Controlling chaos to generate aperiodic orbits

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We show how a chaotic system is able to generate a desired aperiodic orbit by making only small temporal perturbations to an available set of system parameters. The appropriate controls are obtained by applying the method developed by Ott, Grebogi, and Yorke [Phys. Rev. Lett. **64**, 1196 (1990)] to an artificially constructed dynamical system such that when this dynamical system is at its steady state the output of the chaotic system should be the desired aperiodic orbit. We illustrate our method with some numerical examples in which the motion of a chaotic system is converted to two different aperiodic orbits.

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I. INTRODUCTION

In a recent Letter, Ott, Grebogi, and Yorke [1] (OGY) describe a method whereby small time-dependent perturbations are applied to an accessible set of parameters in a chaotic system so as to produce a desired attracting time periodic motion or steady state. The efficacy of this technique has been demonstrated on actual physical systems by Ditto, Rauseo, and Spano [2], and by Azevedo and Rezende [3]. Another related method is described by Shinbrot *et al.* [4] that employs chaos to direct trajectories to targets.

In this paper we wish to enhance the appeal of OGY's pioneering work in the rapidly emerging field of controlling chaos. In particular, we generalize the technique so that by applying *small* temporal perturbations to an available set of system parameters in a chaotic system we generate desired *aperiodic* orbits within some bounded region or, perhaps, even different *chaotic* trajectories. This generalization enables chaotic systems to operate in a wider variety of situations than that originally envisioned. We illustrate our approach with some numerical examples.

The rest of this paper is organized as follows. In Sec. II we outline OGY's method. In Sec. III, we describe how to generate aperiodic orbits. Numerical examples are presented in Sec. IV. We end the paper with conclusions in Sec. V.

II. REVIEW OF OGY'S METHOD

Before presenting our generalization we briefly recapitulate OGY's technique. This method is based on the fact that a chaotic attractor has densely embedded within it an infinite number of unstable periodic orbits and, hence, is extremely sensitive to small perturbations. Therefore, as outlined below, small controls may be applied to the chaotic system so as to stabilize its output about its steady state. Without loss of generality we restrict our review to two-dimensional discrete-time systems that depend on one externally adjustable parameter p,

$$\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n,p)\ , \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^2$ are the system variables, $\mathbf{F}$: $\mathbb{R}^2\to\mathbb{R}^2$, and n is the discrete-time variable. The control parameter p is allowed to vary in a small range about some nominal value p_0, which still maintains the chaotic nature of (1). We arbitrarily set p_0 equal to zero.

In what follows we assume that the chaotic system (1) is controlled so that it remains at a steady state $\mathbf{x}_F(p)$ of (1) corresponding to $p=0$. For instances where it is desired that the output of the chaotic system be a periodic orbit, and for extensions to higher-dimensional continuous-time and discrete-time dynamical systems, the reader is referred to Ref. [1]. Furthermore, as described in Ref. [1], by utilizing delay coordinate embedding the method is applicable to situations where complete *a priori* knowledge of the system dynamics is unavailable.

We now illustrate how to derive the control which will drive the output of the chaotic system back to its fixed point $\mathbf{x}_F(0)$ when the system is perturbed away from it. As the control parameter p is varied slightly from $p_0=0$ to some value $p=\bar{p}$, the fixed point $\mathbf{x}_F(p)|_{p=0}=\mathbf{x}_F(0)$ will shift to some nearby point $\mathbf{x}_F(p)|_{p=\bar{p}}$. Thus the vector $\mathbf{g}$ may be defined as

$$\mathbf{g}\equiv\left.\frac{\partial\mathbf{x}_F(p)}{\partial p}\right|_{p=0}\approx\frac{\mathbf{x}_F(p)-\mathbf{x}_F(0)}{p} \tag{2}$$

for small values of the control p. Note that Eq. (2) allows for an experimental determination of the vector $\mathbf{g}$.

