c)\ . \qquad \text{(A43)}$$

5. Vertical heat current

Finally, we compile the formulas for the laterally averaged vertical convective heat current

$$J_{\text{conv}}(z,t)=\langle w(\mathbf{x},t)T(\mathbf{x},t)-\partial_z\theta(\mathbf{x},t)\rangle_\perp\ . \qquad \text{(A44)}$$

For the fields (A40) and (A41) the conductive part of the temperature field drops out, and using (A35) and (A36) we get

$$J_{\text{conv}}(z,t)=2\pi|A(t)|^2\left[1+\frac{1}{2\pi^2}\sin^2(\pi z)\partial_t\right]\vartheta(t)+\mathcal{O}(\epsilon-\epsilon_c)^{3/2} \qquad \text{(A45)}$$

with $|A(t)|^2$ denoting the absolute square of (A41). In the equilibrated state one finds the z-dependent current

$$J_{\text{conv}}(z,t)=2\pi\frac{\epsilon-\epsilon_c}{g}\left[1+\frac{1}{2\pi^2}\sin^2(\pi z)\partial_t\right]\vartheta(t)+\mathcal{O}(\epsilon-\epsilon_c)^{3/2} \qquad \text{(A46)}$$

for $k=k_c$. Introducing

$$Z(t)=\vartheta(t)/\langle\vartheta(t)\rangle \qquad \text{(A47)}$$

and using the relation (C6) of the nonlinear coefficient g of the amplitude equation to the slope S of the convective heat current, the results (4.2)–(4.4) follow in the normalization

$$j_{\text{conv}}(z,t)=\frac{1}{R_c^{\text{stat}}}J_{\text{conv}}(z,t)\ . \qquad \text{(A48)}$$

Analytical solutions of the time dependences $f_c(t)$, $h(t)$, $\vartheta(t)$, and $Z(t)$ involved in the convective fields and in the convective heat current are given in (2.17) and in Appendix C as far as they enter into the coefficients of the amplitude equation and into the slope of the convective heat current.

APPENDIX B: EXTENSION OF THE METHOD OF NEWELL *et al.* [27–29] TO MODULATED HEATING

In this appendix we present an additional linear theory [30] for ϵ slightly above the convective threshold $\epsilon(k)$. It allows us to identify the quantities τ, ζ, and ξ^2 entering the solvability conditions with derivatives of the linear growth rate $s(\epsilon,k)$, on the one hand, and with matrix elements of linear operators between marginal linear periodic functions, on the other hand.

First, the marginal periodic state function $F(t;k)=F(t;k,\Delta,\omega)$ of the $\mathcal{S}(\mathbf{q})$ mode (A7) of the vertical velocity field w solves the equation

$$\hat{L}(\partial_t,t;\epsilon(k),k)F(t;k)=0 \qquad \text{(B1)}$$

at the stability threshold $\epsilon(k)=\epsilon(k,\Delta,\omega)$, with the operator

$$\hat{L}(\partial_t,t;\epsilon,k)=-q^2(\partial_t+q^2)\left[\frac{1}{\sigma}\partial_t+q^2\right]$$
$$+k^2R_c^{\text{stat}}(1+\epsilon)C(t)\ , \tag{B2a}$$
$$q^2=k^2+\pi^2\ , \tag{B2b}$$

resulting from linear field equations. We note that the Eqs. (2.14), (A21), and (A22) represent (B1) and (B2) at the critical point

$$k=k_c\ ,\quad \epsilon(k_c)=\epsilon_c\ . \tag{B3}$$

Thus, the critical time dependence $f_c(t)$ is given by the marginal function for $k=k_c$,

$$f_c(t)=F(t;k_c)\ , \tag{B4}$$

and the critical linear operator (A22) is related with (B2a) by

$$\hat{\mathcal{L}}_0(\partial_t,t)=\hat{L}(\partial_t,t;\epsilon_c,k_c)\ . \tag{B5}$$

