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Controls of dynamic flows with attractors

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Analytic and numerical results are obtained concerning the entrainment and migration of dynamic systems, which are governed by ordinary differential equations $\dot{x}=E(x)$ ($x\in\mathbb{R}^n=1,2,3$), when they have attracting sets. Using the control $\dot{x}=E(x)+\dot{g}-E(g)$ ($t\geq 0$), the goal dynamics $g(t)$, to which $x(t)$ is entrained, $\lim_{t\to\infty}|x(t)-g(t)|$, is confined to convergent regions of phase space $g(t)\in C_k=\{x\,|\,\|\lambda(x)\delta_{ij}-\partial E_i/\partial x_j\|=0,\ \mathrm{Re}\lambda<0\ \forall\lambda;\ i,j=1,\ldots,n\}$. These regions can be determined analytically, using the Routh-Hurwitz theorem, without explicitly determining the roots $\lambda(x)$ of the characteristic determinant. The control is only initiated when the system is in the basin of entrainment $x(0)\in BE\{(g)\}$, which ensures entrainment. $BE(g_0)$ is proved to exist for any fixed-point goal $g_0\in C_k$. It is conjectured that $BE(\{g(t)\})$ exists for all $g(t)\in C_k$ which are "dynamically limited": $|\dot{g}|<D(\min[\mathrm{Re}\lambda(x)],\max g)$, where the function D is system specific. This dynamic limitation is illustrated for the Duffing oscillator. Basins of entrainment are explicitly determined in one-dimensional flows and for the van der Pol limit cycle ($n=2$) in the Liénard phase space. This example is used to show that convergent regions are not topologically invariant. The convergent regions are obtained for both the Lorenz and Rössler systems ($n=3$). The global character of the basin of entrainment for a class of goals is analytically proved for the Lorenz system. The transfer of systems between different attractors in multiple attractor systems (MAS) is demonstrated both in one-dimensional flows and in the Lorenz system, where the transfers between stable fixed points and from a strange attractor to a stable fixed point are illustrated.

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I. INTRODUCTION

Among the most interesting, and probably the most important, complex dynamic systems are those that have a variety of topologically distinct attractors. Many examples of these multiple-attractor systems (MAS) are known in hydrodynamics [1,2], the heart [3–6], optics [7–9], chemical dynamics, neural network dynamics, and a variety of biological and ecological systems. These MAS may respond to environmental actions (changes) in very different ways, depending on which basin of attraction they are in. Some of these attractors may be destructive, or disabling in some sense, while others may allow hierarchal systems to adapt to environmental changes (e.g., Refs. [10] and [11]). Whatever the role of these attractors may be, it is clearly of great interest to be able to transfer complex systems from one attractor to another in reliable fashion.

Many of the important dynamic models of such "experimental" (E) systems consist of systems of first-order ordinary differential equations (ODE's),

$$\dot{x}\equiv\frac{dx}{dt}=E(x)\quad(x\in\mathbb{R}^n)\ . \tag{1.1}$$

It will be assumed that $E(x)$ is differentiable, so the solutions are uniquely determined by the initial conditions, $x(t=0)\equiv x_0$, and that $E(x)$ satisfies a condition such as $\int_{r_0}^{\infty}M(r)^{-1}dr=\infty$, where $M(|x|)\geq|E(x)|$, so that solutions exist for all $t\geq 0$. Such dynamic systems are frequently referred to as "flows," which is the significance of this terminology in the present study.

This study concerns the control of those flows, which have attracting sets, such as stable fixed points, limit cycles, intermittent attractors, or any strange attractor (possessing a fractional dimension). Of particular (but not exclusive) interest are the multiple-attractor systems (MAS) noted above. The general aspects of this control method have been discussed in Ref. [12] and specifically applied there to one-dimensional maps. The present study illustrates the application of this method to flows in one, two, and three dimensions, with a variety of attractors including MAS.

The control is based on the existence of convergent regions, $C_k(x)$, in phase space ($\mathbb{R}^n$) of such attractor flows. In these convergent regions, all nearby orbits converge along n eigendirections. They are described in more detail in Secs. II and III. One form of control involves the entrainment of the experimental system to an arbitrarily selected "goal dynamics," $g(t)$, which in the simplest cases is entirely contained in some convergent region $g(t)\subset C_k$. "Entrainment" means that

$$\lim_{t\to\infty}|x(t)-g(t)|=0 \tag{1.2}$$

is satisfied for all $x_0\subset\mathrm{BE}(\{g\})$, the basin of entrainment of the dynamics $g(t)$. The determination of the extent of these basins of entrainment is often a considerable challenge, particularly in higher dimensions. Often one needs to revert to numerical methods to estimate the BE($\{g\}$), but sometimes theorems can be proved concerning their extent (see Secs. IV–VI). Considerations of possible dynamic limitation on the goal dynamics that may be required for BE($\{g(t)\}$) to exist are discussed in Sec. III. As discussed by Jackson and Hübler [13] and Jackson [12], we require that BE($\{g\}$) be the convex set of initial states yielding (1.2), in order to ensure experimental reliability of the controls.

The nature of the control of (1.1), which produces the entrainment (1.2), is to apply an action $F(g,\dot{g})$ in the manner

$$\dot{x}=E(x)+F(g,\dot{g})S(t)\ , \tag{1.3}$$

where $S(t)$ is a "switching function," $S(t)=0$ $(t<0)$, and, e.g., $S(t)=1$ $(t\geq 0)$. The time $t=0$ refers to the time that $x_0\in\mathrm{BE}$ is satisfied; if $x(t)$ is not in the basin of entrainment (BE), any "control" (1.3) will have an undesired effect. Aside from the initial information ($x_0\in\mathrm{BE}$), no further information about the state of the system needs to be obtained [F in (1.3) does not depend on x]. In particular, this is not a feedback control method, which is very important for its implementation in complex dynamic systems [14,12]. Other forms of $S(t)$, $(t\geq 0)$ in (1.3), will be explored in Secs. IV–VI. The essential point to note is that, for (1.2) to hold [i.e., for $x(t)=g(t)$ to be a solution of (1.3)], $F(g,\dot{g})$ must be of the form

$$F(g,\dot{g})=\dot{g}-E(g)\ . \tag{1.4}$$

This type of control was first applied to the logistic map and damped nonlinear oscillators by Lüscher and Hübler [15], and by Hübler and Lüscher [16] (also Hübler [14]). The extension of these considerations, to determine the extent of the basins of entrainment, was begun with a study of the logistic map [13] and extended to very general one-dimensional maps by Jackson [12], where the concept of uniform convergent regions was also introduced. The present study extends these considerations to flows in $\mathbb{R}^n$.

Several points should be emphasized about the control (1.4) and (1.3). It is not possible to select any particular solution of the autonomous system to be the goal of the control, $g(t)$. For if $g(t)$ is any solution of the autonomous equations (1.1), then $F(g,\dot{g})\equiv 0$, so there is no control. In particular, if the system has fixed points or limit cycles, these cannot be used for $g(t)$. To obtain these goals, one simply needs to pick a goal within their basin of attraction, and end the control when the system enters this basin. Examples of this will be given in Secs. IV and VII. The second point to note is that some component of the control may be zero, $F_i(g,\dot{g})\equiv 0$, in which case x_i is only influenced indirectly through the control on the other variables. It has been pointed out by Breeden and Hübler [17] that physical requirements may impose this lack of a direct control of some property x_i. In that case the control is limited to the goal that satisfies $\dot{g}_i=E_i(g)$, but the argument of $E_i(g)$ is not an autonomous solution of (1.1) because there is a direct control of some other variable. More complicated forms of direct control may require variables that are functions of the variable x, say $y=H(x)$. The direct control of the variable y is then described by the equations of motion for y. An example of the controls in two related phase spaces $(x,\dot{x})$ and $(y,\dot{y})$ is given for the van der Pol limit cycles, Sec. V.

