Cooper pairing in a soluble one-dimensional many-fermion model

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The dynamical similarities between the three-dimensional (3D) electron fluid "jellium" model and the exactly soluble 1D fermion fluid with attractive δ-function pairwise potentials motivates solving the Cooper-pair problem in this model for all couplings and/or densities. This is accomplished exactly in graphical form. For weak coupling, the essential singularity characteristic of standard 3D low-temperature superconductivity reemerges, and for infinite coupling, BCS theory reproduces the exact ground-state system energy.

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I. INTRODUCTION

An assembly of N ($\gg 1$) fermions of mass m enclosed in a one-dimensional "box" of length L and interacting via an attractive pairwise δ-function potential is described by the Hamiltonian

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{d^2}{dx_i^2}-v_0\sum_{\substack{i=1\\i<j}}^{N}\delta(x_i-x_j),\quad v_0>0\ . \tag{1}$$

Given the number density $\rho\equiv N/L$, introducing the dimensionless coordinates $x_i'\equiv\rho x_i$ leads to the dimensionless Hamiltonian

$$H'\equiv mH/\hbar^2\rho^2=-\tfrac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i'^2}-\lambda\sum_{\substack{i=1\\i<j}}^{N}\delta(x_i'-x_j')\ , \tag{2}$$

where $\lambda\equiv mv_0/\hbar^2\rho$ is a dimensionless coupling parameter, and is clearly the *only* variable upon which the ground-state properties of (2) can depend. Its range of variation is $0\le\lambda<\infty$, and the N-fermion system is seen to have the peculiar dynamics whereby high- (low-) particle density is associated with weak (strong) interaction. This is in marked contrast with many familiar three-dimensional (3D) quantum fluids such as the liquid heliums (both ^{3}He and ^{4}He), nuclear and neutron matter, etc., where the pair interaction $V(r)$ is short ranged attractive and repulsive at even shorter range. Figure 1, top panel, schematically displays the ground-state energy per particle of liquid ^{3}He or nuclear matter as a function of number density ρ.

If the N-fermion system (1) consists of ν distinct species (e.g., $\nu=2$ if we have up and down spins possible), the Bethe ansatz [1] shows that (2) has only *one* bound N-particle state which at zero density has a total energy given [2] simply by

$$E_0(N)=\left[\frac{N}{\nu}\right]E_0(\nu)\ , \tag{3}$$

where $[N/\nu]$ is the nearest integer, from below, to N/ν, and

$$E_0(\nu)\equiv -mv_0^2\nu(\nu^2-1)/24\hbar^2 \tag{4a}$$

$$=-mv_0^2/4\hbar^2\equiv E_0(2)\quad\text{for }\nu=2\ . \tag{4b}$$

Equation (4b) gives the well-known elementary 1D quantum-mechanical result for the (sole) ground-state energy of two particles of mass m (or one of reduced mass $m/2$) interacting with an attractive δ-function potential

of strength v_0. The result (4a) is also the exact total ground-state energy of ν ($\gg 1$) bosons [3] with the Hamiltonian (1) where N is replaced by ν. The energy per boson is *not* extensive since it clearly collapses to $-\infty$ in the thermodynamic limit $\nu \to \infty$. Such a collapse is prevented if the particles are fermions by the Pauli principle. In 3D, however, this is *not* the case. A determinant of plane waves (with wave vectors $\mathbf{k}$ occupying a Fermi sphere in k space of radius k_F) used as a trial function readily gives for high density an expectation value of (1) in 3D of the form $N(A_1\rho^{2/3} - \frac{1}{2}\rho v_0)$, with A_1 a positive constant. This being a rigorous upper bound for the exact $E_0(N)$ (with ρ in fact a variational parameter that can be taken arbitrarily large) proves that the exact ground-state total energy collapses to negative infinity.

The N-fermion wave function is given [4] by

$$F(1,2,\ldots,\nu)F(\nu+1,\nu+2,\ldots,2\nu)\cdots F(N+1-\nu,\ldots,N-1,N)\pm A \tag{5}$$

where

$$F(1,2,\ldots,\nu) \equiv \prod_{\substack{i=1 \\ i<j}}^{\nu} \exp\left[-\frac{m v_0}{2\hbar^2}|x_i - x_j|\right] \times \chi(\sigma_1,\sigma_2,\ldots,\sigma_\nu) \tag{6}$$

is totally antisymmetric, since χ is an antisymmetric "spin" function, and the space part is symmetric, each under the exchange of two-particle indices from the set $1 \le i,j \le \nu$.

Thus, for $\rho = 0$ ($\lambda = \infty$) the system consists of an *ideal gas* of noninteracting $[N/\nu]$ "cluster" (composite) particles, each made up of ν distinct fermion species. It is remarkable that shortly before the word "soliton" was coined [5], McGuire showed [2] by using S-matrix techniques that the clusters of (6) have all the properties of solitons, namely, when they are scattered one from another their constituent fermions may rearrange, and the clusters will be phase shifted, but otherwise they emerge intact (i.e., there is no cluster "breakup" in collisions).