Furthermore, the function $F^\dagger(t;k)=F^\dagger(t;k,\Delta\omega)$ obeys Eqs. (B1) and (B2), replacing ∂_t by $-\partial_t$, so that

$$f_c^\dagger(t)=F^\dagger(t;k_c)\ . \tag{B6}$$

Then, we consider the exponential growth behavior of the amplitude

$$A(t;\epsilon,k)=Ae^{s(\epsilon,k)t} \tag{B7}$$

of the $\mathcal{S}(\mathbf{q})$ mode described by $s(\epsilon,k)=s(\epsilon,k,\Delta,\omega)$. From our linear equation

$$0=\hat{L}(\partial_t,t;\epsilon,k)[A(t;\epsilon,k)F(t;k)] \tag{B8a}$$
$$=A(t;\epsilon,k)\hat{L}(s(\epsilon,k)+\partial_t,t;\epsilon,k)F(t;k) \tag{B8b}$$

the solubility condition requires the disappearance of the time average

$$L(s(\epsilon,k),\epsilon,k)=\langle F^\dagger(t;k)|\hat{L}(s(\epsilon,k)+\partial_t,t;\epsilon,k)|F(t;k)\rangle=0\ , \tag{B9}$$

which determines the linear growth exponent $s(\epsilon,k)$. Using (B1) and (B2), we simplify (B9) to the expression

$$L(s(\epsilon,k),\epsilon,k)=\langle F^\dagger(t;k)|\tilde{L}(\partial_t,t;s(\epsilon,k),\epsilon,k)|F(t;k)\rangle=0\ , \tag{B10}$$

with

$$\tilde{L}(\partial_t,t;s(\epsilon,k),\epsilon,k)=-\frac{q^2}{\sigma}s^2(\epsilon,k)-q_c^{(0)6}\{s(\epsilon,k)L_T(\partial_t;k)-[\epsilon-\epsilon(k)]L_\epsilon(t;k)\}\ , \tag{B11a}$$

$$L_T(\partial_t;k)=\frac{1}{q_c^{(0)6}}q^2\left[\frac{2}{\sigma}\partial_t+\frac{\sigma+1}{\sigma}q^2\right]\ , \tag{B11b}$$

$$L_\epsilon(t;k)=\frac{k^2}{k_c^{(0)2}}C(t)\ . \tag{B11c}$$

Solving the quadratic equation [(B10) and (B11a)] for the characteristic growth exponent $s(\epsilon,k)$ of the amplitude of the $\mathcal{S}(\mathbf{q})$ mode we obtain

$$s(\epsilon,k,\Delta,\omega)=-s_T(k,\Delta,\omega)+\{[\epsilon-\epsilon(k,\Delta,\omega)]s_\epsilon(k,\Delta,\omega)+s_T^2(k,\Delta,\omega)\}^{1/2}\ . \tag{B12}$$

In this expression there enter besides the stability curve $\epsilon(k,\Delta,\omega)$ also the temporal averages

$$s_T(k)=\tfrac{1}{2}\sigma\frac{q_c^{(0)6}}{q^2}\frac{1}{s_F(k)}\langle F^\dagger(t;k)|L_T(\partial_t;k)|F(t;k)\rangle\ , \tag{B13a}$$

$$s_\epsilon(k)=\sigma\frac{q_c^{(0)6}}{q^2}\frac{1}{s_F(k)}\langle F^\dagger(t;k)|L_\epsilon(t;k)|F(t;k)\rangle\ , \tag{B13b}$$

$$s_F(k)=\langle F^\dagger(t;k)|F(t;k)\rangle \tag{B13c}$$

over marginal stable time-periodic functions, where we have omitted the arguments Δ and ω again. Next we consider the total derivatives of the matrix element (B9) of the linear operator $\hat{L}(s(\epsilon,k)+\partial_t,t;\epsilon,k)$, (B2a), between the marginal functions $F^\dagger(t;k)$ and $F(t;k)$, with respect to ϵ and k,

$$\frac{d}{d\epsilon}L(s(\epsilon,k),\epsilon,k)=0\ , \tag{B14a}$$
$$\frac{d}{dk}L(s(\epsilon,k),\epsilon,k)=0\ , \tag{B14b}$$
$$\frac{d^2}{dk^2}L(s(\epsilon,k),\epsilon,k)=0\ , \tag{B14c}$$

