Reversible Hopf bifurcation in four-dimensional maps

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Following a recent work [Sevryuk and Lahiri, Phys. Lett. A **154**, 104 (1991)], we study the bifurcation of four-dimensional reversible maps in which the eigenvalues of the Jacobian of the map at a symmetric fixed point move off the unit circle along a pair of conjugate rays as some parameter ϵ crosses a threshold value. We construct a perturbation scheme to show that, depending on a control parameter γ, the bifurcation can be either "normal" or "inverted" in nature. In the former case, two one-parameter families of elliptic invariant curves passing arbitrarily close to the fixed point (which coexist with Kolmogorov-Arnold-Moser tori) merge together and move away from the fixed point. In the latter case, the families of elliptic invariant curves meet a family of hyperbolic invariant curves. As ϵ is varied, all these invariant curves shrink to the fixed point and are annihilated. The problem of determining whether an invariant curve is elliptic or hyperbolic is related to a tight-binding model on a linear quasiperiodic chain familiar in solid-state theory. Numerical evidence confirming these results is presented. A few areas for further study are indicated.

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I. INTRODUCTION

Much is known by now about the phase-space structures around fixed points of area-preserving maps in two dimensions [1]. By contrast, four-dimensional (4D) volume-preserving maps are not well understood. Unlike the 2D case, a 4D volume-preserving map is not, in general, symplectic (i.e., not derivable in terms of a generating function) and one needs to distinguish between classes of maps forming a sort of spectrum, starting with reversible maps at one end and terminating with symplectic ones at the other. Reversible maps constitute the least restrictive class in this spectrum, possessing at the same time the interesting property that in the vicinity of symmetric (see below) invariant sets, they behave locally like symplectic maps [2] (though in other regions of phase space they can display dissipative behavior as well [3]).

Several studies on 4D reversible and symplectic maps [4] indicate numerous novelties compared to 2D ones, the oldest and most well-known example being, of course, Arnold diffusion [5]. In this paper we shall be concerned, in particular, with an interesting bifurcation phenomenon, to which attention has only recently been drawn [6] (see also an earlier work by Lahiri and Ghoshal [7]). This bifurcation involves a 1:1 resonance near a symmetric elliptic fixed point of a 4D reversible map, in the vicinity of which families of closed invariant curves make their appearance, constituting a special class of invariant sets immersed in families of Kolmogorov-Arnold-Moser (KAM) surfaces which are in general 2-tori. Analogous bifurcation phenomena are well known in Hamiltonian flows (the "Hamiltonian Hopf" bifurcation [8] and in flows generated by reversible vector fields—"reversible Hopf" [2,9]). In the following we present a perturbation scheme for order-by-order calculation of invariant curves (without addressing the small denominator problem involved in the convergence of the perturbation series) and, with reference to this perturbation scheme, present numerical results concerning the phase-space structure, confirming the picture outlined in Ref. [6] by way of a conjecture.

In Sec. II below we introduce notations and summarize the contents of Ref. [6]. The perturbative scheme for the calculation of invariant curves is presented in Sec. III where we relate its principal results with the conjecture presented in Ref. [6]. Section IV contains the results of numerical computations confirming the picture of bifurcation that emerges from Secs. II and III. Section V is devoted to discussions and concluding remarks.

II. INVARIANT CURVES AROUND SYMMETRIC FIXED POINTS IN 4D REVERSIBLE MAPS

A map A is said to be reversible [10] if it satisfies $AGA=G$ for some involution G ($GG=1$), termed as the reversing involution for A. A well-known example is the class of $2m$-dimensional de Vogelaere maps [11]

$$A:\begin{cases} R'=W(R)-S\ , & \text{(1a)}\\ S'=R & \text{(1b)}\end{cases}$$

where R and S are m-dimensional vectors and W is a vector-valued function. These are reversible with respect to the involution

$$G:\begin{cases} R'=S\ , & \text{(2a)}\\ S'=R\ . & \text{(2b)}\end{cases}$$

In the following we set $m=2$ and express the map in the more convenient twice iterated form

$$R_{n+1}-2R_n+R_{n-1}=f(R_n) \tag{3}$$

in terms of successive iterates of a 2D vector

$$R_n \equiv \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

and a vector-valued function f. Setting $S_n=R_{n-1}$ and $W(R)=f(R)+2R$, we get (1a) and (1b) from (3). Reversible mappings arise in numerous physical and mathematical contexts [12].

A fixed point of a reversible map which is simultaneously a fixed point of its reversing involution is termed as a symmetric fixed point of A. The eigenvalues of the Jacobian matrix of the map at a fixed point will be termed multipliers at that point. The multipliers at a symmetric fixed point of a reversible map come in reciprocal pairs [2] $(\lambda_1,\lambda_1^{-1},\lambda_2,\lambda_2^{-1}$, etc.)—the so-called reflexive property. The de Vogelaere map (3) is characterized by the special property that all its fixed points are symmetric.

Because of the special symmetry possessed by a reversible map, its phase space around a symmetric fixed point may contain a Cantor family of invariant curves in accordance with the following proposition.