On the other hand, for $\lambda = 0$ ($\rho = \infty$) the ground-state

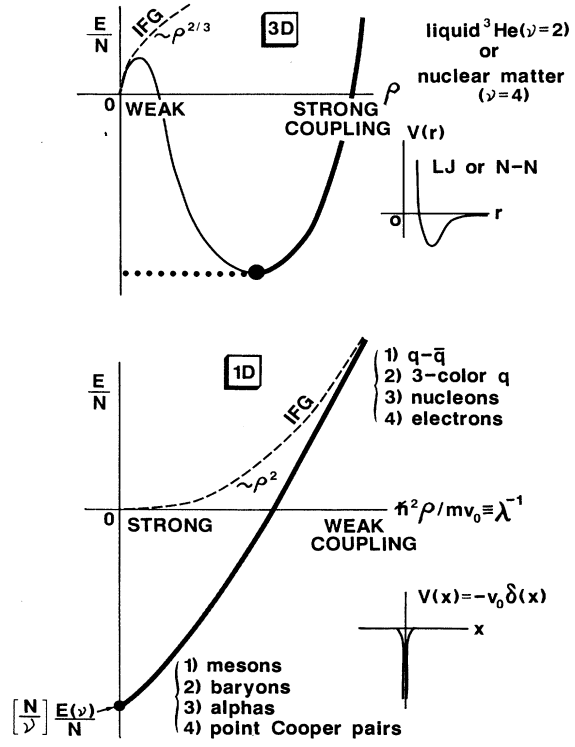

FIG. 1. Comparison of ground-state energy-per-particle vs density curves, or equations of state (EOS), for a typical 3D Fermi liquid (like ^{3}He with two species, or nuclear matter with four) and for the 1D many-fermion fluid defined by Eq. (1) or (2).

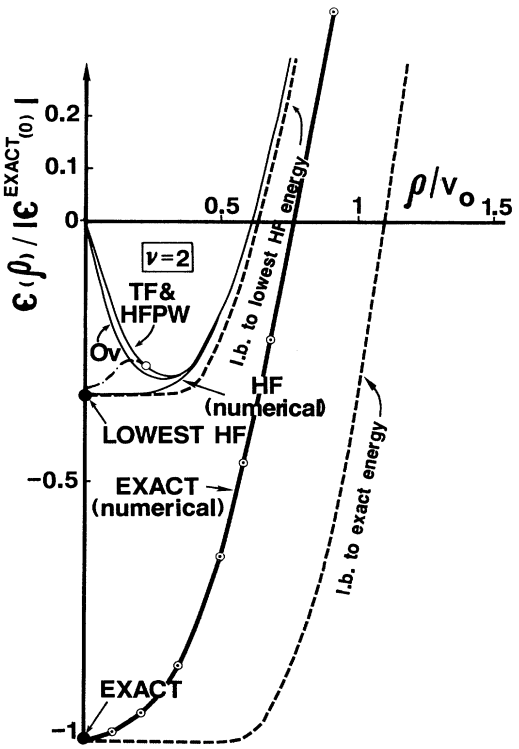

FIG. 2. Exact ground-state EOS (in dimensionless units as defined in Appendix A) of (1) or (2), full thick curve, obtained numerically. Dashed curves refer to rigorous lower bounds [7]; thin full curves to the Hartree-Fock approximation [4,7], and to the Thomas-Fermi calculation discussed in Appendix B.

N-particle energy of (1) is just the ideal Fermi-gas limit

$$E_0(N)=N\frac{1}{3}\frac{\hbar^2k_F^2}{2m}=N\frac{\hbar^2\pi^2}{6m\nu^2}\rho^2 , \tag{7}$$

since the "Fermi sea" consists of the interval in 1D k space given by $[-k_F,k_F]$, and consequently $\rho=N/L=\nu k_F/\pi$. Figure 1, bottom panel, schematically summarizes the expected energy per particle as function of density for the 1D model Hamiltonian (1). The variable $\lambda^{-1}\equiv\hbar^2\rho/m\nu_0$ has a very simple, direct physical meaning: it is proportional to the ratio of the pair diameter [which from (6) is just $4\hbar^2/m\nu_0$] to the average interparticle separation.

For $0<\lambda<\infty$ (implying $\infty>\rho>0$) the ground-state energy as a function of λ can be obtained numerically by solving the so-called Gaudin equations [6] (see Appendix A). In Fig. 2 we plot the resulting dimensionless equation of state (EOS) for the model (thick curve) for $\nu=2$ species. In Fig. 2 $\hbar=m=1$ was used so that the abscissa refers to λ^{-1}. Dashed curves are rigorous *lower* bounds [7] to the energies of both the exact problem [Schrödinger equation (1)] and to the lowest possible energy state in the Hartree-Fock (HF) approximation. The three full thin curves refer to various mean-field results, both Thomas Fermi (see Appendix B) and Hartree Fock, the latter discussed in more detail in Refs. [4] and [7]. Note that both rigorous bounds, the upper (HF) and the lower one, are unsatisfactorily far from the exact (full thick curve) EOS for intermediate values of λ. Before proceeding, we mention several known applications of the model.