and the total differential of the neutral state equation (B1),

$$\frac{d}{dk}[\hat{L}(\partial_t,t;\epsilon(k),k)F(t;k)]=0\ . \tag{B15}$$

Equations (B14) and (B15) are exploited at the critical point (B3) in particular. There, the above total derivatives involve partial derivatives of the linear differential operator $\hat{L}(s(\epsilon,k)+\partial_t,t;\epsilon,k)$, (B2a), relative to s, ϵ, and k, derivatives of the growth exponent $s(\epsilon,k)$, as well as k derivatives of $F(t;k)$ and $F^\dagger(t;k)$, all taken at the minimum of the stability curve (B3). Furthermore we need the relation

$$\partial_k F(t;k)\Big|_{k_c}=2k_c h(t) \tag{B16}$$

between the k derivative of the marginal time dependence $F(t;k)$ at the critical point and the temporal variation $h(t)$ of the vertical velocity field w in order η^2 [(A31a), (A32), and (C10)].

Then, with this method we identify the relations between the characteristic exponent (B12) and the linear coefficients τ, ζ, and ξ^2 (C1)–(C3), of the solvability conditions in order η^2 and in order η^3, thus, of the amplitude equation [(1.4) and (3.9)]: the relaxation time

$$\tau = \frac{1}{\partial_\epsilon s(\epsilon,k)}\bigg|_{\epsilon_c,k_c} \tag{B17}$$

of the amplitude, the quantity

$$\zeta = -i\tau\partial_k s(\epsilon,k)\bigg|_{\epsilon_c,k_c}, \tag{B18}$$

and the coefficient

$$\xi^2 = -\tau\tfrac{1}{2}\partial_k^2 s(\epsilon,k)\bigg|_{\epsilon_c,k_c}. \tag{B19}$$

It turns out that in our case no group velocity arises from the growth exponent (B12):

$$\partial_k s(\epsilon,k)\bigg|_{\epsilon_c,k_c} = 0. \tag{B20}$$

Hence, the term ζ of (3.4) and (A29) vanishes leading to (3.5) and (C2), so that the amplitude $A_0(X,T)$ does not vary on the time scale $T_1=\eta t$ (3.6), and the parameter ϵ_1 of the Poincaré-Lindstedt expansion drops out (3.7). Moreover, (B19) is reduced to the curvature of the stability curve at the critical point (2.12).

Finally, we mention that the comparison of the Taylor expansion

$$s(\epsilon,k) = [(\epsilon-\epsilon_c)\partial_\epsilon + (k-k_c)\partial_k + (k-k_c)^2\tfrac{1}{2}\partial_k^2]s(\epsilon,k)\bigg|_{\epsilon_c,k_c} + \mathcal{O}(\epsilon-\epsilon_c)^{3/2} \tag{B21}$$

of the growth exponent (B12) close to the critical point with the initial behavior (A42) of a time-dependent solution of the amplitude equation

$$s(\epsilon,k) = [\epsilon-\epsilon_c-(k-k_c)^2\xi^2]\frac{1}{\tau} + \mathcal{O}(\epsilon-\epsilon_c)^{3/2} \tag{B22}$$

shows the consistency of the expressions (B17), (B19), and (B20) directly.

APPENDIX C: COEFFICIENTS OF THE AMPLITUDE EQUATION AND SLOPE OF THE CONVECTIVE HEAT CURRENT: SMALL-Δ EXPANSION

In the solvability condition in order η^2 [(3.4) and (A29)] and in order η^3 [(3.8) and (A38)] leading to the amplitude equation [(1.4) and (3.9)] the following coefficients appear: the relaxation time

$$\tau = \frac{1}{E}\langle f_c^\dagger(t)|\mathcal{L}_T(\partial_t)|f_c(t)\rangle \tag{C1}$$

of the slowly varying amplitude, the term

$$\zeta = \frac{1}{E}\langle f_c^\dagger(t)|\mathcal{L}_X(\partial_t,t)|f_c(t)\rangle 2ik_c = 0 \tag{C2}$$