In Sec. II a general analytic method for determining the convergent regions of phase space, C_k, using the Routh-Hurwitz theorem, is presented. Also, the distinction between those regions of phase space in which dynamic volumes contract and the present converging regions for orbits is clarified. It is proved in Sec. III that a basin of entrainment, BE(g_0), exists for all systems when the goal is a fixed point $g(t)=g_0\in C_k$ (i.e., $\dot{g}\equiv 0$). Using the Duffing oscillator as an example, it is shown that BE($\{g(t)\}$) generally only exists if $g(t)$ is dynamically (but not topologically) limited, meaning that $\dot{g}(t)$ may have to be bounded relative to the min[Re$\lambda(x)$], and max$g$, depending on the system (1.1). Section IV is restricted to one-dimensional flows, where the basins of entrainment can be easily determined, and the transfer of the system from one attractor to another can likewise be easily understood. Section V treats the entrainment of the van der Pol limit cycle dynamics, in both the original phase space and the Liénard phase space. The global nature of the basin of entrainment is proved in the latter space, for a class of goals. These cases illustrate that convergent regions are not topologically invariant concepts. These systems are also used to illustrate some effects that arise from using several switch-on functions, $S(t)$ in (1.3). Section VI deals with entraining the Lorenz and Rössler strange attractors. Their convergent regions are determined, and the basins of entrainment of several goals are explored. It is proved that the Lorenz system has a global basin of entrainment for a family of goal dynamics in C. Section VII contains examples of the migration controls applied to the Lorenz system, which permits the transfer between three attractors that occur for certain control parameters. This is a simple example of some types of controls of a multiple attractor system, which are undoubtedly the most common class of systems in truly complex dynamics.

II. THE CONVERGENT REGIONS IN N-DIMENSIONAL PHASE SPACES

As discussed in Ref. [12], the convergent regions of phase space for the system

$$\dot{x}=E(x)\quad (x\in\mathbb{R}^n) \tag{2.1}$$

are defined to be those connected regions C_k in which all the roots $\lambda(x)$ of

$$\left|\left|\delta_{ij}\lambda(x)-\frac{\partial E_i}{\partial x_j}\right|\right|=0\quad (i,j=1,\ldots,n) \tag{2.2}$$

have negative real parts. We write this definition in the form

$$C_k = \left\{ x | x \in \mathbb{R}^n; \left| \left| \delta_{ij}\lambda(x) - \frac{\partial E_i}{\partial x_j} \right| \right| = 0 \ \ \forall \ \mathrm{Re}\lambda(x) < 0 \right\}_{\mathrm{con}} , \tag{2.3}$$

where con refers to the requirement that this set satisfy the connectivity condition: if $(x',x'') \in C_k$, they can be connected by some continuous curve $x = f(\theta) \in C_k$, $1 \geq \theta \geq 0$, $f(0) = x'$, $f(1) = x''$.

The convergent region can be made more restrictive, in order to try to ensure (see below) that nearby orbits converge along eigendirections at least at the rate $e^{-\gamma t}$ for some $\gamma > 0$. This is easily done by noting that if $\mathrm{Re}\lambda < -\gamma$, then $\mathrm{Re}(\lambda + \gamma) < 0$, so if we set $\rho(x) = \lambda(x) + \gamma$, (2.3) can be written in the form

$$C_k(\gamma) = \left\{ x | x \in \mathbb{R}^n; \left| \left| \delta_{ij}\rho(x) - \left\{ \gamma\delta_{ij} + \frac{\partial E_i}{\partial x_j} \right\} \right| \right| = 0 \ \ \forall \mathrm{Re}\rho(x) < 0 \right\}_{\mathrm{con}} . \tag{2.4}$$

Then, within the regions $C_k(\gamma)$ the more-restrictive convergence may occur for nearby orbits. This knowledge may be of both practical and basic interest, as will be noted below. For the present, we will use the form (2.3).

It should be emphasized that if the dynamics is greater than one dimensional, all nearby orbits within a convergent region are not necessarily uniformly attracted toward each other (in the Euclidian sense) even though all $\lambda(x)$ are negative. If the eigenvectors associated with these negative $\{\lambda\}$ are constant in space, and one introduces a norm based on the vector components along these eigenvectors, this norm will decrease uniformly in time (as long as the orbits remain in C). However, the Euclidian norm may not decrease uniformly in this process, even though entrainment will necessarily occur ($x \in C$). The situation is more delicate in those cases where the eigenvectors are not constant in space. It is not presently known whether entrainment, (1.2), necessarily follows from the fact that the orbits $x(t)$ and $g(t)$ remain in a convergent region, but no nonentrainment example has been found to date, despite considerable effort. The fact that $g(t) \in C$ (i.e., entirely in C) is not necessary for entrainment was illustrated in Ref. [13] for the case of maps. The danger is that, if $g(t)$ is not entirely in C, the corresponding basin of entrainment may be disjoint, or even fractal in character, making the initiation of the control unreliable (however, no example of this is known in the case of flows). This lack of necessity will also be illustrated both in the case of the van der Pol (Sec. V) and the Lorenz system (Sec. VI). These uncertainties make it particularly useful to obtain specific entrainment theorems, as limited as they may be. Examples will be given in Secs. V and VI.

The problem of determining the roots $\lambda(x)$ of (2.2) when $n \geq 3$ can become tedious at best. Fortunately what is required to determine C_k is not these roots, but simply the region in which they satisfy $\mathrm{Re}\lambda(x) < 0$. The answer to this problem is well known, and does not require a knowledge of the roots $\lambda(x)$. The characteristic equation (2.2) is a polynomial equation of order n in λ,

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0 \quad (a_0 = 1) . \tag{2.5}$$

A necessary and sufficient condition for all roots of (2.5) to have negative real parts is that

$$a_{2k} > 0, \quad \Delta_{2k+1} > 0 \quad (k = 0, 1, \ldots)$$

or

$$a_{2k+1} > 0, \quad \Delta_{2k} > 0 , \tag{2.6}$$

where $\Delta_i(a_1, \ldots, a_i)$ are the so-called Hurwitz determinants of order i. These conditions are referred to as the stability criterion of Liénard and Chipart (see, e.g., Ref. [18]) and will be used in later sections.

Before considering special examples of control, we will discuss the relationship between convergent regions and the classic concept of the change in the volume of a phase-space region, as defined by Poincaré's integral

$$I(t) = \int_{\Omega(t)} dx_1 \ldots dx_n .$$

Here, $\Omega(t)$ is a domain whose points move according to (2.1). It is not difficult to prove (see Ref. [19], Appendix C) that quite generally,

$$\frac{dI}{dt} = \int_{\Omega(t)} \nabla \cdot E(x) dx_1 \ldots dx_n .$$

Note this holds for "dissipative" or autocatalytic systems, not just Hamiltonian systems (the usual Liouville theorem). We now introduce the terminology "contractive," "expansive," and "conservative" for the local region of phase space where any $\Omega(t)$ has the respective properties

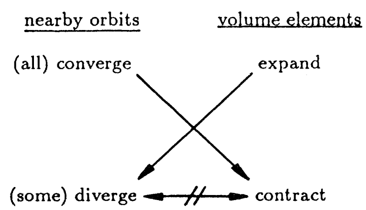
$$\begin{aligned} &\nabla \cdot E(x) < 0 \left[\frac{dI}{dt} < 0 \right] , \\ &\nabla \cdot E(x) > 0 \left[\frac{dI}{dt} > 0 \right] , \\ &\nabla \cdot E(x) = 0 \left[\frac{dI}{dt} = 0 \right] . \end{aligned} \tag{2.7}$$

Note that the terminology contractive and expansive, are adjectives being applied to regions to describe the behavior of contained volumes, whereas convergent and divergent, associated with (2.3), are adjectives applied to regions relative to the behavior of (all or some) nearby orbits. The convergent condition (2.3) requires the satisfaction of n conditions (2.6), whereas the region of phase space in which contraction occurs satisfies only one condition, (2.7).

A simple relation can be established between these two concepts by considering a_1 in (2.5). This is the coefficient of λ^{n-1} in the expansion of the determinant (2.2). It is not difficult to see that

$$a_1 = -\nabla \cdot E \equiv -\sum_{i=1}^{n} \frac{\partial E_i}{\partial x_i} \tag{2.8}$$

and the necessary condition $a_k > 0$ $(k = 1, \ldots, n)$ shows that a necessary condition for convergence is $-\nabla \cdot E > 0$, or $\nabla \cdot E < 0$. According to (2.10) this implies contraction, so convergence implies contraction, which is rather obvious. Moreover, expansion $\nabla \cdot E > 0$ implies nonconvergence, equivalent to divergence (of some orbits). Thus the implications are limited as shown:



Given knowledge of the lower levels, no implications can be drawn about the other column. That is, a nonconvergent region may have any sign of dI/dt for Ω in this region. Conversely, a contractive region of phase space may have some or no diverging nearby orbits.