Let a $2N$-dimensional reversible map A possess, at a symmetric fixed point O, a pair of multipliers $\exp(\pm i\phi)$ on the unit circle (the remaining $2N-2$ multipliers may also be of modulus unity). Then under certain genericity (nondegeneracy and nonresonance) assumptions (which we omit) the map A near O admits a one-parameter Cantor family of closed invariant curves. On approaching O, the rotation numbers of these curves tend to $\phi/2\pi$ [13].

This proposition can be generalized [13] to the effect that for m pairs of multipliers on the unit circle [the remaining $(N-m)$ pairs may also be of modulus unity] the map A possesses near O an m-parameter family of invariant m tori on which A induces quasiperiodic motion. Applying the above results to the case of a symmetric *elliptic* fixed point O of a 4D reversible map A (for which there are two pairs of multipliers both on the unit circle) we are led to the existence of two one-parameter Cantor families of invariant curves immersed in a two-parameter Cantor family of 2 tori. The question we now address is, what happens to the families of invariant curves near a 1:1 resonance of the fixed point? More precisely, let A_ϵ depend smoothly on a parameter ϵ such that for $\epsilon<0$, $\epsilon=0$, and $\epsilon>0$ the dispositions of multipliers in the complex plane are as in Figs. 1(a)–1(c), respectively. This constitutes an alternative generic route through which the fixed point becomes hyperbolic, compared to the other known cases where a pair of multipliers move off the unit circle at -1 or $+1$. The bifurcation depicted in Figs. 1(a)–1(c) is obviously ruled out in 2D maps. Our main concern here is to inquire into the nature of this bifurcation.

As already mentioned, corresponding bifurcation phenomena in Hamiltonian and reversible vector fields are well studied. By invoking a correspondence between reversible maps and fixed time flow maps of reversible vector fields, the answer to the above question was put forward in Ref. [6] in the form of a conjecture which we restate here as follows.

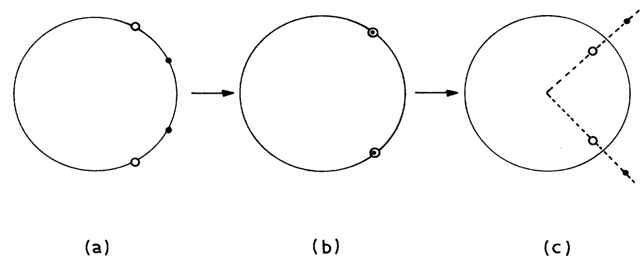

FIG. 1. Disposition of the multipliers at a symmetric fixed point corresponding to (a) $\epsilon<0$, (b) $\epsilon=0$, and (c) $\epsilon>0$.

Let the multipliers of A_ϵ about O be of the form $\exp(\pm i\phi_1)$, $\exp(\pm i\phi_2)$, with $\phi_{1,2}(\epsilon)$ given by

$$\phi_{1,2}=\omega\pm k(-\epsilon)^{1/2}+O(\epsilon) \tag{4}$$

where $0<\omega<\pi$, $k>0$, and the functions $O(\epsilon)$ are real valued for $\epsilon<0$. Thus at $\epsilon=0$ there correspond a Jordan block of order 2 corresponding to each of the eigenvalues $\exp(\pm i\omega)$. Then the nature of bifurcation at $\epsilon=0$ of invariant curves around O is determined by a certain parameter γ that can be determined from A_0 (see Sec. III) and may be summarized as follows.

(a) Let $\gamma>0$ (superthreshold or normal bifurcation). Then there exist two one-parameter Cantor families of elliptic invariant curves near O for $\epsilon<0$ with rotation number $\mu<\phi_2/2\pi$ and $\mu>\phi_1/2\pi$, respectively. At $\epsilon=0$ these merge into a single family of invariant curves passing arbitrarily close to O. As ϵ becomes positive the (merged) family of invariant curves moves away from O.

(b) Next consider $\gamma<0$ (subthreshold or inverted bifurcation). In this case, for $\epsilon<0$, there exist two one-parameter families of elliptic invariant curves, but now with rotation numbers μ satisfying $\phi_2/2\pi<\mu<\phi_1/2\pi$. However, the situation here is slightly more involved. There exist certain $\tilde{\phi}_{1,2}$, $\phi_2<\tilde{\phi}_2<\tilde{\phi}_1<\phi_1$, such that the elliptic invariant curves mentioned above correspond to rotation numbers $\phi_2/2\pi<\mu<\tilde{\phi}_2/2\pi$ and $\tilde{\phi}_1/2\pi<\mu<\phi_1/2\pi$. In addition there exists a family of *hyperbolic* invariant curves with rotation numbers $\tilde{\phi}_2/2\pi<\mu<\tilde{\phi}_1/2\pi$. All these invariant curves shrink to O as $\epsilon\to 0$ and get annihilated at $\epsilon=0$ so that there exists no invariant curve near O for $\epsilon>0$.