A. Hadronic and nuclear physics

Table I summarizes some recent applications of the 1D N-fermion model to hadronic [8] and nuclear [9] physics. If the N-fermion system is composed of two species, quark and antiquark $(q,\bar{q})$ when $\lambda=0$ ($\rho=\infty$), the resulting $\lambda=\infty$ ($\rho=0$) clusters are *mesons*. If we begin with three-color quarks (q_1,q_2,q_3) as ideal fermion-gas constituents, color-neutral *baryons* form as coupling increases. Finally, if the fermion gas is noninteracting nuclear matter with protons and neutrons, each of spin up or down, then $\nu=4$ and *alpha particles* are formed as the coupling is increased. The usefulness of the 1D N-fermion model in hadronic physics is clear from the fact that quark interactions also share with the soluble model the same peculiar dyanmics, namely, they interact weakly (strongly) when close together (far apart).

B. Solid-state physics

The simplest model of a 3D metal is the familiar jellium model [10] consisting of N-point electrons moving in a rigid uniform background of positive charge. The Hamiltonian is

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2+V_{bb}+V_{be}+V_{ee} , \tag{8}$$

where the first term is the N-electron kinetic energy and the three remaining terms are the (positive) background-background repulsive potential energy V_{bb}, the (negative) background-electron attraction V_{be}, and the (positive) electron-electron repulsion $V_{ee}\equiv\sum_{\substack{i=1\\i<j}}^{N}e^2/r_{ij}$. The Hamiltonian (8) can be reduced [11] to only two terms

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2+\sum_{\substack{i=1\\i<j}}^{N}{}'\,e^2/r_{ij} , \tag{9}$$

where the prime on the second summation means that a Fourier decomposition of the interaction e^2/r_{ij}, the zero-momentum component [the (diverging) volume integral of the pair interaction] is put equal to zero. Letting $\rho\equiv N/L^2\equiv[(4\pi/3)r_0^3]^{-1}$, we can define dimensionless coordinates $(x',y',z')\equiv(x/r_0,y/r_0,z/r_0)$ and $r'\equiv r/r_0$ to introduce the dimensionless Hamiltonian

$$H'\equiv a_0r_s^2H/e^2=-\tfrac{1}{2}\sum_{i=1}^{N}\nabla_{r_i'}^2+r_s\sum_{\substack{i=1\\i<j}}^{N}{}'\,r_{ij}'^{-1} , \tag{10}$$

where $a_0\equiv\hbar^2/me^2$ is the Bohr radius, and the dimensionless coupling parameter r_s is defined as

$$0\le r_s\equiv r_0/a_0\equiv me^2/\hbar^2\left[\frac{4\pi}{3}\rho\right]^{1/3}<\infty . \tag{11}$$

Evidently, we again encounter the same dynamics as in the 1D N-fermion model, namely, strong (weak) coupling [large (small) e^2] is associated with low (high) density ρ. The 1D N-fermion model may thus also be useful in tracing the evolution [12] from weakly coupled, overlapping "Cooper pairs" of electrons to the strong-coupling extreme of tightly bound "local pairs" (or "bipolarons") which appear to play an increasingly important role in recent theories [13] of high-temperature superconductivity. Table II summarizes this application of the model, first considered by Takahashi [14], which we address in the next section.

TABLE I. Constituent fermions ($\lambda=0$; $v_0=0$ and/or $\rho=\infty$) and ν-fermion clusters ($\lambda=\infty$, $v_0=\infty$ and/or $\rho=0$) of the many-body Hamiltonian (1) or (2).

ν (No. of species)	Fermions	Clusters
2	$q,\bar{q}$	mesons
3	three-color q's	baryons
4	nucleons	alphas

TABLE II. Same as Table I but for "electrons."

ν (No. of species)	Fermions	Clusters
2	electrons	dimers: Cooper pairs ($\lambda\to0$); Local pairs ($\lambda\to\infty$) (bipolarons)

II. COOPER PAIRING

A. BCS model interaction in 3D

The Cooper fermion-pair problem consists of solving the Schrödinger equation (in the momentum representa-

tion) for *two* fermions interacting via the potential $V(r)$, and which cannot scatter by phonon exchange into single-fermion states already occupied by the $N-2$ background fermions. The two fermions have opposite spins and zero center-of-mass motion. Thus, if $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{K}\equiv\mathbf{k}_1+\mathbf{k}_2=\mathbf{0}$ so that $\mathbf{k}_1=-\mathbf{k}_2\equiv\mathbf{k}$, and the intrinsic space wave function is $\psi(\mathbf{r})=\sum_{\mathbf{k}} C_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$. The coefficients $C_{\mathbf{k}}$ then satisfy

$$2\varepsilon_k^0 C_{\mathbf{k}} + \sum_{\mathbf{k}'} C_{\mathbf{k}'} V_{\mathbf{k}',\mathbf{k}} = E C_{\mathbf{k}} ,$$
$$C_{\mathbf{k}} \equiv 0 \quad (k < k_F) , \tag{12}$$

where $\varepsilon_k^0 \equiv \hbar^2 k^2/2m$, E is the pair energy eigenvalue and, if L^D is the D-dimensional system volume, and we have defined

$$V_{\mathbf{k}',\mathbf{k}} \equiv L^{-D} \int d^D r e^{-i\mathbf{k}'\cdot\mathbf{r}} V(r) e^{-i\mathbf{k}\cdot\mathbf{r}} . \tag{13}$$