Near the fixed point $\mathbf{x}_F(0)$ and for small values of the control p, a linear approximation for the map (1) is given by

$$[\mathbf{x}_{n+1}-\mathbf{x}_F(p)]\approx\underline{M}\cdot[\mathbf{x}_n-\mathbf{x}_F(p)]\ , \tag{3}$$

where $\underline{M}$ is a 2×2 Jacobian matrix of $\mathbf{F}$ evaluated at $\mathbf{x}_F(0)$. Let λ_u and λ_s denote the unstable and stable eigenvalues of the matrix $\underline{M}$, respectively, $(|\lambda_u|>1>|\lambda_s|)$. These can be determined experimentally if the system dynamics are not known. Thus $\underline{M}\mathbf{e}_u=\lambda_u\mathbf{e}_u$ and $\underline{M}\mathbf{e}_s=\lambda_s\mathbf{e}_s$, where $\mathbf{e}_u$ and $\mathbf{e}_s$ are the unstable and stable eigenvectors of unit magnitude of the matrix $\underline{M}$, respectively. Furthermore, let $\mathbf{f}_u$ and $\mathbf{f}_s$ denote the contravariant basis vectors defined by

$$\mathbf{f}_s\cdot\mathbf{e}_s=\mathbf{f}_u\cdot\mathbf{e}_u=1,\quad \mathbf{f}_s\cdot\mathbf{e}_u=\mathbf{f}_u\cdot\mathbf{e}_s=0\ . \tag{4}$$

Note that $\mathbf{e}_u$ and $\mathbf{e}_s$ are column vectors and $\mathbf{f}_u$ and $\mathbf{f}_s$ are row vectors.

Hence Eq. (3) may be expressed [5] as

$$[\mathbf{x}_{n+1}-\mathbf{x}_F(p)]\approx(\lambda_u\mathbf{e}_u\mathbf{f}_u+\lambda_s\mathbf{e}_s\mathbf{f}_s)\cdot[\mathbf{x}_n-\mathbf{x}_F(p)]\ . \tag{5}$$

This equation relates $\mathbf{x}_n$ to $\mathbf{x}_{n+1}$ from which the temporally determined control parameter p_n at iteration n can be computed. Upon observing $\mathbf{x}_n$, the control parameter p is adjusted to p_n so that $\mathbf{x}_{n+1}$ falls near the fixed point $\mathbf{x}_F(0)$. Observing from Eq. (2) that

$$\mathbf{x}_F(p)\approx p\mathbf{g}+\mathbf{x}_F(0)\ , \tag{6}$$

Eq. (5) can be rewritten as

$$\mathbf{x}_{n+1}\approx p_n\mathbf{g}+\mathbf{x}_F(0) +(\lambda_u\mathbf{e}_u\mathbf{f}_u+\lambda_s\mathbf{e}_s\mathbf{f}_s)\cdot[x_n-p_n\mathbf{g}-\mathbf{x}_F(0)]\ . \tag{7}$$

Since it is desired that $\mathbf{x}_{n+1}$ falls on the stable manifold of $\mathbf{x}_F(0)$, the control p_n is chosen so that

$$\mathbf{f}_u\cdot\mathbf{x}_{n+1}=0\ . \tag{8}$$

Thus, taking the inner product of Eq. (7) and $\mathbf{f}_u$, and solving for the control p_n, we get

$$p_n=\frac{(1-\lambda_u)\mathbf{f}_u\cdot\mathbf{x}_F(0)+\lambda_u\mathbf{f}_u\cdot\mathbf{x}_n}{(\lambda_u-1)\mathbf{f}_u\cdot\mathbf{g}}\ . \tag{9}$$

We assume that $\mathbf{f}_u\cdot\mathbf{g}\neq\mathbf{0}$. Observe that if $\mathbf{x}_F(0)=\mathbf{0}$ then the control p_n given by Eq. (9) is identical to that shown in Ref. [1].