We now present a perturbation scheme for order-by-order construction of invariant curves and relate the results to the picture conjectured above.

III. PERTURBATIVE CONSTRUCTION OF INVARIANT CURVES

We consider a class of maps given by Eq. (3) with

$$R_n=\begin{bmatrix} x_n \\ y_n \end{bmatrix} .$$

It turns out that terms of degree three must be retained in $f(R_n)$ to describe the bifurcation under consideration. We choose for convenience a particularly simple form for f such that (3) reduces to

$$x_{n+1}-2x_n+x_{n-1}=px_n+y_n \ , \tag{5a}$$

$$y_{n+1}-2y_n+y_{n-1}=-\epsilon x_n+py_n+\alpha x_n^2+\beta x_n^3 \ , \tag{5b}$$

which expresses the 4D maps A_ϵ in terms of two second-order difference equations parametrized by p, α, β (in addition to ϵ). The linear terms have been chosen such that the Jacobian at origin O (which is a symmetric fixed point and on which we shall henceforth concentrate our attention) has the required Jordan normal form. In addition, the nonlinear terms have been so chosen that the map can be expressed as a single fourth-order difference equation of sufficient simplicity

$$(x_{n+2}+x_{n-2})-2(p+2)(x_{n+1}+x_{n-1}) +(p^2+4p+6+\epsilon)x_n=\alpha x_n^2+\beta x_n^3 \ . \tag{6}$$

This implies that trajectories of the map can be described in terms of successive iterates of x_n of a single coordinate. It can be easily verified that, if p satisfies $-4<p<0$ the multipliers of A_ϵ at the origin O for small $|\epsilon|$ are arranged in the complex plane as in Figs. 1(a)–1(c) for $\epsilon<0$, $\epsilon=0$, and $\epsilon>0$, respectively, and that, in particular, for $\epsilon<0$ (subthreshold side of the bifurcation) the multipliers are of the form $\exp(\pm i\phi_1)$, $\exp(\pm i\phi_2)$ with $\phi_{1,2}$ given by Eq. (4) [14], where

$$\omega=\cos^{-1}(1+p/2) \ , \tag{7a}$$

$$k=(2\sin\omega)^{-1} \ . \tag{7b}$$

The existence of an invariant curve with rotation number μ is equivalent to the requirement that x_n can be expressed as a Fourier series

$$x_n=a+[b\exp(in\phi)+\text{c.c.}]+[c\exp(2in\phi)+\text{c.c.}] +[d\exp(3in\phi)+\text{c.c.}]+\cdots \tag{8a}$$

where $\phi=2\pi\mu$. In the following we shall characterize the members of the family of invariant curves (wherever it exists) in terms of a parameter $\hat{\phi}$, such that $\hat{\phi}/2\pi$ denotes the deviation of rotation number of the family measured from $\omega/2\pi$ [vide Eq. (7a)]. Consequently, our perturbation scheme will include the small parameters ϵ and $\hat{\phi}$, and is as follows.

We substitute (8a) into (6), equate coefficients of $\exp(in\phi)$ $(n=0,1,2,\ldots)$ on both sides and try to solve for the coefficients $a,b,c,\ldots,$ seeking the solution for each coefficient (say a) in the form of a series ($a=\sum_i a_i$, etc.) such that consecutive terms of the series are of successively higher orders of smallness. A trivial solution would, of course, be $a=b=c=\cdots=0$. But in certain circumstances a *second,* bifurcating, solution (depending parametrically on $\hat{\phi}$) is found to exist. As an example retaining only the leading contributions (a_0,b_0,c_0) in a,b,c and ignoring the coefficients d onwards (which results in a consistent approximation as may be verified from below) we find

$$x_n\simeq a_0+2b_0\cos n\phi+2c_0\cos 2n\phi \tag{8b}$$

where

$$p^2a_0=2\alpha b_0^2 \ , \tag{8c}$$

$$b_0[\epsilon+4(\cos\phi-\cos\omega)^2]=2\alpha(a_0b_0+b_0c_0)+3\beta b_0^3 \ , \tag{8d}$$

$$c_0=\alpha b_0^2/[\epsilon+4(\cos2\phi-\cos\omega)^2] \ , \tag{8e}$$

and

$$\phi=\omega+\hat{\phi} \ . \tag{8f}$$

So far as the invariant curve (rather than the actual trajectory i.e., the successive iterates of x_n) is concerned, the phase of b_0 is redundant and we have chosen b_0 to be real. As may be easily verified from (8c)–(8f), the nontrivial solution for b_0 (existing under conditions to be specified below) is given by

$$b_0^2=\frac{1}{\gamma}[\epsilon+(4\sin^2\omega)\hat{\phi}^2] \tag{9a}$$

corresponding values of a_0,c_0 being obtained by substitution in Eqs. (8c) and (8e), respectively, where

$$\gamma=\frac{2[1+2(p+3)^2]}{p^2(p+3)^2}\alpha^2+3\beta \tag{9b}$$

and where we have assumed the nonresonance condition

$$(\cos2\omega-\cos\omega)^2\gg\epsilon$$

to hold (i.e., ω should be sufficiently away from $2\pi/3$) since otherwise c_0 would not be small compared to b_0, causing the perturbation scheme to break down. Equations (9a) and (9b) provide us with surprisingly rich information concerning the bifurcation, most of which is obtained from the simple requirement that the right-hand side of (9a) is to be positive for the invariant curve to exist.