Cooper [15] employed what is now called the "BCS model interaction": $V_{\mathbf{k}',\mathbf{k}} = -V$ if $E_F < \varepsilon_k^0$, $\varepsilon_{k'}^0 < E_F + \hbar\omega_D$ and $=0$, otherwise, where $\hbar\omega_D$ is the maximum energy a phonon can have, and $V>0$. This simplification leads him to the eigenvalue equation

$$1 = V \sum_{\mathbf{k}}{}' \frac{1}{2\varepsilon_k^0 - E} , \tag{14}$$

where the prime on the summation sign means the $\mathbf{k}$ sum is restricted so that $E_F < \varepsilon_{\mathbf{k}}^0 < E_F + \hbar\omega_D$. The sum can be converted to an energy integral if the density of states $g(\epsilon)$ $[\equiv (L/2\pi)^D d^D k/d\epsilon]$ is introduced, so that (14) becomes

$$1 = V \int_{E_F}^{E_F+\hbar\omega_D} d\epsilon \frac{g(\epsilon)}{2\epsilon - E} \simeq V g(E_F) \int_{E_F}^{E_F+\hbar\omega_D} \frac{1}{2\epsilon - E} d\epsilon , \tag{15}$$

where the last step follows in 3D from the empirical fact that in metals $\hbar\omega_D \ll E_F$. We note that this essentially reduces the problem to a 2D one, where $g(\epsilon)$ is a constant independent of ϵ. The remaining integral is elementary and gives

$$E = 2E_F - \frac{2\hbar\omega_D}{e^{1/\lambda}-1} \xrightarrow[\lambda\to 0]{} 2E_F - 2\hbar\omega_D e^{-1/\lambda} , \tag{16}$$

where $\lambda \equiv g(E_F)V/2$.

Although in arriving at (16) we have started with the problem in the 3D form, it can be shown [16] that the essential singularity $e^{-1/\lambda}$ at $\lambda=0$ *also emerges in 1D.* This means that it is *unrelated* to the 2D property $\hbar\omega_D \ll E_F$ of the BCS interaction model—which restricts the interaction to the immediate vicinity of the Fermi surface—contrary to what might be expected from ordinary 2D quantum binding [17] where a similar essential singularity appears, and which seems to be the conventional wisdom.

B. Attractive δ-potential model in 1D

We wish to solve the same problem in 1D for the interaction model (1), namely, $V(r) = -v_0\delta(x)$, with $v_0>0$ and $x \equiv x_1 - x_2$. Then, (13) gives $-v_0/L$ and instead of (14) we have

$$1 = \frac{v_0}{L} \sum_k{}' \frac{1}{2\varepsilon_k^0 - E} , \tag{17}$$

with the prime now meaning *only* that $E_F < \epsilon_k^0 < \infty$. If this restriction were removed, then (17) becomes for $E<0$

$$1 = \frac{v_0}{L} \frac{L}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\hbar^2 k^2/m + |E|} dk = \frac{v_0}{2\hbar} \sqrt{m/|E|} , \tag{18}$$

thus giving the exact two-body Schrödinger result (4b) for the eigenvalue E, as expected.

But in the corresponding Cooper problem, instead of (15) we now have exactly

$$1 = \frac{v_0}{L} C_1 \int_{E_F}^{\infty} d\epsilon \frac{\epsilon^{-1/2}}{2\epsilon - E} , \tag{19}$$

where $g(\epsilon) = C_1 \epsilon^{-1/2}$ is the 1D density of states with $C_1 \equiv \sqrt{m/2}L/\pi\hbar$. Introducing the dimensionless $\varepsilon \equiv E/2E_F$, the integral in (19) gives

$$\frac{1}{2\sqrt{E_F}} \frac{1}{\sqrt{\varepsilon}} \ln\left[\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right] \quad \text{if } \varepsilon>0 , \tag{20}$$

$$\frac{1}{\sqrt{E_F}} \frac{1}{\sqrt{|\varepsilon|}} \left[\frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{|\varepsilon|}}\right] \quad \text{if } \varepsilon<0 . \tag{21}$$

The Fermi energy $E_F \equiv \hbar^2 k_F^2/2m$ is related to the number density $\rho = N/L = 2k_F/\pi$, and since $\lambda \equiv m v_0/\hbar^2\rho$ the prefactor in (19) is then just $2\lambda\sqrt{E_F}/\pi^2$, so that (19)–(21) lead to the *two* implicit transcendental equations for ε, namely,

$$\pi^2\sqrt{\varepsilon}/\lambda = \ln\left[\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right] \quad \text{if } \varepsilon>0 , \tag{22}$$

$$\pi^2\sqrt{|\varepsilon|}/\lambda = \pi - 2\tan^{-1}\frac{1}{\sqrt{|\varepsilon|}} \quad \text{if } \varepsilon<0 . \tag{23}$$