Since the control p_n depends on $\mathbf{x}_n$, a new control must be calculated at each iteration n. This control is applied by adjusting the tunable parameter of the chaotic system by an amount p_n which then drives the output of the chaotic system to the stationary point $\mathbf{x}_F(0)$.

The control policy we employ is given by

$$p_n=\begin{cases}p_*, & p_n\geq p_*\\ -p_*, & p_n\leq -p_*\\ p_n, & -p_*<p_n<p_*\ .\end{cases} \tag{10}$$

We imagine that this control policy comes into effect only after the initial transients have died.

Note that if the control p_n determined from Eq. (9) is such that $|p_n|>p_*$, then the output $\mathbf{x}_n$ of the dynamical system will behave in a chaotic fashion and will wander from $\mathbf{x}_F(0)$. However, the orbit of the chaotic attractor will, within some finite amount of time, return to an arbitrarily small neighborhood of $\mathbf{x}_F(0)$. When this happens a control of the appropriate magnitude may then be applied to the chaotic system so as to drive it back to $\mathbf{x}_F(0)$. Thus an arbitrarily small set of controls that will drive the chaotic system output to the desired fixed point can always be found.

III. GENERATING APERIODIC ORBITS

With this background, we now proceed to illustrate how the OGY technique may be employed to generate aperiodic orbits. The key to generating aperiodic trajectories from a chaotic system is to formulate a related chaotic system and then apply OGY's method on this artificially constructed system. This system is such that when it is at its steady state the output of the chaotic system should be the desired orbit. This artificial device is employed only in a theoretical sense to derive the appropriate controls used to regulate the chaotic system and is not physically realized.

To obtain these appropriate controls so that the output of the chaotic system is the desired aperiodic orbit we consider the related chaotic system whose output $\mathbf{e}_{n+1}$ is a measure of the error between the desired aperiodic motion $\mathbf{r}_{n+1}$ and the output of the original chaotic system $\mathbf{x}_{n+1}$; that is, $\mathbf{e}_{n+1}=\mathbf{r}_{n+1}-\mathbf{x}_{n+1}$, where $\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n,p)$ and p is the controllable parameter. Noting that $\mathbf{x}_n=\mathbf{r}_n-\mathbf{e}_n$, and that $\mathbf{r}_n$ depends on n, $\mathbf{e}_{n+1}$ may be written as

$$\mathbf{e}_{n+1}=\mathbf{h}(\mathbf{e}_n,n,p)\ . \tag{11}$$

We now apply OGY's method to determine the controls which, when applied to the chaotic system (1), drive the system (11) to its steady state. Because (11) is a nonautonomous system, a time-invariant steady state $\mathbf{e}_F(p)|_{p=0}$ may not exist as required by OGY's method. Instead, the steady state will trace out an arc in phase space over time. Consequently, the error between the given reference signal and the output of the controlled chaotic system $\mathbf{x}_n$ will not remain constant, much less be zero. On the other hand, if the desired reference signal does not vary much over time, system (11) behaves in a manner similar to that of an autonomous system over the region of interest and allows for the application of OGY's method. As a result, the steady state $\mathbf{e}_F(0)$ is almost time invariant, implying that the error between the reference signal and the output of the controlled chaotic system $\mathbf{x}_n$ can be made as small as possible. This leads to near perfect generation of the reference orbit by the controlled chaotic system (1).

In general, however, the given reference orbit will not be as described above. Therefore, if we want to employ OGY's technique we first have to appropriately transform the given reference signal so that it meets the above criteria. This transformed signal $\mathbf{r}_n$ is obtained by an appropriate scaling and translation of the given reference signal $\bar{\mathbf{r}}_n$. If $\mathbf{T}$: $\mathbb{R}^2\to\mathbb{R}^2$ represents this transformation, $\mathbf{r}_n=\mathbf{T}(\bar{\mathbf{r}}_n)$. OGY's method is then employed to determine the controls that must be applied to the chaotic system (1) so that it generates the transformed orbit $\mathbf{r}_n$ instead of the original reference signal $\bar{\mathbf{r}}_n$. The desired reference orbit $\bar{\mathbf{r}}_n$ can be reconstructed by applying the inverse transformation to the output of the chaotic system. That is $\bar{\mathbf{r}}_n=\mathbf{T}^{-1}(\mathbf{x}_n)$.