We next consider some of the possible dynamic limitations that may be required of the goal dynamics, $g(t)$, in order for a basin of entrainment to exist.

III. POSSIBLE DYNAMIC LIMITATIONS OF GOAL DYNAMICS

All of the present control methods are based on the existence of a basin of entrainment, BE($\{g(t)\}$), associated with a dynamic "goal," $g(t) \in C_k$, contained in some convergent region of the phase space. It is easy to prove that such a basin exists if $g(t)$ is a fixed point; $g(t) \equiv g_0$. From (1.3) and (1.4), the controlled system is generally [if $S(t) = 1$]

$$\dot{x} = E(x) + \dot{g} - E(g) \quad (x, g \in \mathbb{R}^n) . \tag{3.1}$$

Setting $u = x - g$, and assuming that $E(x)$ is smooth, one obtains

$$\dot{u} = E(u+g) - E(g) = \frac{\partial E(g)}{\partial g_i} u_i + \frac{1}{2} \frac{\partial^2 E(g)}{\partial g_i \partial g_j} u_i u_j + \ldots , \tag{3.2}$$

where repeated indices are summed (1 to n). Entrainment (1.2) is now characterized by the fact that the solutions of (3.2) satisfy $u \to 0$. If $g(t) = g_0$, the coefficients of the u-factors in (3.2) are constants, and the linear equations are

$$\dot{u}_j = \left[\frac{\partial E_j}{\partial g_i} \right]_{g_0} u_i \quad (j = 1, \ldots, n) . \tag{3.3}$$

Since g_0 is in some convergent region of the phase space, $g_0 \in C_k$, the fixed point $u = 0$ of (3.3) is asymptotically stable (all eigenvalues have negative real parts). Then, by a famous theorem of Lyapunov [29] (see also [18]), $u = 0$ is also asymptotically stable for the nonlinear equation (3.2). Thus the basin of entrainment BE(g_0) exists, but its extent in the phase space is of course not determined by such a linear analysis. Other more global methods must be employed for this purpose, which are system specific, and will be illustrated in the following sections.

The interesting and much more difficult problem is to prove that BE($\{g(t)\}$) exists for some family of dynamic goals $g(t) \in C_k$ $(\dot{g} \neq 0)$. In this case, the coefficients in (3.2) are nonconstant, and theorems concerning the conditions for the asymptotic stability of $u = 0$ are much more difficult to establish. It is to be expected that, if $g(t)$ does not change "too rapidly" in time, BE($\{g\}$) will always exist. We now will consider some of these issues.

If the goals are periodic functions, the coefficients of u in (3.3) are likewise periodic. The corresponding linear equations have been widely studied, to which the names of Floquet, Mathieu, Hill, and Lyapunov are often associated. There is no fundamental difficulty in studying the stability of any specific system, but general statements obviously cannot be made. What should be emphasized is that Lyapunov also established the relationship between the stability of these linear systems with periodic coefficients and the nonlinear equations (3.3), also with periodic coefficients. For a discussion of this result see Gantmacher (Ref. [18], Vol 2, p. 120). Thus it is quite possible to prove whether or not BE($\{g(t)\}$) exists for periodic $g(t)$ for any specific system.

To illustrate what is involved, we outline such an analysis for the Duffing oscillator (whose whole phase space is convergent):

$$\ddot{x} + \mu\dot{x} + x + x^3 = 0 . \tag{3.4}$$

The control equation (3.2) is then

$$\ddot{u} = -\mu\dot{u} - (1 - 3g_x^2)u - 3g_x u^2 - u^3 . \tag{3.5}$$

The damping term can be eliminated by setting $u = e^{-\mu t/2} w$, yielding

$$\ddot{w} = -[1 - 3g_x^2 - (\mu/2)^2]w - 3g_x w^2 e^{-\mu t/2} - w^3 e^{-3\mu t/2} .$$

If all solutions of the linear equation

$$\ddot{w} = -[1 - 3g_x^2 - (\mu/2)^2]w \tag{3.6}$$

have a growth rate less than $(\mu/2)$, for some $g_x(t)$, the solutions of (3.5) will satisfy $\lim_{t\to\infty} u(t) = 0$. In this case entrainment to $g(t) \in \mathbb{R}^2$ is proved for sufficiently small $u(0)$.

For periodic $g_x(t)$, (3.6) is an example of Hill's equation. To illustrate, consider the case

$$g_x(t) = A \sin(\omega t) . \tag{3.7}$$

Note that migrational goals can also be of this form, with "large" A, small ω, and $S(t) \neq 0$ only for $0 \le t \le \pi/\omega$. We can now readily transform (3.6) into the canonical

Mathieu equation

$$\frac{d^2w}{d\tau^2}+[a-2q\cos(2\tau)]w=0\ , \tag{3.8}$$

where $\tau=\omega t$, $q=3A^2/4\omega^2$, and $a=[1+(3A^2/2)-(\mu/2)^2]/\omega^2$. It is well known that the stability of w only occurs in bands within (a,q) space. In the present case we are interested in the character of the solutions along the parameter region $a \gtrsim 2_q$. For small a, the solutions are stable, and as $a \simeq 2q$ is increased from zero, they pass through bands of instability. If the growth rate for w is ν, $u(t)$ will grow as $\nu\omega-\mu/2$. It can be shown [30] that $\max\nu(a\simeq 2q)$ in these bands decreases rapidly as A/ω increases, so $\nu\omega-\mu/2$ becomes negative if A/ω and μ/ω are both sufficiently large. Therefore, if either A/ω is sufficiently small or A/ω and μ/ω are sufficiently large, the solutions of (3.5) are entrained, provided that $u(0)$ is also sufficiently small. For example, if $|u(0)|<0.1$, then numerical solutions indicate that both $(A=5,\omega=1)$ and $(A=1,\omega=5)$ are stable when $\mu=0.1$. One can, of course, expect a band structure in this stability, as a function of $a\simeq 2q$, but these details are not presently known.

While this stability is necessary for entrainment, it is not sufficient for practical controls, because rarely would one find the system in a state that satisfies $|x(0)-g(0)|<0.1$. It is much more important to know whether the fixed-point attractor of (3.4), $P_F=(x=0,\dot{x}=0)$, lies in the basin of entrainment $\mathrm{BE}(\{g\})$, which ensures (1.2). If this is so, the control can be initiated any time the system is near its attractor, and it will be entrained. Unfortunately no analytic criterion is known, which ensures that $P_F\subset \mathrm{BE}$ for the present system. However, numerical studies indicate that $P_F\subset\mathrm{BE}(\{g\})$ if (A,ω) in (3.7) satisfy (conservatively) $A\omega<0.7$. This type of dynamical constraint still allows for large-scale migrations of the system (large A), provided the speed is not too large (small ω). Also, it will be shown in later sections that analytic criteria can often be obtained for the basins of entrainment.

Finally, the present example is a good place to illustrate the physical conditions implied by these controls. For (3.4) one of the controlled equations is $\dot{x}=y+\dot{g}_x-g_y$ and, if $\dot{x}$ is the velocity of some mechanical mass, it is difficult to apply any direct control to $\dot{x}$. Thus the most physically meaningful control may require that $\dot{g}_x-g_y\equiv 0$, which defines g_y, given by (3.7). The corresponding "force" $\mathcal{F}$ that must be applied is determined by

$$\dot{y}=-\mu y-x-x^3+\dot{g}_y+\mu g_y+g_x+g_x^3$$

(namely, force means g-dependent terms) or, in the case of the goal (3.7),

$$\mathcal{F}=(1-\omega^2)A\sin(\omega t)+\mu\omega A\cos(\omega t)+A^3\sin^3(\omega t)\ .$$

This very nonlinear, and "custom-structured" force is the reason that the nonlinear system tends toward such a simple goal dynamics, (3.7).