Consider first the case $\gamma>0$. Equation (9a) implies that, for $\epsilon<0$, invariant curves exist for $\hat{\phi}$ satisfying [vide (7a) and (7b)] $\hat{\phi}<-k(-\epsilon)^{1/2}$, $k(-\epsilon)^{1/2}<\hat{\phi}$, i.e., for rotation number μ precisely satisfying the conditions stated in (a), Sec. II. We also note that in this case b_0^2 can be made arbitrarily small by choosing $\hat{\phi}^2$ close enough to $-k^2\epsilon$. At $\epsilon=0$ the upper and lower limits of the rotation number for the two families coincide. For $\epsilon>0$ on the other hand, we have a single family of invariant curves corresponding to arbitrary (but small enough) values of $\hat{\phi}$ and, for any given ϵ, there is a minimum value of $b_0\approx(\epsilon/\gamma)^{1/2}$, i.e., the family of invariant curves moves away from O with increasing ϵ.

Next we consider $\gamma<0$. Then (9a) shows that, for $\epsilon>0$ no nontrivial solution for b_0 exists while for $\epsilon<0$, there again exists a family of invariant curves, but now with $\hat{\phi}$ lying in the range $-k(-\epsilon)^{1/2}<\hat{\phi}<k(-\epsilon)^{1/2}$, i.e., with rotation numbers μ satisfying precisely the condition $\phi_2/2\pi<\mu<\phi_1/2\pi$ as conjectured in (b), Sec. II. We further note that as $\epsilon\to0$ from below, $b_0\to0$ for all $\hat{\phi}$ in the above range, i.e., all the invariant curves shrink to O, again verifying the conjecture presented in the preceding section. Not all the invariant curves obtained by varying $\hat{\phi}$ in the above range are, however, of the same nature. In fact, they are demarcated into two families of *elliptic* and another family of *hyperbolic* invariant curves. We shall present below a rough estimate for the transition values

$(\tilde{\phi}_{1,2})$ of $\hat{\phi}$ marking the transition from elliptic to hyperbolic invariant curves, but before that several other observations are in order.

The results presented above have been obtained by ignoring the third and higher harmonics in the Fourier expansion of x_n and even the coefficients of the lower harmonics have been evaluated in the leading order only. This defect can be removed, in principle, by calculating terms up to any harmonics and any finite order, though at the cost of rapidly escalating complexity of calculation. Considerable simplification is achieved by taking $\alpha=0$, noting at the same time that the quadratic term (αx_n^2) in (5b) is not essential in characterizing the bifurcation. Indeed, as we have seen, the nature of the bifurcation is governed by the signature of γ. As seen from (9b), we cannot change the signature of γ by taking $\beta=0$ and changing α, while the reverse is certainly true. This is entirely analogous to the situation obtaining in reversible Hopf bifurcation for vector fields where the normal form for the bifurcation does not contain quadratic terms [2,6].

With $\alpha=0$ in (5b) we can, with much greater ease, extend our calculations to include higher harmonics and up to higher orders of smallness. Indeed, an exact result in this case is that the coefficients of *all* even harmonics $\exp(2ikn\phi)$ $(k=0,1,2,\ldots)$ vanish, i.e., $a=c=\cdots=0$ to all orders of smallness. As an example, ignoring the fifth and higher harmonics in the expansion (8a) we find

$$x_n \approx 2(b_0+b_1)\cos n\phi+2d_0\cos 3n\phi \tag{10a}$$

where we have retained the first nonleading contribution in b and only the leading contribution in d (which may easily be seen to be a consistent approximation) and where

$$b_0^2=(\epsilon+4\sin^2\omega\hat{\phi}^2)/3\beta\ , \tag{10b}$$

$$d_0=\beta b_0^3/4(\cos 3\omega-\cos\omega)^2\ , \tag{10c}$$

$$b_1=-\tfrac{1}{2}d_0\ . \tag{10d}$$

Equations (10a)–(10d) constitute an order of magnitude refinement over (8b)–(8e), (9a), and (9b) in so far as the computation of an invariant curve for given ϵ and $\hat{\phi}$ is concerned. However, the expression (9b) for γ is exact (even for $\alpha\neq 0$) since the bifurcation actually involves the limit $\epsilon\to 0$. An alternative derivation for γ leading to the same result is to be found in Ref. [6].