IV. NUMERICAL EXAMPLES

We elucidate our technique of having a chaotic system generate a given aperiodic trajectory by considering the logistic map

$$x_{n+1}=(\mu_0+p)x_n(1-x_n)\ , \tag{12}$$

which is chaotic when $3.57<\mu_0+p<4$, where p is the controllable parameter. In our numerical computations we have set $\mu_0=3.9$.

Let $\bar{r}_{n+1}=t(n)$ denote a generic time-dependent aperiodic reference signal that we wish to generate and let r_{n+1} denote a scaled and translated version of $\bar{r}_{n+1}$. Recall that r_n is the orbit that will actually be generated by (12). Thus, from Eq. (11), the error e_{n+1} is given by

$$\begin{aligned} e_{n+1}&=r_{n+1}-x_{n+1} \\ &=r_{n+1}-\mu x_n(1-x_n)\ , \end{aligned} \tag{13}$$

where $\mu=\mu_0+p$. Substituting (r_n-e_n) for x_n yields

$$e_{n+1}=r_{n+1}-\mu(r_n-e_n)[1-(r_n-e_n)]\ . \tag{14}$$

The fixed points $e_F(0)$ for the system (14) are determined as

$$e_F=\frac{1-\mu+2\mu r_n}{2\mu}-\frac{[(\mu-1)^2+4\mu(r_n-r_{n+1})]^{0.5}}{2\mu} \tag{15}$$

and

$$e_F=\frac{1-\mu+2\mu r_n}{2\mu}+\frac{[(\mu-1)^2+4\mu(r_n-r_{n+1})]^{0.5}}{2\mu}\ . \tag{16}$$

If Eq. (16) is taken to be the fixed point, then it is possible for $g\approx 0$, which from Eq. (2) will result in impractically large values of the control parameter p. Hence, for our computations we have selected the fixed point $e_F(0)$ given by Eq. (15).

The dependence of $e_F(0)$ on r_n and r_{n+1} means that the value of the aperiodic orbit at time $(n+1)$ is required in order to compute the control p_n at time n. To ensure that the argument under the square root in Eq. (15) is non-negative, the difference between the reference trajectory at time n and at time $(n+1)$ should be small in relation to $(\mu-1)^2/4\mu$. Furthermore, as described above, one way in which the effect of the discrete-time variable n on the steady state $e_F(0)$ is reduced is if the reference trajectory r_n varies only slightly within some bounded region.

In general, the original reference trajectory $\bar{r}_n$ will not satisfy these conditions. Hence $\bar{r}_n$ may have to be scaled by the factor β, where β is a small number, and then translated. To obtain the appropriate translation and the appropriate scaling factor β, we first have to get an approximation of $e_F(0)$ from Eq. (15). If $\bar{r}_n$ is going to be transformed as illustrated above, then $(r_n-r_{n+1}) \ll (\mu-1)^2/4\mu$. Hence $e_F(0)$ may be approximated as

$$e_F(0)\approx r_n-\left[1-\frac{1}{\mu}\right]\ . \tag{17}$$

In order that the output of the chaotic system x_n closely conforms to the reference trajectory r_n, $|e_F(0)|$ should be as small as possible. For this reason, we select r_n as