While the above analysis shows that there may be dynamic limitations on the goal dynamics, in order to achieve entrainment of some systems (as illustrated by the Duffing oscillator), a number of entrainment theorems for other systems, illustrated in the following sections, do not exhibit similar dynamic limitations. The reason for this difference in different systems should be explainable by a study of their associated parametric equations (3.2).

We will now consider examples of the application of these and other concepts to flows in one, two, and three dimensions.

IV. CONTROLLED ONE-DIMENSIONAL FLOWS

The dynamics of autonomous one-dimensional flows is of limited interest, because of their very restricted "topological repertoire" (phase portraits). On the other hand, nonautonomous one-dimensional flows are capable of very complicated dynamics (see, e.g., section (4.6) of Ref. [19]). In controlled systems, the goal dynamics are not typically complicated, but this option means that even one-dimensional controlled systems are generally by no means trivial dynamic systems. In this section we will explore a limited range of these issues.

Let $g(t)$ be the desired goal dynamics, so the controlled system is of the form

$$\dot{x}=F(x)+\dot{g}-F(g)\quad (x\in\mathbb{R})\ , \tag{4.1}$$

which we write in the form

$$\frac{d}{dt}(x-g)=(x-g)\lambda(x,g)\ , \tag{4.2}$$

$$\lambda(x,g)\equiv[F(x)-F(g)]/(x-g)\ . \tag{4.3}$$

We assume that $F(x)$ is differentiable, in which case $\lambda(x,t)$ is defined for all (x,g). The convergent regions of (3.1) are simply the connected regions $(k=1,2,\dots)$ where $dF/dx<0$,

$$C_k=\left\{x\,\middle|\,\frac{dF}{dx}<0\right\}_{\mathrm{con}}\quad (k=1,2,\dots)\ , \tag{4.4}$$

where the con index refers to the connected property discussed in Sec. II. To ensure entrainment,

$$\lim_{t\to\infty}|x(t)-g(t)|=0\ , \tag{4.5}$$

it is simplest to require that $\lambda(x,g)$ be negative for all $x(t)$ sufficiently near $g(t)$, in which case $g(t)$ must be in a goal region, G_k, within some C_k,

$$g(t)\in G_k\subset C_k\ . \tag{4.6}$$

Other $g(t)$, which only satisfy (4.6) during a majority of the time, may also produce entrainment, but (4.6) is a simple and useful sufficient condition.

If the goal dynamics is simply a fixed point, $g(t)=g_0$, then g_0 must be in some convergent region, and we can immediately determine its (maximum) basin of entrainment.

$$\mathrm{BE}(g_0)=\{x\,|\,\lambda(x,g)<0\}_{\mathrm{con}}\ . \tag{4.7}$$

This region is particularly transparent in a graph of

$F(x) - F(g_0)$ vs $(v - g_0)$, as shown in Fig. 1, where $g_0 \in C_4$. It can be seen in this case that $BE(g_0)$ contains the two convergent regions C_3 as well as C_4.

It is easy to see from this that if $g_0 \subset C_k$, then $C_k \subset BE(g_0)$; that is, the basin of entrainment is always larger than the associated convergent region (the same is not necessarily true of maps [12]). This figure also illustrates a number of other features. This particular system has two (autonomous) attractors labeled A_1 and A_4 that necessarily fall within convergent regions. Indeed, for the same reasons as above, these convergent regions are contained in their basins of attraction, $C_k \subset BA_k (k = 1,4)$. There are, however, many convergent regions not associated with such attractors (e.g., C_2 and C_3). Each of these regions offers the opportunity of selecting goals that are in very different regions from the "natural" (autonomous) attractors.

This system is also a simple example of a multiple attractor system discussed in Ref. [12]. The graphical method in Fig. 1 readily shows that simple entrainment goals can be used to transfer this system from one attractor to another. Specifically, if

$$g_0 \in C_1 \text{ and } g_0 < A_1 ,$$

then (4.8a)

$$A_4 \in BE(g_0) ,$$

which follows from the fact that $F(A_4) - F(g_0) < 0$, and if

$$g_0 \in C_4 \text{ and } g_0 > A_4 ,$$

then (4.8b)

$$A_1 \in BE(g_0) ,$$

since $F(A_1) - F(g_0) > 0$. This means that, in the case of (4.8a), the goal g_0 will entrain the system that is initially in A_4 (or near) and once $x(t) \in BA$, the control can be terminated, and $x(t)$ will tend to A_1. This will produce the transfer $A_4 \to A_1$. Similarly the g_0 of (4.8b) can be used to produce the transfer $A_1 \to A_4$.

We note also that it is simple to determine the basin of uniform entrainment, which satisfies

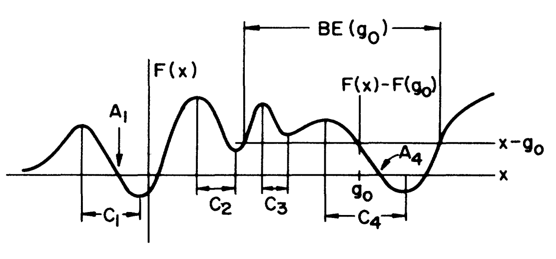


FIG. 1. The four convergent regions of $\dot{x} = F(x)$, with the two attractors A_1 and A_4. The basin of entrainment, $BE(g_0)$, of $g_0 \in C_4$ is easily determined using the coordinates $F(x) - F(g_0)$ vs $(x - g_0)$.

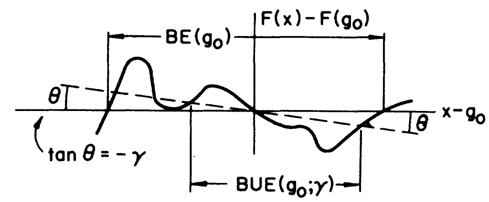


FIG. 2. The basin of uniform entrainment at a rate not less than γ, $BUE(g_0;\gamma)$ is determined graphically.

$$|x(t) - g_0| \le |x(0) - g_0| e^{-\gamma t} \quad (\gamma > 0; \forall t \ge 0)$$

simply by using (4.2) and (4.3), which yield

$$BUE(g_0;\gamma) = \{x | \lambda(x,g_0) < -\gamma\} . \tag{4.9}$$

The corresponding graphical construction is shown in Fig. 2.

Now turning to the general time-dependent $g(t)$, which satisfies (4.6), we note that $\lambda(x,g) < 0$ for all $x \in C_k$. Therefore, using (4.2) we obtain a minimal basin of entrainment:

$$BE(g(t) \in C_k) = C_k \quad \text{(minimal)} . \tag{4.10}$$

The maximal BE can be obtained from $BE(g_{max})$ and $BE(g_{min})$ defined by (4.7), namely

$$BE(g(t)) = BE(g_{max}) \cap BE(g_{min}) ,$$
$$g_{max} = \max(g(t)) \quad g_{min} = \min(g(t)) . \tag{4.11}$$

This can be readily seen by varying g_0 in Fig. 1 to g_{max} and g_{min}. One can obviously also obtain $BUE(g(t),\gamma)$ similarly:

$$BUE(g(t),\gamma) = BUE(g_{max},\gamma) \cap BUE(g_{min},\gamma) \tag{4.12}$$

Possibly one of the surprising aspects of this control is that, regardless of how rapidly $g(t)$ is changed, the system $x(t)$ nonetheless satisfies the entrainment requirement (4.5).

We next consider some important autocatalytic examples of limit cycles in two-dimensional flows.

V. ENTRAINMENT OF LIMIT CYCLES IN TWO DIMENSIONS

Limit cycles in two dimensions have historically been of great interest, and continue to be an area of active research (see, e.g., Ref. [20]). In this section we will consider the entrainment of the classic van der Pol dynamics in two phase spaces: the original van der Pol phase space and the Liénard phase space. This will clarify the distinction between contraction and convergent regions, illustrate how a simple entrainment theorem can be obtained in the Liénard phase space, and consider some aspects of "switching on" the control using different types of functions $S(t)$. It will also make clear the fact that convergent regions are not topologically invariant.