The validity of the above perturbation scheme up to any finite order is subject to *nonresonance* conditions, as we have already seen in our leading-order calculations. The choice $\alpha=0$ does away with the first nonresonant condition $(\cos 2\omega\neq\cos\omega)$ but the next one, $\cos 3\omega\neq\cos\omega$ is essential for the validity of (10a)–(10d) since otherwise the third harmonic would compete with the first. In general with $\alpha=0$, the formal validity of the perturbation series requires the nonresonant conditions

$$\cos(2k+1)\omega\neq\cos\omega \tag{11a}$$

with $k=1,2,3,\ldots$, i.e.,

$$\omega\neq\frac{p}{q}\pi\quad (p<q;\ p,q,\ \text{positive integers}) \tag{11b}$$

[recall that we have already required ω to lie in the range $0<\omega<\pi$; notice also that $\omega\neq 2\pi/3$ is included in (11b) though $\cos 2\omega\neq\cos\omega$ has been made redundant by the choice $\alpha=0$].

However, two questions remain even when the nonresonant conditions are taken into consideration. The first, of course, concerns the problem of *small denominators.* Even when ω/π is irrational, an invariant curve with irrational rotation number μ for which the perturbation series is formally defined may in fact be nonexistent owing to the predominance of some sufficiently large harmonics that vitiates the convergence of the series. In the absence of a formal criterion of convergence we restate the conjecture, put forward in Ref. [6] with a supporting plausibility argument, that only the invariant curves with "strongly irrational" [6] rotation numbers exist, the rotation numbers having a Cantor-set-like structure on the real line.

This brings up the second question, namely, that of the nature of trajectories in the *resonant gaps.* This is another area of the problem we leave unexplored in the present paper, presenting in Sec. IV some preliminary numerical evidence for *island formation* in low-order resonances. A few more observations on the above two questions concerning resonant gaps will be included in the concluding section.

We now turn to the problem of estimating the values $(\tilde{\phi}_{1,2})$ of $\hat{\phi}$ marking the transition from elliptic to hyperbolic invariant curves on the subthreshold side $(\epsilon<0)$ for the inverted bifurcation $(\gamma<0)$. For this we again take $\alpha=0$ and ignore b_1,d_0 in (10a)–(10d). Denoting the trajectory (10a) by $\bar{x}_n$, we linearize (6) about this trajectory by taking

$$x_n=\bar{x}_n+\xi_n\ , \tag{12a}$$

which gives

$$(\xi_{n+2}+\xi_{n-2})-2(p+2)(\xi_{n+1}+\xi_{n-1}) +(p^2+4p+6+\epsilon-3\beta\bar{x}_n^2)\xi_n=0\ . \tag{12b}$$

Equation (12b) can be fruitfully discussed in terms of the tight-binding model [15] (TBM) well-known in solid-state theory. In the present paper we confine ourselves to a rough estimate in which we replace $3\beta\bar{x}_n^2$ by (a) its maximum value, namely, $12\beta b_0^2$ [refer to Eq. (8b) in which we put $a_0=c_0=0$ corresponding to $\alpha=0$] and (b) its average value, namely, $6\beta b_0^2$. The condition for the invariant curve being elliptic may be shown to reduce in the above two cases [for each of which Eq. (12b) corresponds to a linear chain with constant on-site potentials and second-nearest-neighbor hopping] to

(a) $\hat{\phi}^2>3|\epsilon|/(4\sin\omega)^2$

and

(b) $\hat{\phi}^2>2|\epsilon|/(4\sin\omega)^2$,

respectively. As seen in numerical experiments (see Sec. IV), these two estimates do give us useful information about the transition from elliptic to hyperbolic invariant curves in the case of the inverted bifurcation. More detailed numerical work on (12b) treated as a tight-binding model with quasiperiodic on-site potential is planned to be reported in a forthcoming communication. We mention here that all the principal features of the bifurcation presented in Secs. II and III parallel closely the known features in reversible Hopf and Hamiltonian Hopf bifurcations in vector fields [8,9].

IV. NUMERICAL EVIDENCE

All the results and conclusions of the preceding section are completely borne out in numerical iterations of Eq. (6). In each numerical experiment, ϵ, p, α, and β were given preassigned values (with ϵ small, and p in the range $-4<p<0$) and iterations were performed for a series of initial conditions. Each initial condition in the series consisted of four successive values x_j ($j=1,2,3,4$) chosen suitably. The results of the iterations were plotted as a two-dimensional projection of the 4D trajectories with $u_n(\equiv x_n - x_{n-1})$ plotted against x_n. In most of our experiments we chose $\alpha=0$. An initial set of iterations were performed by referring to the leading order of perturbation results [Eqs. (8b)–(9b)], and x_j were taken as [vide Eq. (7a)]

$$x_j = 2b_0\cos[j(\omega+\hat{\phi})] , \quad (13a)$$

$$b_0 = |(\epsilon + 4\sin^2\omega\hat{\phi}^2)/3\beta|^{1/2} , \quad (13b)$$

($j=1,2,3,4$), where $\hat{\phi}$ was given suitable values to represent different initial conditions. For those values of ϵ, β, and $\hat{\phi}$ for which the perturbation theory predicts the existence of invariant curves, Eq. (13) corresponds to initial conditions chosen on the theoretically predicted curve, and the validity of the theory requires that the subsequent iterates do actually lie on the predicted curve. In reality, however, this is expected to happen only when the curve concerned is elliptic; for hyperbolic curves the accumulating numerical errors in iteration would lead the computed trajectory away from the curve. For values of the parameters for which no invariant curve is predicted in the perturbation theory we expect that trajectories started from the initial conditions (13) may either blow up or wind upon KAM 2 tori depending on circumstances (see below).