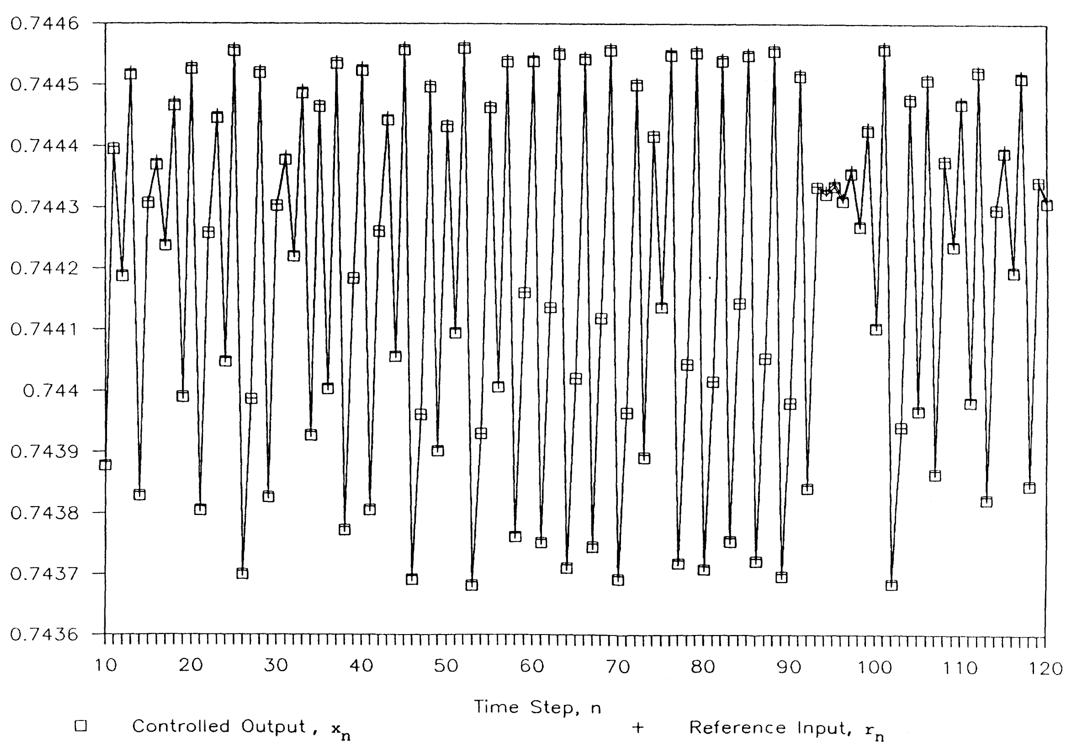

FIG. 1. Generation of $\bar{r}_{n+1}=\bar{r}_n\exp[3.5(1-\bar{r}_n)]$.

$$r_n = \beta \bar{r}_n + \left[1 - \frac{1}{\mu} \right] . \tag{18}$$

To compute the necessary control given by Eq. (10) we need to determine g and the unstable eigenvale λ_u. Since the dynamics of the system (14) are fully known, g and λ_u can be determined analytically. Furthermore, since we are considering a one-dimensional system, the quantity f_u will cancel out of Eq. (10) and is therefore unnecessary to calculate.