The first form of the van der Pol equation that will be considered is

$$\ddot{x}+k(x^2-1)\dot{x}+x=0 \quad (k>0)\ , \tag{5.1a}$$

which we write in the form ($x_1 \equiv x$)

$$\dot{x}_1=x_2,\quad \dot{x}_2=k(1-x_1^2)x_2-x_1 \tag{5.1b}$$

so the phase space is (x_1,x_2). As discussed in Sec. II, if $\dot{x}=E(x)$ ($x \in \mathbb{R}^n$), the contractive region of phase space is given by $\nabla\cdot E<0$. In the case (4.1), we obtain

$$\nabla\cdot E=k(1-x_1^2)<0 \quad (\text{contraction})\ . \tag{5.2}$$

The convergent region, on the other hand, is obtained from the determinant (2.2), which in the case (4.1) yields the polynomial in λ:

$$\lambda^2+k(x_1^2-1)\lambda+1+2kx_1x_2=0\ . \tag{5.3}$$

This is a special case of (2.5), and the Hurwitz condition (2.6) simply requires that the necessary and sufficient conditions for all $\mathrm{Re}(\lambda)<0$ are

$$a_1\equiv k(x_1^2-1)>0,\quad a_2\equiv 1+2kx_1x_2>0\ . \tag{5.4}$$

This can, of course, be trivially verified by obtaining the explicit roots $\lambda(x)$ of (5.3), but that is not necessary to determine the convergent regions. We note that (5.4) defines two (disconnected) convergent regions $C^{\pm}$, in which $\pm x_1>1$, respectively. These are illustrated in Fig. 3. This also gives a simple illustration of the difference between the convergent regions and contractive regions, (5.2). If the goal dynamics is $(g_1(t),g_2(t))$, the controlled system is (4.1) or

$$\begin{aligned}\dot{x}_1&=x_2+(\dot{g}_1-g_2)S\ ,\\ \dot{x}_2&=k(1-x_1^2)x_2-x_1+[\dot{g}_2-k(1-g_1^2)g_2+g_1]S\ .\end{aligned} \tag{5.5}$$

The Liénard phase plane is also frequently used to describe the dynamics (5.1). Denoting this phase plane by (y_1,y_2), the equations of motion are

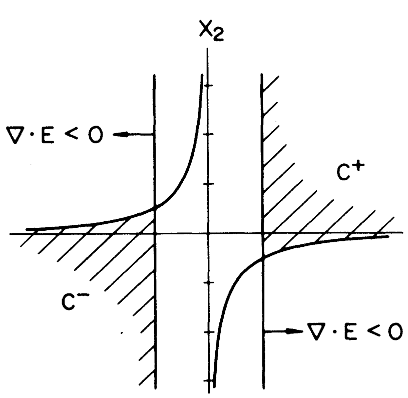


FIG. 3. The convergent regions, $C^{\pm}$, of the van der Pol equation in the phase space (x_1,x_2). $C^{\pm}$ are subregions of the contractive regions, $\nabla\cdot E<0$, as shown.

$$\dot{y}_1=y_2+k(y_1-\tfrac{1}{3}y_1^3),\quad \dot{y}_2=-y_1 \quad (k>0)\ . \tag{5.6}$$

One can readily verify, by considering $\ddot{y}_1$, that y_1 satisfies (5.1a), so $y_1=x_1$; hence $x_2=y_2+k(x_1-\frac{1}{3}x_1^3)$ completes the homeomorphism $(x_1,x_2)\leftrightarrow(y_1,y_2)$ connecting these phase spaces.

Since the contractive region (5.2) does not depend on x_2, it remains the same in the Liénard plane,

$$\nabla\cdot E=k(1-y_1^2)<0\ . \tag{5.7}$$

However the polynomial equation for $\lambda(y_1,y_2)$ is now simply

$$\lambda^2+k(y_1^2-1)\lambda+1=0\ . \tag{5.8}$$

Hence one of the conditions for convergence, $a_2\equiv 1>0$, is always satisfied, leaving only the condition $a_1\equiv k(y_1^2-1)>0$, which is now identical with (5.7). Thus, in the Liénard plane, the convergent and contractive regions are identical. To obtain the convergent region $C_k(\gamma)$, (2.4), where nearby orbits converge at least as fast as $e^{-\gamma t}$, one needs to consider the polynomial

$$\rho^2+[k(y_1^2-1)-2\gamma]\rho+1+\gamma^2=0 \tag{5.9}$$

instead of (5.8). We see that

$$C^{\pm}(\gamma)=\{\pm y_1>[1+(2\gamma/k)]^{1/2}\}\ .$$

Let the goal dynamics in the Liénard phase space be denoted as (h_1,h_2). Then the controlled system is

$$\begin{aligned}\dot{y}_1&=y_2+k(y_1-\tfrac{1}{3}y_1^3)+[\dot{h}_1-h_2-k(h_1-\tfrac{1}{3}h_1^3)]S(t)\ ,\\ \dot{y}_2&=-y_1+(\dot{h}_2+h_1)S(t)\ .\end{aligned} \tag{5.10}$$

We can now prove that the goal dynamics that is confined to the region $h_1^2>4$ has a global basin of entrainment.

Theorem 1. If $h_1^2(t)>4$, the solutions of (5.10) ($S=1, t\geq 0$) satisfy

$$\lim_{t\to\infty}[(y_1-h_1)^2+(y_2-h_2)^2]=0$$

for any $(y_1(0),y_2(0))$. The proof easily follows by noting that (5.10) can be written in the form

$$\dot{u}_1=\lambda_1u_1+u_2,\quad \dot{u}_2=-u_1 \quad (u_k\equiv y_k-h_k)\ , \tag{5.11}$$

where

$$\lambda_1=-k(h_1^2-1+h_1u_1+\tfrac{1}{3}u_1^2)\ .$$

Since $k>0$, $\lambda_1<0$ for any u_1 provided that $h_1^2>4$. On the other hand, (5.11) yields

$$\frac{d}{dt}(u_1^2+u_2^2)=2\lambda_1u_1^2$$

and by (5.11), $u_1(t)\not\equiv 0$ unless $u_2\equiv 0$. Since $u_1^2+u_2^2$ is non-negative, the last equation shows that, if $\lambda_1<0$, $u_1^2\to 0$, and therefore $u_1^2+u_2^2\to c$ (a constant), so $u_2^2\to c$. However, from (5.11) this constant must be zero, proving the theorem.

To illustrate this entrainment, Figs. 4(a) and 4(b) show a goal dynamics

$$h_1 = A + B\sin(\omega t), \quad h_2 = -B\omega\cos(\omega t) \tag{5.12}$$

with $A=3$, $\omega=9$, and $B=0.45$, which has a flow that is counterclockwise, in contrast to the clockwise autonomous flow. The frequency has also been taken much larger than the natural frequency of the system (about 0.94, when $k=1$), making the goal quite "unnatural." A and B were selected such that $h_1 > 2.5$, placing the goal well inside the convergent region. Since the basin of entrainment is global, by *Theorem 1*, the control can be switched on at any time. Note that (5.12) is not constrained by any of the dynamic considerations of the Duffing system (section III).

A potentially important issue in this process is the manner in which the control is switched on, characterized by $S(t)$ in (5.10). This may be important because the low-dimensional models may represent a high-dimensional system's motion on an attracting manifold ("inertial manifold" for ODE's). A violent action on these variables may activate the other degrees of freedom, thereby invalidating the dynamic model. This aspect of control has been emphasized by Lüscher and Hübler [15] and Hübler [14]. For the present we will simply illustrate the system's response to a "hard" and "soft" switching function $S(t)$.

In Fig. 4(a), the switching function was the step function

$$S(t) = H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t \ge 0 \end{cases} \tag{5.13}$$

and one can see the rather violent response of the system. The entrainment in this case consists of a descending spiral motion toward $h(t)$. In Fig. 4(b), the switching function was taken to be

$$S(t) = 1 - \exp(-\lambda t) \tag{5.14}$$

and λ was adjusted to give the illustrated "smooth" entrainment ($\lambda = 0.0025$). Such a "soft" form of entrainment may be of great importance in applications. Another example will be noted below, in the Lorenz system.

When a symmetric goal is selected [$A=0$ in (5.12)], the goal dynamics is outside of the convergent region at least part of the time. If $B<1$, then it is outside of the convergent region all of the time, and the typical lack of control is illustrated in Fig. 5 ($B=0.45$). When B is increased beyond $B=1$, the fact that the goal is in $C^\pm$ for longer periods of time can overcome the divergent properties outside of $C^\pm$ and again yield entrainment. A detailed investigation of this form of entrainment will be considered in a subsequent investigation.