Actual trajectories in our experiments with initial conditions given by (13) did indeed conform to the above expectations. However, the expected invariant curves were found to have a slight spread owing to the fact that (8b)–(8e), (9a), and (9b) do not constitute a good enough approximation to the actual curves.

In the next set of experiments we chose x_j ($j=1,2,3,4$) as in (10a), with b_0 given by (13b) [for the sake of numerical iterations, we replaced $\sin^2\omega\hat{\phi}^2$ by the more accurate expression $(\cos\phi-\cos\omega)^2$], d_0, b_1, as in (10c) and (10d), and $\phi=\omega+\hat{\phi}$. It was found that the spread of the invariant curves mentioned above *vanished abruptly* and the expected invariant curves did appear as 1D objects in our 2D (u_n versus x_n) projections even up to more than 10^7 iterations (all our numerics were performed in double precision with an accuracy $\sim 10^{-15}$). This observation showed that our next-to-leading-order calculations do constitute a reasonably good approximation to study the phase-space structure. We now exhibit a few computed trajectories by way of illustration.

Figure 2(a) shows a typical set of invariant curves with $\beta>0$, $\epsilon>0$ (normal bifurcation, superthreshold side) which are found to exist for all sufficiently small $\hat{\phi}$. Though the origin is a hyperbolic fixed point in this case, KAM 2 tori are preserved in the phase space in regions slightly away from the origin, and invariant curves are sort of immersed in the 2 tori. Figure 2(b) shows the projection of three of the invariant curves of 2(a) together with the projection of one such KAM torus. The initial condition for the latter was chosen with x_j ($j=2,3,4$) as in (13a) and (13b) with a particular value of $\hat{\phi}$, but with a slightly shifted x_1.

Figure 3 shows a family of invariant curves on the subthreshold side in the normal bifurcation for a set of values for $\hat{\phi}$ satisfying $\hat{\phi} > k(-\epsilon)^{1/2}$, together with the results of iterating with two other initial conditions satisfy-

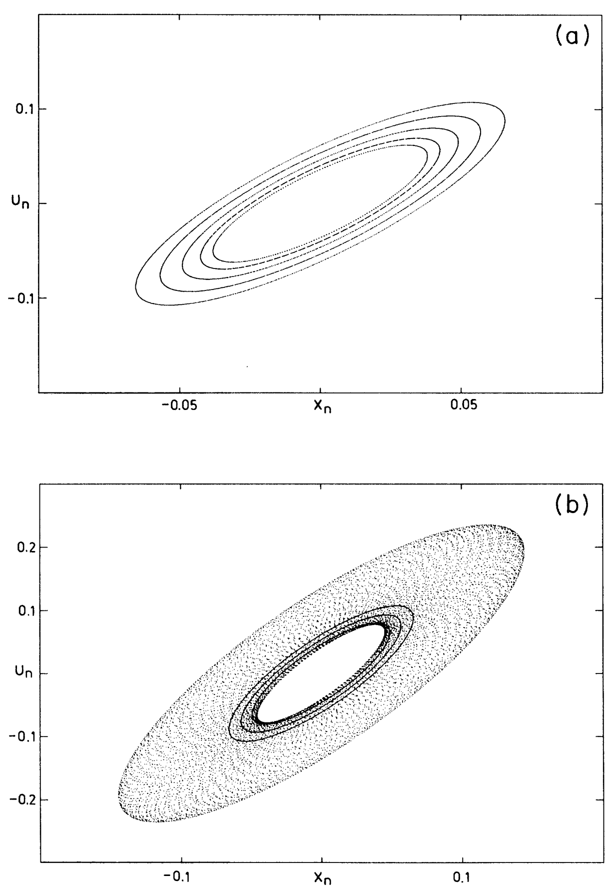

FIG. 2. (a) Projection [$u_n(\equiv x_n - x_{n-1})$ vs x_n] of a set of invariant curves on the superthreshold side in normal bifurcation, $\epsilon=0.001$ with $\alpha=0$, $\beta=1$ and rotation number $\mu_0=\omega/2\pi=(3.313\,579)^{-1}$ corresponding to five values of $\hat{\phi}$ equally spaced between 0.0005 and 0.0025. (b) The three innermost invariant curves of (a) immersed in a KAM torus, obtained with slightly different initial conditions (see text).