Recalling the $\mu = \mu_0 + p$, g is easily shown to be

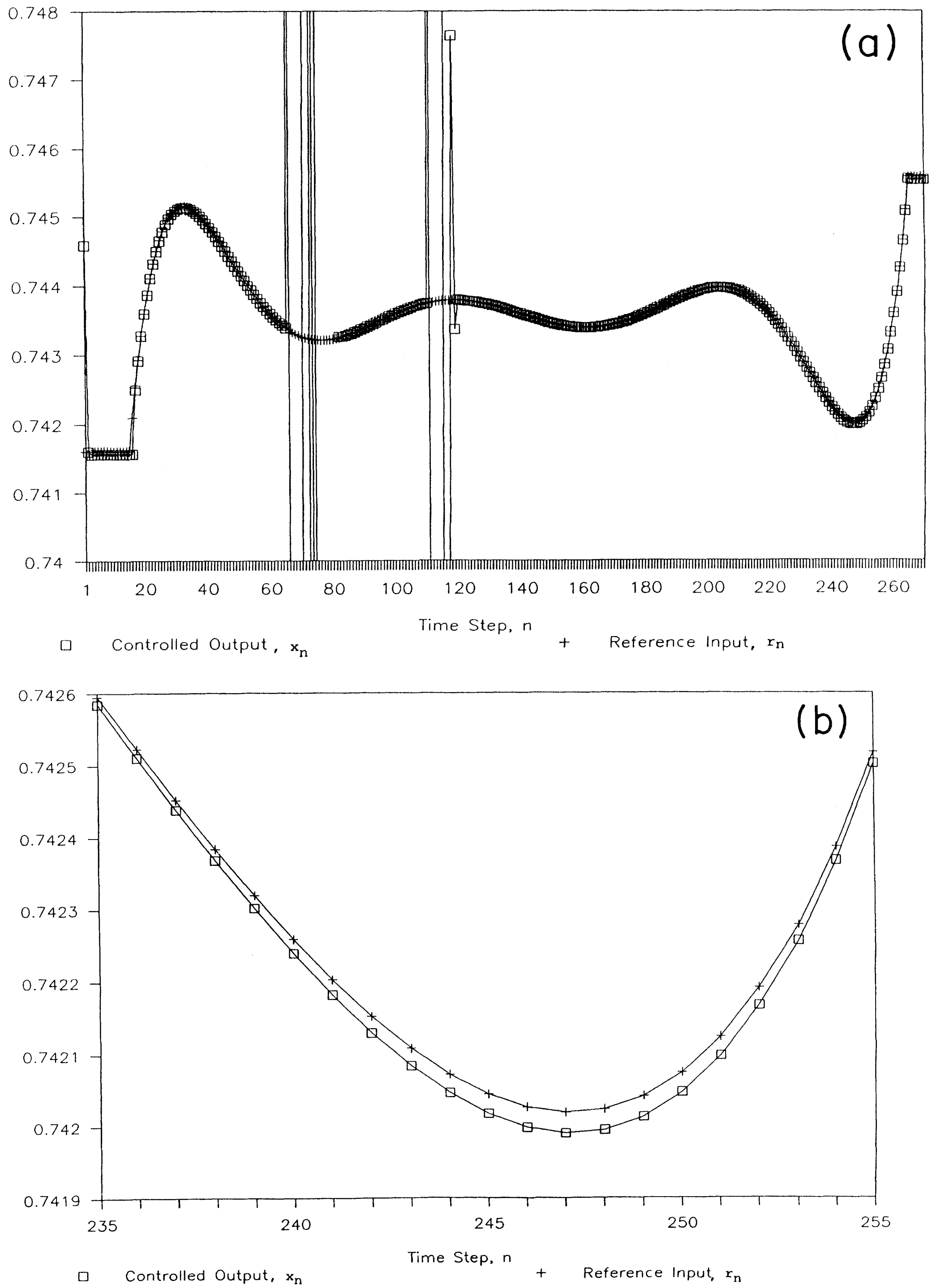

FIG. 2. (a) Generation of $\bar{r}_n = (n-20)(n-60)(n-100)(n-140)(n-180)(n-220)(n-260)$. (b) Detail of (a) for $235 \le n \le 255$.

$$g=-\frac{1}{2\mu_0^2}-\frac{1}{2}\left[\frac{2(\mu_0-1)+4(r_n-r_{n+1})}{2\mu_0[(\mu_0-1)^2+4\mu_0(r_n-r_{n+1})]^{0.5}}-\frac{[(\mu_0-1)^2+4\mu_0(r_n-r_{n+1})]^{0.5}}{\mu_0^2}\right], \tag{19}$$

and the eigenvalue λ is determined as

$$\lambda=\mu-2\mu[r_n-e_F(0)] . \tag{20}$$

If r_n is obtained from the original reference signal $\bar{r}_n$ as shown above, then it is easy to see that λ is an unstable eigenvalue as $|\lambda|>1$. Hence, for the logistic map considered here, the control p_n, from (10) takes the following form:

$$p_n=\frac{(1-\lambda)e_F(0)+\lambda e_n}{(\lambda-1)g} , \tag{21}$$

where $e_n=r_n-x_n$.