Both right-hand side (rhs) members behave like $2\sqrt{|\varepsilon|} + O(|\epsilon|^{3/2})$ for small $|\varepsilon|$, but (22) is concave up in ϵ while (23) is concave down in $|\epsilon|$. Consequently, there will be *nontrivial* ($|\varepsilon|>0$) *solutions* (NTS) whenever

$$0<\lambda<\pi^2/2 \quad \text{for } \varepsilon>0 ,$$
$$\pi^2/2<\lambda \quad \text{for } \varepsilon<0 , \tag{24}$$

with the value $\lambda=\pi^2/2$ corresponding to $\varepsilon=0$. Figure 3 illustrates the graphical solutions, where solid curves refer to the rhs (thick curve) and left-hand side (lhs) (thin curve) of (22), while dashed curves refer to the rhs (thick dashed curve) and lhs (thin dashed curve) of (23). The dot-dashed straight line is the asymptote $2\sqrt{|\varepsilon|}$ of both rhs members of (22) and (23) for small $|\varepsilon|$.

Consider the two "Leggett extremes" [12] of $\lambda\to 0+$ and $\lambda\to\infty$ are, respectively, the weakly bound, strongly overlapping Cooper-pair limit and the tightly bound (point boson) "dimer" limit. Let $\Delta \equiv 2E_F - E$ be the (positive) binding energy of the weakly bound Cooper pair. Defining $\delta \equiv \Delta/2E_F$, (22) can be expanded for $1-\varepsilon \equiv \delta \to 0+$ and yields $\pi^2/\lambda \simeq -\ln(\delta/4)$ or

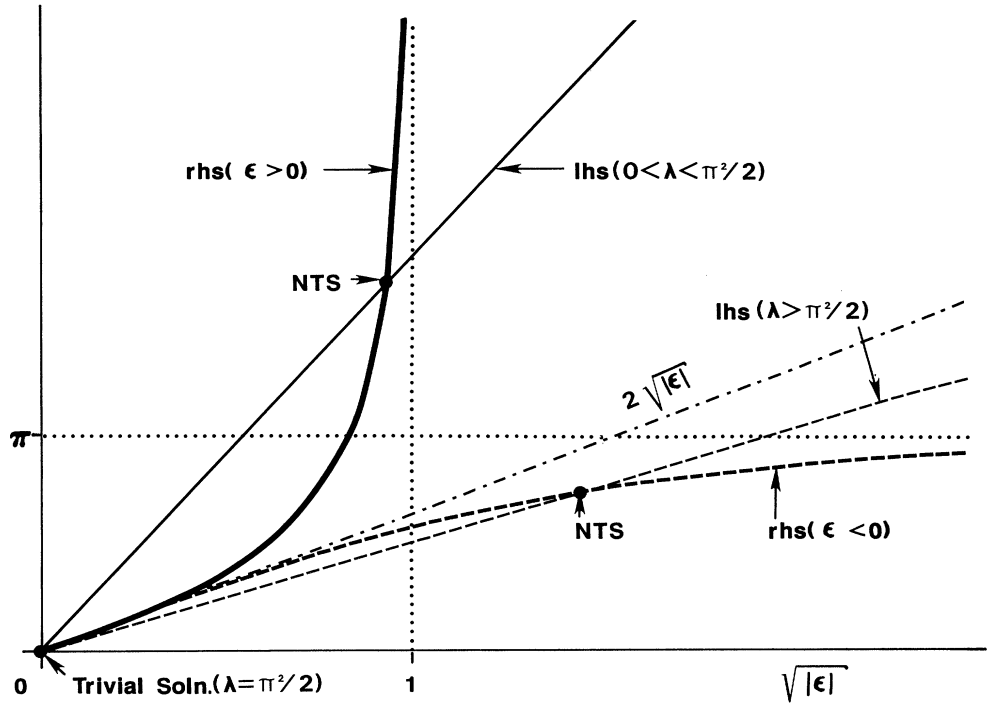

FIG. 3. rhs and lhs of (22) (full curves) and of (23) (dashed curves), plotted against $\sqrt{|\varepsilon|}$. Intersections denote NTS for both positive and negative $\varepsilon \equiv E/2E_F$ values. The dot-dashed line is the small $\sqrt{|\varepsilon|}$ asymptote of the rhs of both (22) and (23).

$$\delta = 4e^{-\pi^2/\lambda} \text{ as } \lambda \to 0 , \tag{25}$$

meaning that

$$2E_F - E \equiv \Delta \simeq 8E_F e^{-\pi^2\hbar^2\rho/mv_0} \neq \Delta_0 + \Delta_1 v_0 + \Delta_2 v_0^2 + \cdots . \tag{26}$$

The same type essential singularity in coupling (25) also appears in the 3D Cooper-pair problem [15], in the 3D many-electron BCS theory gap parameter and, finally, in the BCS superconductive transition temperature [11]. On the other hand, $\lambda \equiv mv_0/\hbar^2\rho \to \infty$ is equivalent to finite v_0 and ρ (or E_F)$\to 0$, and thus should correspond to treating the two fermions *in a vacuum*. In this case (23) becomes, since $|\varepsilon| \equiv E/2E_F \to \infty$,

$$\pi^2\sqrt{|\varepsilon|}/\lambda = \pi - \frac{2}{\sqrt{|\varepsilon|}} + \cdots \quad \text{or} \quad \sqrt{|\varepsilon|} \simeq \lambda/\pi , \tag{27}$$

which upon squaring again gives $E = E_0(2)$ of (4b), as it should.