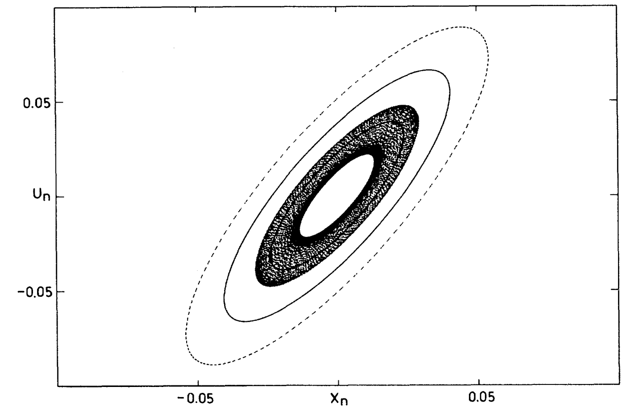

FIG. 3. Invariant curves on the subthreshold side, $\epsilon=-0.001$, for three values of $\hat{\phi}$ between 0.02 and 0.03, other parameters being the same as in Figs. 2(a) and 2(b). Also shown are iterates for two other values of $\hat{\phi}$ (0.010, 0.015) for which there are no invariant curves and the trajectories open up into 2 tori.

ing $|\hat{\phi}|<k(-\epsilon)^{1/2}$. Our perturbation theory tells us that no invariant curves are to exist in the latter case and we indeed find the trajectories to open up into two KAM tori. All the invariant curves shown in Figs. 2 and 3 are elliptic.

Figure 4(a) illustrates a family of elliptic invariant curves on the subthreshold side of the inverted bifurcation ($\epsilon<0$, $\beta<0$) obtained with a set of $\hat{\phi}$ values satisfying $|\hat{\phi}|<k(-\epsilon)^{1/2}$ together with a KAM torus obtained with $|\hat{\phi}|>k(-\epsilon)^{1/2}$ for which the perturbation theory tells us that no invariant curve is to exist. Figure 4(b), on the other hand, provides evidence for a *hyperbolic* invariant curve [with $|\hat{\phi}|<k(-\epsilon)^{1/2}$] where we notice that the trajectory gradually deviates from some closed curve and then moves off due to the unstable nature of the curve. In all our iterations we found that there exists some $\tilde{\phi}$ such that the transitional values ($\tilde{\phi}_{1,2}$) of $\hat{\phi}$ mentioned in (b), Sec. II are given by $\tilde{\phi}_{1,2}=\pm\tilde{\phi}$, and $\tilde{\phi}^2$ is of the same order of magnitude as the values estimated at the end of Sec. III.

All iterations on the superthreshold side ($\epsilon>0$) of the

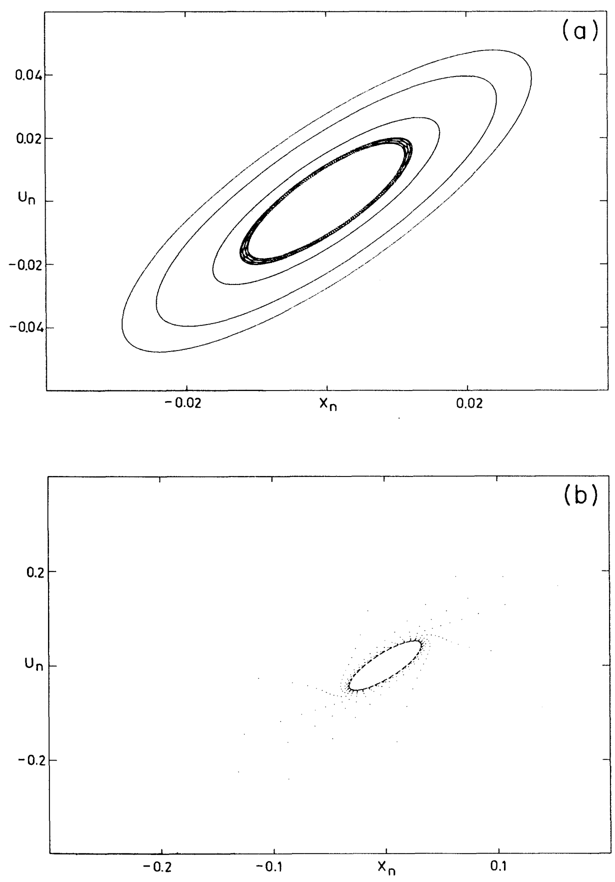

FIG. 4. (a) Elliptic invariant curves on the subthreshold side of inverted bifurcation $\epsilon=-0.001$, $\beta=-1$, for three values of $\hat{\phi}$ between 0.0010 and 0.0015; also shown is trajectory with $\hat{\phi}=0.001\,75$, opening up into a torus. (b) A hyperbolic invariant curve for $\epsilon=-0.001$, $\beta=-1$ for $\hat{\phi}$ (=0.008) below the transition value (see text).