We demonstrate the performance of our method by generating two different aperiodic trajectories. In both cases we plot graphs that show only how the output x_n of the chaotic system generates the aperiodic orbit r_n. If x_n is to reconstruct the original aperiodic orbit $\bar{r}_n$, we need to apply the inverse transformation $\underline{T}^{-1}$, which shifts x_n by $-(1-1/\mu)$ and scales it by $1/\beta$.

The first aperiodic orbit that we generate is given by

$$\bar{r}_{n+1}=\bar{r}_n\exp[3.5(1-\bar{r}_n)] , \tag{22}$$

where $\bar{r}_0=0.753\,59$. The scaling factor β was 10^{-3}. Note that this system is also chaotic. However, its attractor is different from the uncontrolled attractor of (12). In this particular example a variance of 0.77% was allowed on the adjustable parameter μ by imposing a limit of $|p_*|=0.03$ on the control once the initial transients had died. For all subsequent p_n, however, $|p_n|\le p_*$, demonstrating that it is possible for one chaotic system to generate an attractor of another chaotic system using only *small* controls. The results are shown in Fig. 1.

The second aperiodic orbit that we generate is the seventh order polynomial.

$$\bar{r}_n=\begin{cases}-1.981\times10^{13}, & n<15\\ (n-20)(n-60)(n-100)(n-140)(n-180)(n-220)(n-260), & 15\le n\le 265\\ 1.981\times10^{13}, & n>265 .\end{cases} \tag{23}$$

The scaling factor β was 10^{-17}. The control that could be applied to the chaotic system was limited in magnitude to 0.04. That is, $|p_n|\le 0.04$. At time steps 65 and 110 the chaotic system was given a large disturbance. As can be seen from Fig. 2(a), the dynamical system initially loses track of the reference signal r_n but, due to its inherent chaotic nature, is able to recover proper orbit generation.

Since $|e_F(0)|\approx 0$, the output generated by the controlled chaotic system is a near perfect reproduction of the aperiodic orbit, making it very difficult in both examples to differentiate between the desired wave form and that produced by the chaotic system. Figure 2(b) illustrates that, for the second example, the desired reference trajectory and the aperiodic orbit obtained from the chaotic system do actually vary, however slightly.

V. CONCLUSION

In conclusion the infinite complexity inherent in a chaotic system enables it to be controlled so that it will generate orbits of arbitrary higher and lower orders. By exploiting this complexity, it is possible to design simple and extremely flexible systems which will generate an infinite variety of trajectories yet require only minimal changes to the control system itself. We believe that such an approach may prove useful in many applications, particularly in situations where it is necessary that the system track an exogenous signal.

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[2] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990).

[3] A. Azevedo and S. M. Rezende, Phys. Rev. Lett. **66**, 1342 (1991).

[4] T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990).

[5] Observe that

$$\underline{M}=[\mathbf{e}_u\mathbf{e}_s]\begin{bmatrix}\lambda_u & 0\\ 0 & \lambda_s\end{bmatrix}[\mathbf{e}_u\mathbf{e}_s]^{-1} .$$

Furthermore,

$$\begin{bmatrix}\mathbf{f}_u\\ \mathbf{f}_s\end{bmatrix}[\mathbf{e}_u\mathbf{e}_s]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

implying that

$$[\mathbf{e}_u\mathbf{e}_s]^{-1}=\begin{bmatrix}\mathbf{f}_u\\ \mathbf{f}_s\end{bmatrix} .$$

Thus $\underline{M}$ may be written as

$$\underline{M}=[\mathbf{e}_u\mathbf{e}_s]\begin{bmatrix}\lambda_u & 0\\ 0 & \lambda_s\end{bmatrix}\begin{bmatrix}\mathbf{f}_u\\ \mathbf{f}_s\end{bmatrix}=(\lambda_u\mathbf{e}_u\mathbf{f}_u+\lambda_s\mathbf{e}_s\mathbf{f}_s) .$$