The Leggett question [12] of "what happens to a Cooper pair as density is decreased" is here trivially but explicitly answered: it becomes a point bound dimer (or "bipolaron"). More significantly, this "dimerization" can also be seen to occur when *all N* fermions (not just Cooper-pair partners) are treated on an equal footing, as in the BCS many-fermion theory to which we now turn.

III. BCS GAP AND NUMBER EQUATIONS

For the derivation of the two fundamental equations of the BCS theory of an interacting many-fermion system in any dimension we follow the notation of Fetter and Walecka's text [11]. That discussion is more general than most treatments in that it does not assume the chemical potential μ to coincide with the value E_F (which depends on ρ but not on v_0), a result occurring only for weak coupling. For the Hamiltonian (1) the interaction matrix elements $\langle k_1 k_2 | V | k_3 k_4 \rangle$ are simply $(v_0/L)\delta_{k_1+k_2,k_3+k_4}$, and the HF single-particle energies ε_k then become

$$\varepsilon_k = \varepsilon_k^0 - \tfrac{1}{2}\rho v_0 , \tag{28}$$

where as before $\varepsilon_k^0 \equiv \hbar^2 k^2/2m$. Setting $\xi_k \equiv \varepsilon_k - \mu$, the gap parameter Δ is defined as

$$\Delta \equiv \frac{v_0}{L}\sum_k v_k (1 - v_k^2)^{1/2} , \tag{29}$$

where the occupation probability v_k^2 in a state k (for *one*-spin direction) is given by

$$v_k^2 \equiv \tfrac{1}{2}(1 - \xi_k/E_k) , \tag{30}$$

$$E_k \equiv \sqrt{\Delta^2 + \xi_k^2} . \tag{31}$$

Thus, (29) simplifies to the *gap equation*

$$1 = \frac{v_0}{2L}\sum_k \frac{1}{E_k} \to \frac{v_0}{4\pi}\int_{-\infty}^{\infty} dk \left[\left[\frac{\hbar^2 k^2}{2m} - \frac{1}{2}\rho v_0 - \mu \right]^2 + \Delta^2 \right]^{-1/2} . \tag{32}$$

Simultaneously, the *number equation* (which determines μ) becomes

$$N=2\sum_k v_k^2=\sum_k(1-\xi_k/E_k)\rightarrow\frac{L}{2\pi}\int_{-\infty}^{\infty}dk\left[1-\frac{\varepsilon_k-\mu}{[(\varepsilon_k-\mu)^2+\Delta^2]^{1/2}}\right]. \tag{33}$$

The two coupled, implicit equations (32) and (33) must be solved (numerically, at worst) to determine both Δ and μ as functions of the dimensionless coupling parameter λ. This would ultimately allow the total energy per particle (which for $\lambda\rightarrow 0$ is well known [11] to be quadratic in Δ) to be calculated in the BCS approximation for all $0<\lambda<\infty$. This computation is in progress, but we report the result for the two extremes $\lambda=0$ and ∞ which is straightforward.

For $\rho\rightarrow 0$ (33) implies that $v_k\rightarrow 0$ which from (29) means that $\Delta\rightarrow 0$. The gap equation (32) then becomes an implicit equation for μ alone,

$$1=\frac{v_0}{2\pi}\int_0^{\infty}dk\,(\hbar^2k^2/2m-\mu)^{-1}, \tag{34}$$

which, if $\mu=-|\mu|$, gives the same type of integral as in (18), so that we immediately find

$$\mu=-\tfrac{1}{2}|E|\equiv-\tfrac{1}{2}\frac{mv_0^2}{4\hbar^2}=\tfrac{1}{2}E_0(2), \tag{35}$$

i.e., μ in this limit is *one-half* the energy of a single cluster $E_0(2)$ as given by (4b). One can further see that the exact $\rho\rightarrow 0$ ($\lambda\rightarrow\infty$) total-energy result (3) is fully reproduced in the BCS approximation. This comes from the fact that the BCS μ given by (35) is indeed the *exact* value of μ in this limit. The latter assertion follows from (3), (A7), and the Hugenholtz-van Hove theorem [18] stating that the ground-state energy per particle $E_0(N)/N$ of an interacting many-fermion system at zero pressure ($P\equiv\rho^2\partial[E_0(N)/N]/\partial\rho=0$) is identical with its chemical potential μ. (A result equivalent to the first equality in (35) has been obtained [19] in the zero-density limit of the BCS theory of a 2D fermion gas with arbitrary short-ranged pair interactions.) Finally, in the other (trivial) extreme $\lambda\rightarrow 0$, or $v_0\rightarrow 0$, from (29) Δ again vanishes. This reduces v_k^2 in (30) to the step function $\Theta(\mu-\hbar^2k_F^2/2m)$, which in (33) with $\rho\equiv N/L=2k_F/\pi$ integrates to give $\mu=\hbar^2k_F^2/2m\equiv E_F$, as expected.