proportional to the vanishing group velocity, the curvature

$$\xi^2 = \frac{1}{E}[\langle f_c^\dagger(t)|\mathcal{L}_{X^2}(\partial_t)|f_c(t)\rangle + \langle f_c^\dagger(t)|4k_c^2\mathcal{L}_X(\partial_t,t)|h(t)\rangle] \tag{C3}$$

of the stability curve in the critical point, and the nonlinear coefficient

$$g = \frac{\pi k_c^2}{q_c^{(0)6}}\frac{1}{E}\langle f_c^\dagger(t)|\vartheta(t)|f_c(t)\rangle = \frac{\pi}{R_c^{\mathrm{stat}}}\frac{\langle f_c^\dagger(t)|\vartheta(t)|f_c(t)\rangle}{\langle f_c^\dagger(t)|C(t)|f_c(t)\rangle}, \tag{C4}$$

with the normalization by

$$E = \langle f_c^\dagger(t)|\mathcal{L}_\epsilon(t)|f_c(t)\rangle. \tag{C5}$$

The slope S of the convective heat current and the nonlinear coefficient g of the amplitude equation are related by

$$S = \frac{\pi}{R_c^{\mathrm{stat}}}\frac{\langle\vartheta(t)\rangle}{g} = \frac{\langle f_c^\dagger(t)|C(t)|f_c(t)\rangle}{\langle f_c^\dagger(t)|Z(t)|f_c(t)\rangle}, \tag{C6}$$

where we make use of (A47). The above scalar products arise after integrating the fast spatial coordinates x and z away so that the evaluation of averages on the time scale t remains. Whereas the linear coefficients τ, ζ, and ξ^2 are given by matrix elements of critical operators between critical fields, the nonlinear coefficient g and the slope S result from the coupling of the $\mathcal{S}(2\pi)$ mode of the temperature field to the vertical velocity field. Therefore, g and S are generated by temporal averages over these modes. In the definitions of the coefficients of the amplitude equation we introduce the following linear operators at the critical point ϵ_c, k_c:

$$\mathcal{L}_T(\partial_t) = \frac{1}{q_c^{(0)6}}q_c^2\left[\frac{2}{\sigma}\partial_t + \frac{\sigma+1}{\sigma}q_c^2\right], \tag{C7a}$$

$$\mathcal{L}_X(\partial_t,t) = \frac{1}{q_c^{(0)6}}\left[q_c^2\frac{\sigma+1}{\sigma}\partial_t + 2q_c^4 - \frac{\pi^2}{q_c^2}R_c^{\mathrm{stat}}(1+\epsilon_c)C(t)\right], \tag{C7b}$$

$$\mathcal{L}_{X^2}(\partial_t) = \frac{1}{q_c^{(0)6}}4k_c^2\left[\frac{\sigma+1}{\sigma}\partial_t + 3q_c^2\right], \tag{C7c}$$

$$\mathcal{L}_\epsilon(t) = \frac{k_c^2}{k_c^{(0)2}}C(t). \tag{C7d}$$

The definitions (C1)–(C3) of the linear coefficients τ, ζ, and ξ^2 are consistent with (B17)–(B19) and (2.12) obtained by an argumentation starting from the linear growth rate (B12). Since $\zeta=0$, we outline in the following the evaluation of τ, ξ^2, g, and S.

To this end we solve the differential equations governing the temporal response of the low-frequency modes of the vertical velocity field w and of the temperature field θ, i.e., we seek the periodic solutions $f_c(t)$, $f_c^\dagger(t)$, $h(t)$, and $\vartheta(t)$ of (A21), (A22), (A25)–(A27), (A32), and (A35). For weak modulation we expand into powers of the modulation amplitude Δ. Then the critical time dependence $f_c(t)$ of the velocity field w reads

$$f_c(t;\Delta)=1+\Delta f^{(1)}(t)+\Delta^2 f^{(2)}(t)+\Delta^3 f^{(3)}(t)+\mathcal{O}(\Delta^4)\ , \tag{C8}$$