The fact that convergent regions are not topologically invariant can be illustrated by using the map (homeomorphism) between the van der Pol and Liénard phase planes,

$$x_1 = y_1, \quad x_2 = y_2 + k(x_1 - \tfrac{1}{3}x_1^3) = y_2 + k(y_1 - \tfrac{1}{3}y_1^3) . \tag{5.15}$$

The convergent regions in (y_1, y_2) are where $y_1^2 > 1$, and any y_2. These Liénard regions map onto the regions $x_1^2 > 1$ (any x_2) in the van der Pol phase plane, which is not the convergent region in that space, (5.4). Therefore convergent regions are not topologically invariant.

To illustrate this last fact, in Figs. 6(a) and 6(b) we show the dynamics in the van der Pol plane, using the goals

$$g_1 = A + B\sin(\omega t), \quad g_2^\pm = \pm A - B\omega\cos(\omega t) . \tag{5.16}$$

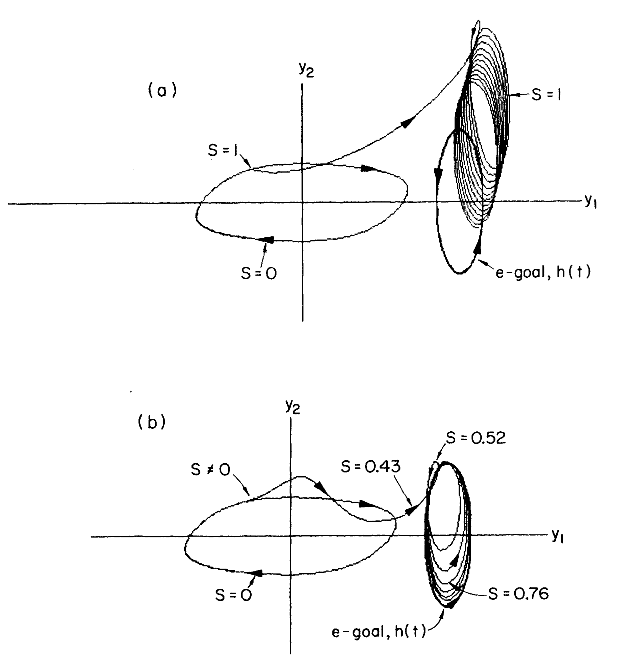


FIG. 4. (a) The entrainment of the van der Pol oscillator in the Liénard phase space (y_1, y_2), to the fast "counter-rotating" entrainment goal (EG) (5.12). The "hard" switch, (5.13), cause the rather violent response of the system. (b) The same conditions as in 4(a) except that the control is initiated using the "soft" switch (5.14), with $\lambda = 0.0025$. The response of the system is much less violent.

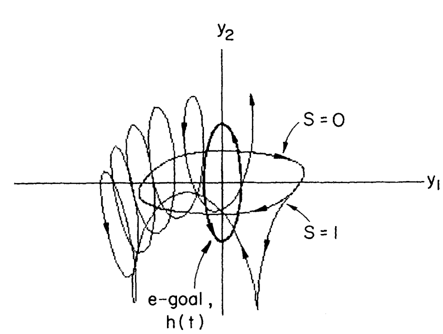


FIG. 5. The response of the van der Pol oscillator to an EG which is outside of the convergent regions $y_1^2 < 1$. The system continues to spiral in a solenoidal-appearing manner about the origin.

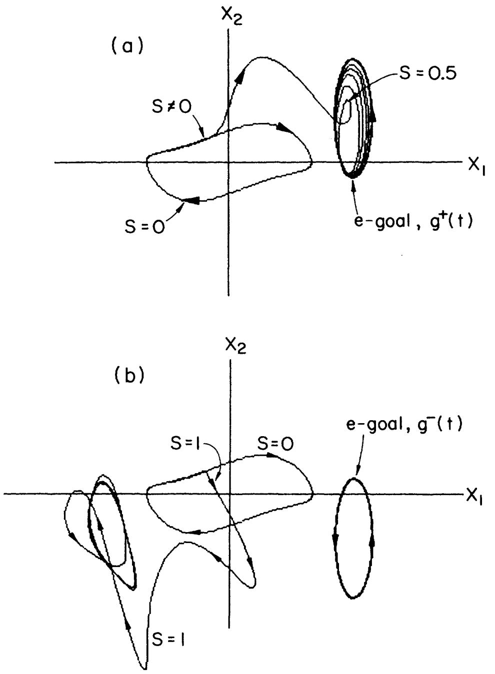


FIG. 6. (a) The entrainment in the van der Pol phase space, using the EG (g_1, g_2^+), (5.16) and the soft switch (5.14). (b) The same as (a) except the EG (g_1, g_2^-) is used, which does not lie in the convergent region (see Fig. 3). The response of the system is to go to another attractor.

These goals are modified forms of (5.12), with the same values of x_1, but differing values of x_2. In Fig. 6(a) the goal g_2^+ is used, which lies in the convergent region (5.4) and the "soft-switch," (5.14), was employed. The system clearly has a large basin of entrainment. In Fig. 6(b), g_2^- was used, so that much of $g(t)$ lies outside the convergent region. The switch (5.13) was used to limit the scale of the motion. The response of the system is to go to some undesired attractor. These examples clearly show the importance of the convergent region, and its lack of invariance under (5.15).

We next will consider some classic attractors in $\mathbb{R}^3$.

VI. ENTRAINMENT OF THE RÖSSLER AND LORENZ DYNAMICS

In this section two classic dynamic systems will be examined from the point of view of entraining their dynamics. The convergent regions of the Rössler and Lorenz dynamics are both infinite, but they differ in a number of respects, which makes them interesting to compare and contrast.

We begin with one of the many chaotic systems introduced by Rössler [21]:

$$\dot{x} = -y - z , \tag{6.1a}$$

$$\dot{y} = x + ay , \tag{6.1b}$$

$$\dot{z} = b + z(x - c) . \tag{6.1c}$$

The convergent region of any system is determined by the determinant (2.2), which yields the polynomial equation (2.5). For the system (6.1) the polynomial equation (2.5) has the coefficients ($a_0 = 1$):

$$\begin{aligned} a_1 &= c - a - x , \\ a_2 &= a(x - c) + z + 1 , \\ a_3 &= c - x - az . \end{aligned} \tag{6.2}$$

The necessary and sufficient conditions for all $\mathrm{Re}\lambda(x,y,z) < 0$ are given by (2.6),

$$a_1 > 0, \quad a_1 a_2 - a_3 > 0, \quad a_3 > 0 . \tag{6.3}$$

These conditions only involve (x,z), and not y [nor the constant b in (6.1)]. Specifically, the necessary and sufficient conditions for convergence are [(2.6)]

$$a_1 = c - a - x > 0 \quad \text{(contractive region)} , \tag{6.4a}$$

$$a_3 = c - x - az > 0 , \tag{6.4b}$$

$$a_1 a_2 - a_3 = (c - x)z - a(c - x)^2 + a^2(c - x) - a > 0 . \tag{6.4c}$$

Note that (6.4a), or $a_1 > 0$, is the region of contracting dynamic volumes, (2.8), whereas two more conditions are required to ensure convergence.

Because (6.4) is independent of y, the convergent region in $\mathbb{R}^3$ can be represented simply by the intersection of its boundary with the (x,z) plane. Figure 7 shows the three boundaries, (6.4), when $a = b = 0.2$ and $c = 5.7$ (one of the cases discussed by Rössler [21]). The contracting region lies to the left of the vertical line $x = c - a$, and the convergent region is a subset of this contractive region, as discussed in Sec. II. The lower boundary of the convergent region is given by (6.4c) and the upper boundary is given by (6.4b). These two conditions imply (6.4a), as can be seen by their intersection at the vertical contractive boundary.