IV. CONCLUSIONS

The exactly soluble many-fermion system with pairwise attractive δ-function interactions in 1D possesses the same qualitative dynamics as the 3D electron fluid jellium model, and is found useful to study the highly correlated electron fluid.

The Cooper-pair problem can be solved for the model exactly for all coupling and density in graphical form. The weak-coupling and/or high-density extreme is characterized by the same type essential singularity of the 3D Cooper problem with the BCS model interaction, of the many-electron BCS theory gap parameter and of the BCS critical-temperature formula of low-temperature superconductivity.

It will be interesting to solve, numerically if necessary, the BCS theory equations to compute the ground-state energy and compare with the exact results for all values of coupling and/or density. However, one already notes that perfect coincidence occurs at *both* extremes, namely (a) of weakly bound, strongly overlapping Cooper pairs, and (b) of tightly bound, point (bipolaronic) dimers. Low-temperature superconductivity is characterized by the large, overlapping pairs exemplified in extreme (a), whereas high-temperature suprconductivity empirically reveals the presence of much smaller pairs, approaching extreme (b).

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APPENDIX A: GAUDIN EQUATIONS—VIRIAL EXPANSION AND NUMERICAL SOLUTION

In the notation of Sec. I with $\tilde{\varepsilon}(\lambda)\equiv E_0(N)/[N/2]|E_0(2)|$, the exact ground-state energy per particle for $\lambda=\infty$ ($\rho=0$ and/or $v_0=\infty$) in dimensionless form is simply $\tilde{\varepsilon}(\infty)=-1$ (Fig. 2, lowest dot on ordinate axis). The Gaudin [6] equations allow one to obtain $\tilde{\varepsilon}(\lambda)$ for any $0\le\lambda<\infty$. In dimensionless form they are

$$F(x)=2-\frac{K}{\pi}\int_{-1}^{1}dy\frac{F(y)}{1+K^2(x-y)^2}, \tag{A1}$$

$$\frac{1}{\lambda}=\frac{K}{\pi}\int_{-1}^{1}dy\,F(y), \tag{A2}$$

$$\tilde{\varepsilon}(\lambda)=-1+\frac{4}{\pi}K^3\lambda\int_{-1}^{1}dy\,y^2F(y), \tag{A3}$$

where $K\equiv Q/v_0$ is a non-negative parameter to be eliminated from (A3) with the help of (A1) and (A2).

A "virial" (or low-density) expansion of $\tilde{\varepsilon}(\lambda)$ can be easily generated. For $\rho\rightarrow 0$ ($\lambda\rightarrow\infty$) (A1) gives $F(x)\rightarrow 2$. Since $Q\rightarrow 0$ we can expand the integrand in (A1) with $F(y)=2$ and get

$$\begin{aligned}F(x)&=2-\frac{2K}{\pi}\int_{-1}^{1}dy[1-K^2(x-y)^2+\cdots]\\&=2-\frac{4}{\pi}K+\frac{4}{3\pi}K^3+\frac{4}{\pi}K^3x^2+O(K^5).\end{aligned} \tag{A4}$$

TABLE III. Ground-state energy per particle of (1) or (2) in dimensionless form, as defined in Appendix A, obtained by numerical integration of Gaudin equations (A1)–(A3) (exact), and by the low-density three-term expansion (A7) (virial).

λ^{-1}	Exact $\tilde{\varepsilon}(\lambda)$ [Eq. (A3)]	Virial $\tilde{\varepsilon}(\lambda)$ [Eq. (A7)]
0.00	−1.0000	−1.0000
0.05	−0.9978	−0.9978
0.1	−0.9909	−0.9910
0.15	−0.9784	−0.9787
0.2	−0.9595	−0.9605
0.25	−0.9333	−0.9357
0.3	−0.8989	−0.9038
0.35	−0.8551	−0.8640
0.4	−0.8011	−0.8158
0.5	−0.6588	−0.6916
0.6	−0.4653	−0.5263
0.7	−0.2159	−0.3149
0.8	+0.0929	−0.0525
0.9	+0.4633	+0.2658
1	+0.8970	+0.6449

Substituting this in (A2) gives

$$\frac{1}{\lambda}=\frac{4K}{\pi}\left[1-\frac{2K}{\pi}+\frac{4}{3\pi}K^3\right]+O(K^6) \tag{A5}$$

or

$$K=\frac{\pi}{4}\frac{1}{\lambda}+\frac{\pi}{8}\frac{1}{\lambda^2}+O(1/\lambda^3)\ . \tag{A6}$$

Inserting (A6) into (A4) and (A3) gives

$$\tilde{\varepsilon}(\lambda)=-1+\frac{\pi^2}{12}\frac{1}{\lambda^2}-\frac{\pi^2}{24}\frac{1}{\lambda^3}+O(1/\lambda^4) \tag{A7}$$

since $\lambda\equiv mv_0/\hbar^2\rho$ is a series in powers of ρ beginning with ρ^2.