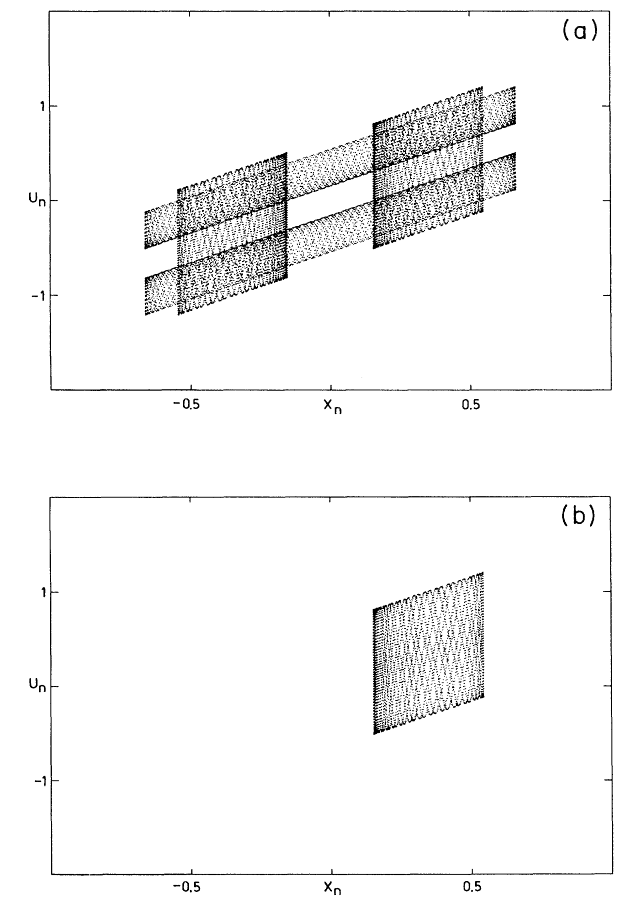

FIG. 5. (a) Decay of a torus resulting in breakup into four islands due to a large perturbation [$\epsilon=0.04$, $\alpha=0$, $\beta=0.25$, $\mu_0=(4)^{-1}$ corresponding to $p=-2$, x_j (j=2,3,4) chosen as in Eqs. (13a) and (13b) with $\hat{\phi}=0.05$ and x_1 shifted by 0.05]. (b) One of the four islands of (a).

inverted bifurcation resulted in the trajectories quickly escaping away from the origin, showing that there exist neither invariant curves nor KAM tori in the vicinity of the fixed point.

Figure 5(a) illustrates the *breakup* of an invariant curve at a resonance, namely, $\omega=\pi/2$ ($p=-2$) where a trajectory initiated with x_j ($j=2,3,4$) as in (13a) and (13b) but with x_1 different is seen to result in a highly distorted pattern with a 1:4 island structure. In Fig. 5(b) we show one of the four islands separately by plotting every fourth iterate and deleting the three succeeding ones. This strong resonance effect is, however, restricted to a rather narrow gap $\Delta\hat{\phi}\simeq O(|\epsilon|^{1/2})$ away from which our perturbation calculations are found to give remarkably good results.

V. CONCLUDING REMARKS

In summary, we have presented a perturbation scheme for the order-by-order construction of invariant curves near a "reversible Hopf" bifurcation in 4D maps, the results of which support the conjectures made in Ref. [6], and have exhibited numerical evidence confirming these results.

As already mentioned, we have not obtained criteria for convergence of the perturbation series that can support the conjecture that the invariant curves are organized in Cantor families. Indeed, as mentioned in Sec. II, the family of maps represented by (3) appears to be intermediate between reversible and symplectic ones in that the linearization about *any* fixed point leads to multipliers occurring in reciprocal pairs (since all fixed points are symmetric ones) and so the invariant curves may possibly be organized more densely than a Cantor family with reference to rotation numbers. However, the fact that resonant gaps exist almost everywhere is empirically observed in numerical experiments where even a slight variation in $\hat{\phi}$ results in a reversal of the direction in which an invariant curve is described during iteration.

Another area left unexplored in the present paper is that of direct confirmation of the existence of hyperbolic invariant curves for $\epsilon<0$, $\gamma<0$ and that of accurately identifying the transitional values $\tilde{\phi}_{1,2}$ [see (b), Sec. II]. Work in this regard is in progress along the line indicated in Ref. [6] and that mentioned at the end of the preceding section.

Finally, we mention that on the superthreshold side of the inverted bifurcation ($\epsilon>0$, $\gamma<0$) the phenomenon of *intermittency* is likely to be encountered. Intermittency in 2D reversible maps has been studied in Refs. [16], where exact solutions to the relevant RG equations have been obtained. Intermittency in the present case is expected to present new features. Results of investigations are planned to be reported in a future paper.

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[14] For a general reversible map near a symmetric elliptic fixed point, if $\exp[\pm i\phi_1(\epsilon)]$ and $\exp[\pm i\phi_2(\epsilon)]$ denote the two pairs of multipliers then their coalescence on the unit circle is described by the condition $\phi_1(\epsilon)=\phi_2(\epsilon)$, and thus the bifurcation under study is of codimension one. Indeed, Eq. (4) represents a generic way in which a pair of roots of a real equation approach each other as some parameter (ϵ) is varied, and so the scenario depicted in Fig. 1 is not special to the particular family [Eqs. (5a) and (5b)] considered here. The situation here is essentially the same as in reversible Hopf bifurcation of vector fields.

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