Because of the simplicity of representing the Rössler convergent region, due to its y-independence, it is also possible to simply illustrate the more restrictive convergent regions, $C(\gamma)$. These are given by (2.4), and its associated polynomial for ρ. If we write this as

$$b_0 \rho^n + b_1 \rho^{n-1} + \cdots + b_n = 0 \quad (b_0 \equiv 1)$$

one finds that

$$\begin{aligned} b_1 &= c - a - x - 3\gamma , \\ b_2 &= (a + 2\gamma)(x - c) + z + 1 + \gamma(2a + 3\gamma) , \\ b_3 &= (c - x - \gamma)(1 + \gamma a + \gamma^2) - z(a + \gamma) . \end{aligned} \tag{6.5}$$

This generalizes the result (6.2), and one again finds that the upper and lower boundaries of the convergent regions, $C(\gamma)$, are given by $b_3 > 0$ and $b_1 b_2 - b_3 > 0$, respectively. These are also illustrated in Fig. 7, for $\gamma = 0$, 0.5, and 0.8. It can be seen that $C(\gamma = 0.8)$ is a very narrow (but infinite) region. This weak convergence of the Rössler system is apparent in the entrainment process,

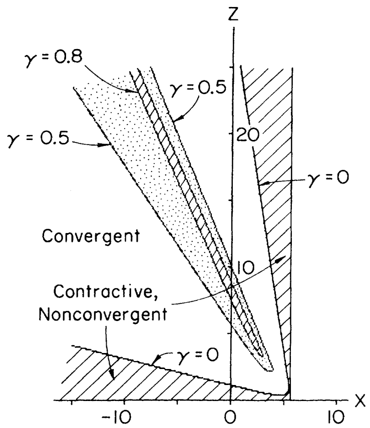


FIG. 7. The contractive region of the Rössler system (6.1) is $x < c - a$. The convergent region [(6.4b), (6.4c)], which is independent of y, is illustrated for $c = 5.7$, $a = b = 0.2$. The boundaries of the convergent regions, $C(\gamma)$, are illustrated for $\gamma = 0$, 0.5, 0.8.

which we illustrate next.

The relationships of the dynamics of the Rössler strange attractor and the convergent region C, (6.4), is illustrated in Fig. 8 ($a = b = 0.2$, $c = 5.7$). If a control with a goal $g(t)$ is used on this system, the controlled system is governed by

$$\dot{x} = -y - z + (\dot{g}_x + g_y + g_z) S(t) \ , \tag{6.6a}$$

$$\dot{y} = x + ay + (\dot{g}_y - g_x - a g_y) S(t) \ , \tag{6.6b}$$

$$\dot{z} = b + z(x - c) + [\dot{g}_z - b - g_z(g_x - c)] S(t) \ . \tag{6.6c}$$

One goal that might appear to be reasonable is to have the system change from a chaotic state (say $c = 5.7$) to a nonchaotic Rössler state (characterized by another c value, say $c_g = 3$). In this case, $(g_x, g_y, g_z) \equiv (x, y, z)$ would satisfy (6.1) with this new value, and the coefficients of $S(t)$ in (6.6a) and (6.6b) would vanish [yielding (6.1a) and (6.1b)], whereas (6.6c) would become

$$\dot{z} = b + z(x - c) + (c - c_g) g_z S(t) \ . \tag{6.7}$$

However, since this goal does not lie inside the convergent region (6.4), if it is used in (6.7), then a typical response of the system at some time is to go to infinity, as illustrated in Fig. 9.

To obtain a viable control, the goal $g(t)$ must generally be in the convergent region. A very "unnatural" goal dynamics,

$$g_x = -5 - 2\sin(3\pi t) \ ,$$
$$g_y = 3\cos(\pi t) \ , \tag{6.8}$$
$$g_z = 8 + 2\cos(\pi t)$$

is illustrated in Fig. 10. The control is turned on ($S = 1$) only when the system enters the convergent region. It can be seen that the system reacts violently to this control, and does not approach $g(t)$ uniformly. This violent reaction can be greatly reduced by using a switching function such as $S(t) = [1 - \exp(-t/\tau)]$ with, (e.g., $\tau = 20$) as was done in the van der Pol example. In any case, the system soon tends toward $g(t)$ (also illustrated in Fig. 10), and all numerical examples indicate that entrainment occurs, although no proof of entrainment has yet been found for this Rössler system. However, a proof can be obtained for a class of goals in the Lorenz system, as will be shown below.

Next we consider the more complicated dynamics of the Lorenz system [22],

$$\dot{x} = \sigma(y - x) \ ,$$
$$\dot{y} = rx - y - xz \ , \tag{6.9}$$
$$\dot{z} = -bz + xy \quad (\sigma, b, r > 0) \ .$$

The determinant (2.2) yields the polynomial equation

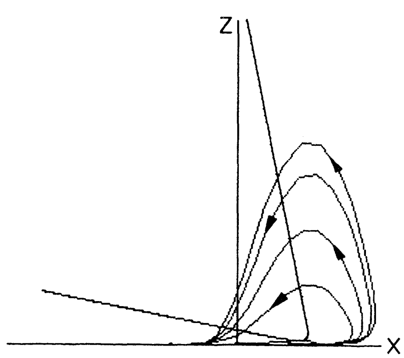


FIG. 8. Illustrates a limited portion of the Rössler dynamics and its relationship to the convergent region. Note that a large fraction of the attractor lies outside this convergent region.

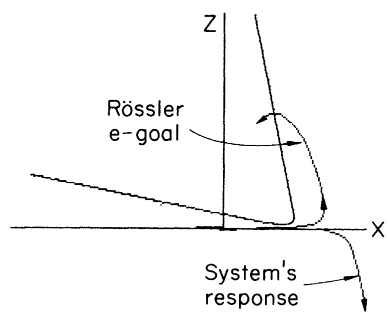


FIG. 9. Because the Rössler dynamics lies outside the convergent region much of the time, when it is used for the EG, the "controlled" systems response is to go to infinity after some time.

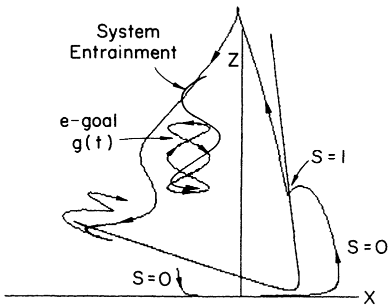


FIG. 10. The initial violent response of the Rössler system to the "hard" switch, (5.13), using a complicated EG, (6.8), in the convergent region. Also shown is the developing entrainment a short time later.

(2.5), for λ with the coefficients ($a_0=1$),

$$a_1=1+\sigma+b \ ,$$
$$a_2=\sigma(1-r+z)+x^2+b(1+\sigma) \ , \qquad (6.10)$$
$$a_3=b\sigma(1-r+z)+\sigma(x+y)x \ .$$

The convergent regions for (6.9) are where $\mathrm{Re}\lambda(x,y,z)<0$, and the necessary and sufficient conditions for this to hold is again determined by (6.3), which also implies that $a_2>0$. The contractive condition $a_1>0$ is independent of $(x,y.z)$, and is satisfied for the usual values $\sigma\geq 0, b\geq 0$. The conditions $a_3>0$ and $a_1a_2>a_3$ are, respectively given by

$$bz^*+x(x+y)>0 \ , \qquad (6.11)$$

$$\sigma(1+\sigma)z^*+(1+b)x^2-\sigma xy+b(1+\sigma)(1+\sigma+b)>0 \ , \qquad (6.12)$$

where $z^*=z-r+1$ is the z distance from the constant-z plane containing two fixed points of (6.9). Obviously these fixed points are in the convergent region (i.e., z^* can be negative at $x=y=[b(r-1)]^{1/2}$ in (6.12)) only when they are stable, $r>\sigma(\sigma+b+3)/(\sigma-b-1)$ (see, e.g., Ref. [23]).

The boundary of the convergent region is illustrated in Fig. 11, when $\sigma=10, b=8/3$. The convergent region lies above this surface, satisfying (6.11) and (6.12). This surface depends on r only through the variable z^*, and hence scales in a simple manner in the phase space (x,y,z). A simple consequence of this is that a Lorenz system can only be entrained to another Lorenz dynamics provided that the latter has a sufficiently larger value of r. This contrasts with the Rössler system in which the "control," (6.7), does not provide entrainment.