where

$$f^{(1)}(t)=\tfrac{1}{2}f_1^{(1)}e^{-i\omega t}+\text{c.c.}\ , \tag{C9a}$$

$$f^{(2)}(t)=\tfrac{1}{2}f_2^{(2)}e^{-2i\omega t}+\text{c.c.}\ , \tag{C9b}$$

$$f^{(3)}(t)=\tfrac{1}{2}f_3^{(3)}e^{-3i\omega t}+\tfrac{1}{2}f_1^{(3)}e^{-i\omega t}+\text{c.c.}\ , \tag{C9c}$$

so that $\langle f^{(n)}(t)\rangle=0$. The expansion of the adjoint function $f_c^\dagger(t)$ is analogous. The function $h(t)$ in the form

$$h(t;\Delta)=\Delta h^{(1)}(t)+\Delta^2 h^{(2)}(t)+\Delta^3 h^{(3)}(t)+\mathcal{O}(\Delta^4)\ , \tag{C10}$$

with $h^{(1)}(t)$, $h^{(2)}(t)$, and $h^{(3)}(t)$ similar to (C9), is an effect of modulation—$h(\Delta=0)=0$. Finally, we need the expansion of $\vartheta(t)$ in the nonlinear part of the temperature field θ,

$$\vartheta(t;\Delta)=\vartheta^{(0)}[1+\Delta\vartheta^{(1)}(t)+\Delta^2\vartheta^{(2)}(t)+\Delta^3\vartheta^{(3)}(t)+\mathcal{O}(\Delta^4)]\ , \tag{C11}$$

with $\vartheta^{(1)}(t)$ and $\vartheta^{(3)}(t)$ as (C9a) and (C9c) and

$$\vartheta^{(0)}=\frac{1}{\pi}\frac{R_c^{\text{stat}}}{q_c^{(0)2}}\ , \tag{C12a}$$

$$\vartheta^{(2)}(t)=\tfrac{1}{2}\vartheta_0^{(2)}+\tfrac{1}{2}\vartheta_2^{(2)}e^{-2i\omega t}+\text{c.c.} \tag{C12b}$$

The time dependence $Z(t)$ of the convective heat current (4.3) follows then from (A47) and shows the form (C8) and (C9). The functions $f_c(t)$, $f_c^\dagger(t)$, $h(t)$, $\vartheta(t)$, and $Z(t)$ display a superposition of basic and higher harmonics of the external drive. The following first-order coefficients depending on the frequency ω enter into the Δ^2 shifts of the coefficients τ, ξ^2, and g of the amplitude equation and of the slope S of the convective heat current:

$$f_1^{(1)}=\frac{i}{\omega\tau^{(0)}}\frac{\Gamma(\omega)}{\beta(\omega)}\ , \tag{C13a}$$

$$f_1^{(1)\dagger}=-\frac{i}{\omega\tau^{(0)}}\frac{\Gamma(\omega)}{\beta^*(\omega)}\ , \tag{C13b}$$

$$h_1^{(1)}=\frac{2i}{\omega}\frac{\sigma}{\sigma+1}\frac{\Gamma(\omega)}{\beta(\omega)}\left[1-\frac{1}{2\beta(\omega)}\right]\ , \tag{C13c}$$

$$Z_1^{(1)}=\vartheta_1^{(1)}=\frac{2i}{\omega\tau^{(0)}}\frac{b(\omega)\Gamma(\omega)}{\beta(\omega)d(\omega)}\ , \tag{C13d}$$

where $\Gamma(\omega)$ is defined in Eq. (2.15c) and

$$\beta(\omega)=1-\frac{i\omega}{(\sigma+1)q_c^{(0)2}}\ ,\quad b(\omega)=1-\frac{i\omega}{2\sigma q_c^{(0)2}}\ ,\quad d(\omega)=1-\frac{i\omega}{(2\pi)^2}\ . \tag{C14}$$

The second-order coefficient $\vartheta_0^{(2)}$ is given in (3.15a). Further coefficients of $f_c(t)$ and $Z(t)$ are calculated in Ref. [25] to discuss the time dependence with higher accuracy including order Δ^3.