The Gaudin equations (A1)–(A3) were integrated numerically to obtain $\tilde{\varepsilon}(\lambda)$ (full thick curve in Fig. 2). This was accomplished by discretizing the integrals and solving the ensuing algebraic equations. The parameter K was adjusted iteratively until the desired value of λ was reproduced. Table III lists some values of the exact $\tilde{\varepsilon}(\lambda)$, some of which are plotted in Fig. 2 (full thick curve). Also given are values of the virial expansion (A7), which is seen to be quite adequate up to about $\lambda^{-1}=0.3$. (Exact values listed in the table replace erroneous numbers reported and graphed in Ref. 4, although both sets of data being qualitatively similar would not alter the conclusions there.)

APPENDIX B: SEMICLASSICAL MEAN-FIELD APPROXIMATION—THOMAS-FERMI SOLUTION

Following the method proposed by Dagens [20] and developed in Refs. [21] and [22] one can solve the Hamiltonian (1) in a self-consistent mean-field approximation using a semiclassical expansion in powers of $\hbar$. In coordinate space, for a two-body interaction $v(x_i,x_j)$ which does not depend upon spin variables, the local Hartree potential is

$$U_H(x_i)=\int dx_j v(x_i,x_j)\rho(x_j)\ , \tag{B1}$$

and the averaged exchange potential can be written as

$$U_{\rm ex}(x_i)=-\int dx_j \frac{v(x_i,x_j)\rho^2(x_i,x_j)}{\rho(x_i)}\ , \tag{B2}$$

where $\rho(x_j)$ and $\rho(x_i,x_j)$ are the diagonal and the off-diagonal parts of the density matrix. Following Refs. [22] and [23], (B1) can be solved self-consistently using the semiclassical approach for the diagonal matter density (to order $\hbar$ giving Thomas Fermi, or to order $\hbar^2$ giving the Wigner-Kirkwood approximation). For the off-diagonal density we use the Slater approximation and this allows us to express $\rho(x_i,x_j)$ in terms of the diagonal density $\rho(x_i)$. For the one-dimensional case this yields

$$\rho(x_i,x_j)=\frac{\nu k_F}{\rho}j_0(k_F[x_i-x_j])$$

where $k_F=\pi\rho/\nu$ and ν is the number of distinct fermion species. Using the Thomas-Fermi approach for the diagonal density, (B1) becomes

$$U_H^{(n+1)}(x_i)=\frac{\nu}{\pi\hbar}\int_{x_-}^{x_+}dx_j v(x_i,x_j) \times\{2m[\mu^{(n)}-U_H^{(n)}(x_j)]\}^{1/2}\ , \tag{B3}$$

where x_+ and x_- are the two turning points, the index n refers to the order of iteration, and $\mu^{(n)}$ is the chemical potential at the nth iteration, calculated by requiring that the integral of $\rho(x_j)$ gives the correct number of particles. In order to find the self-consistent mean field we iterate up to self-consistency.

If the two-body interaction is a δ potential as in (1) the self-consistent mean field is given by

$$U=-v_0\rho(1-1/\nu)\ , \tag{B4}$$

and the chemical potential is then

$$\mu=\frac{\hbar^2}{8m}\frac{\rho^2}{\nu^2}-v_0\rho(1-1/\nu)\ . \tag{B5}$$

The total energy for a system of N particles is given by

$$\langle p^2/2m\rangle+\frac{N}{2}U\ ,$$

where the first term is the Thomas-Fermi kinetic energy, namely,

$$\langle p^2/2m\rangle=\frac{\nu}{6\pi\hbar m}\int_{x_-}^{x_+}(\mu-U)^{3/2}dx\ . \tag{B6}$$

Thus the total energy per particle is

$$\frac{E(\rho)}{N}=\frac{\hbar^2\pi^2\rho^2}{6m\nu^2}-\frac{v_0\rho}{2}(1-1/\nu)\ , \tag{B7}$$

which in the limit $\rho\to\infty$ coincides with the ideal Fermi-gas average energy $\hbar^2k_F^2/6m$. As expected, the Thomas-Fermi result (B7) coincides with the Hartree-Fock plane-wave (HFPW) solution for an infinite system (curve la-

beled HFPW in Fig. 2).

In some cases it is possible to solve (B3) using the Wigner-Kirkwood approach for the density, going to order $\hbar^2$ in the semiclassical approach [21,22]. However, if the two-body interaction is $\delta(x_i - x_j)$ the mean field is always proportional to the density of the ground state to all orders in $\hbar^{2n}$. Then, as shown in [21], the matter density to order $\hbar^{2n}(n \geq 1)$ is a distribution in the mathematical sense and can only be used in computing expectation values.

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