We decompose the scalar products contained in τ, ξ^2, g, and S by

$$\langle f_c^\dagger(t)|\mathcal{L}_T(\partial_t)|f_c(t)\rangle =\tau^{(0)}\left[1+\Delta^2\left[E_1+\tfrac{4}{3}k_c^{(2)}+\frac{1}{q_c^{(0)2}}\frac{2}{\sigma+1}E_2\right]+\mathcal{O}(\Delta^4)\right]\ , \tag{C15a}$$

$$\langle f_c^\dagger(t)|\mathcal{L}_{X^2}(\partial_t)|f_c(t)\rangle =\frac{4}{q_c^{(0)2}}\frac{k_c^2}{k_c^{(0)2}}[1+\Delta^2(E_{1+}\tfrac{2}{3}k_c^{(2)}+\tfrac{1}{3}\tau^{(0)}E_2)+\mathcal{O}(\Delta^4)]\ , \tag{C15b}$$

$$\langle f_c^\dagger(t)|\mathcal{L}_X(\partial_t,t)|h(t)\rangle =\frac{1}{q_c^{(0)2}k_c^{(0)2}}\left[\Delta^2\frac{1}{3}\left[\frac{\sigma+1}{\sigma}E_4-8k_c^{(2)}\right]+\mathcal{O}(\Delta^4)\right]\ , \tag{C15c}$$

$$\langle f_c^\dagger(t)|\mathcal{L}_\epsilon(t)|f_c(t)\rangle =\frac{k_c^2}{k_c^{(0)2}}\langle f_c^\dagger(t)|C(t)|f_c(t)\rangle \tag{C15d}$$

$$=\frac{k_c^2}{k_c^{(0)2}}[1+\Delta^2(E_1-2\epsilon_{c1}^{(2)})+\mathcal{O}(\Delta^4)] \tag{C15e}$$

$$=1+\Delta^2(E_1-2\epsilon_{c1}^{(2)}+2k_c^{(2)})+\mathcal{O}(\Delta^4)\ , \tag{C15f}$$

$$\langle f_c^\dagger(t)|\vartheta(t)|f_c(t)\rangle =\vartheta^{(0)}[1+\Delta^2(\vartheta_0^{(2)}+E_1+s^{(2)})+\mathcal{O}(\Delta^4)]\ , \tag{C15g}$$

$$\langle f_c^\dagger(t)|Z(t)|f_c(t)\rangle=1+\Delta^2(E_1+s^{(2)})+\mathcal{O}(\Delta^4)\ , \tag{C15h}$$

where we identify the time averages

$$E_1=\langle f^{(1)\dagger}(t)|f^{(1)}(t)\rangle\ , \tag{C16a}$$

$$E_2=\langle f^{(1)\dagger}(t)|\partial_t|f^{(1)}(t)\rangle\ , \tag{C16b}$$

$$-\epsilon_{c1}^{(2)}=\langle f^{(1)\dagger}(t)|c(t)\rangle=\langle c(t)|f^{(1)}(t)\rangle\ , \tag{C16c}$$

$$E_4=\langle f^{(1)\dagger}(t)|\partial_t|h^{(1)}(t)\rangle\ , \tag{C16d}$$

$$\frac{4k_c^{(2)}}{q_c^{(0)2}}=\langle c(t)|h^{(1)}(t)\rangle\ , \tag{C16e}$$

$$s^{(2)}=\langle f^{(1)\dagger}(t)|\vartheta^{(1)}(t)\rangle+\langle\vartheta^{(1)}(t)|f^{(1)}(t)\rangle \tag{C16f}$$

over functions of first order in Δ so that only E_1, E_2, E_4, and $s^{(2)}$ remain to be calculated. With (3.12) and (4.5) we find the $\Delta=0$ results (3.13) and the Δ^2 shifts [(3.14) and (3.15)] of the coefficients of the amplitude equation and of the slope of the convective heat current induced by the modulation. These shifts $\tau^{(2)}(\omega)$, $\xi^{2(2)}(\omega)$, $g^{(2)}(\omega)$, and $S^{(2)}(\omega)$ are discussed in Secs. III and IV and illustrated in Figs. 4, 5, and 7.

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