It is worth pointing out how the convergent region of the phase space is manifested in the autonomous dynamics of a system. For example, Nese [31] made an interesting study of the local divergence rate on the Lorenz attractor. He characterized various regions of the attractor (his Fig. 7) by the fact that their trajectories are "very predictable," "predictable," "unpredictable," and "very unpredictable." The first two of Nese's regions on the attractor correspond to the portion of the attractor that lies in the convergent region of the phase space (above the surface in Fig. 11). Thus this geometrical region of the phase space, which is defined without reference to any specific dynamics, is reflected in this autonomous dynamics (as well, of course, as the controlled dynamics considered here).

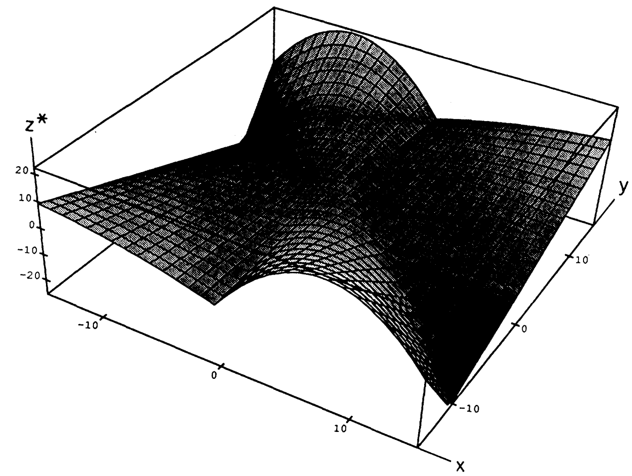


FIG. 11. Shows the boundary of the convergent region for the Lorenz system in the coordinates $(x,y,z^*=z-r+1)$, when $(b=8/3, \sigma=10)$. Above this surface the convergent conditions (6.11) and (6.12) are satisfied.

The control equations for the goal dynamics (g_x,g_y,g_z) are

$$\dot{x}=\sigma(y-x)+[\dot{g}_x-\sigma(g_y-g_x)]S(t) \ ,$$
$$\dot{y}=rx-y-xz+(\dot{g}_y-rg_x+g_y-g_xg_z)S(t) \ , \qquad (6.13)$$
$$\dot{z}=-bz+xy+(\dot{g}_z+bg_z-g_xg_y)S(t) \quad (\sigma,b,r>0) \ ,$$

where $S(t)=0$ for $t<0$. An example of the entrainment is shown in Fig. 12 ($\sigma=10, b=\frac{8}{3}$). In this case the experimental system has the value $r=10$, so that the two fixed points at $z=r-1$ are stable. The entrainment-goal (EG) dynamics is taken to be the Lorenz dynamics when $r=30$, and hence tending to a strange attractor. The figure shows a small segment of autonomous dynamics of the experimental system, which is tending toward its stable fixed point. When the control is turned on [$S=1$ in (6.13) and $t=0$), the system responds by making a wide arc and then rapidly approaching the EG dynamics. Actually, for these values of r, the goal dynamics does not have a strange attractor that is entirely in the convergent region of the system. Nonetheless, the entrainment appears to be achieved. This illustrates that $g(t)\in C$ is not always necessary for entrainment, as discussed in Sec. II.

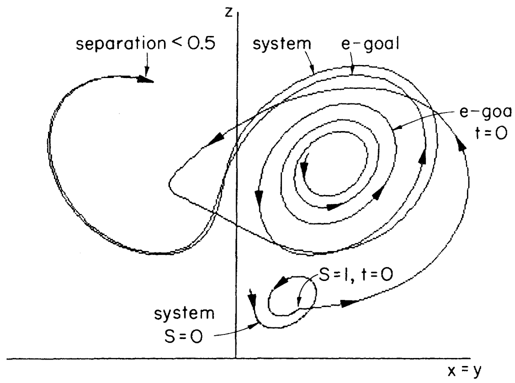


FIG. 12. The entrainment of the Lorenz system with $r+10$ to the EG of the chaotic Lorenz dynamics $r=30$.

Other examples of entrainment that take the system from a chaotic state (strange attractor) to an ordered, periodic motion can easily be achieved. The periodic goal can either be selected in some arbitrary fashion, as in the example used in the Rössler system (6.8) (but, of course, now placed in the convergent region of the Lorenz system), or by using a periodic solution of the Lorenz system. For example, the chaotic Lorenz dynamics (say $r=30$) can be entrained to the limit cycle associated with $r=350$, but with these extreme differences in r the entrainment can cause the system to react very violently. As noted in Sec. IV, such a violent reaction may excite in the real physical system other degrees of freedom that are not represented in the low-dimensional "inertial manifold" model (here the Lorenz system). In this case the model dynamics are no longer accurate, and the predictions of entrainment by the model may not occur in the physical system. This violent reaction of the system can be "softened" by again using a switching function such as (5.14). Little is yet known about methods of "optimizing" the selection of such soft switches, or of proving entrainment when they are employed. The knowledge to date is only from numerical examples, and it would be useful to extend this understanding.

From many numerical examples, it appears that the basin of entrainment of any goal in the convergent region (6.11) and (6.12) is global. That is, all solutions of (6.13) with $S=1$ satisfy $\lim_{t\to\infty}|x(t)-g(t)|=0$, provided that $g(t)$ lies in the convergent region (6.11) and (6.12). If this is true, then the control can be activated at any time [$S(t)$ set equal to 1]. To try to prove such a fact, it is useful to rewrite (6.13), $S=1$, in the form [replacing (x,y,z) with (1,2,3)]

$$\begin{aligned}\dot u_1&=\sigma(u_2-u_1)\,,\\ \dot u_2&=ru_1-u_2-u_1u_3-u_1g_3-g_1u_3\,,\\ \dot u_3&=-bu_3+u_1u_2+u_1g_2+u_2g_1\,,\end{aligned}\tag{6.14}$$

where $u_k\equiv x_k-g_k$. One would then like to determine if all solutions of (6.14) go to the origin of u-space, provided that $g\in C$. A step in this direction is to note that (6.14) yields

$$\begin{aligned}\frac{d}{dt}(\rho u_1^2+u_2^2+u_3^2)&=(\rho\sigma+r-g_3)u_1u_2-\rho\sigma u_1^2\\ &\quad -u_2^2-bu_3^2+g_2u_1u_3\\ &\equiv-(\alpha u_1-u_2)^2-b(\beta u_1-u_3)^2\end{aligned}\tag{6.15}$$

provided that $\dot\rho=0$ and (ρ,α,β) satisfy

$$\alpha^2+b\beta^2=\rho\sigma,\quad 2\alpha=\rho\sigma+r-g_3,\quad 2b\beta=g_2\ .$$

One finds that these conditions require that

$$\begin{aligned}&\alpha=1\pm[1+g_3-r-(g_2^2/4b)]^{1/2}\quad\text{(one root)}\,,\\ &\beta=g_2/2b,\quad \rho=(\alpha^2+b\beta^2)/\sigma\ .\end{aligned}\tag{6.16}$$

For real α, and $\dot\rho=0$,

$$\begin{aligned}&g_3>r-1+(g_2^2/4b)\,,\\ &\alpha(t)\dot g_3-g_2\dot g_2/2b=0\ .\end{aligned}\tag{6.17}$$

Moreover, from (6.16) we note the important fact that $\rho>0$.

Therefore we have the following result: the solutions of (6.14) satisfy

$$\frac{d}{dt}(\rho u_1^2+u_2^2+u_3^3)=-(\alpha u_1-u_2)^2-b(\beta u_1-u_3)^2\qquad(\rho>0)\,,$$

provided that (6.16) and (6.17) are satisfied. Since $\rho>0$, one can conclude in a straightforward fashion (similar to the reasoning in Sec. IV) that all solutions of (6.14) satisfy $\lim_{t\to\infty}|u|=0$. This can also be expressed in terms of the controlled equations (6.13).

Theorem 2. If

$$g_z>r-1+g_y^2/4b\tag{6.18a}$$

and one of the following conditions is satisfied:

$$[1\pm(1-r+g_z-g_y^2/4b)^{1/2}]\dot g_z-g_y\dot g_y/2b=0\ ,\tag{6.18b}$$

then all solutions of (6.13) (with $S=1$) are entrained to $g(t)$,

$$\lim_{t\to\infty}|x(t)-g(t)|=0\ .$$

In other words, the basin of entrainment of (6.18) is global ($\mathbb{R}^3$). The conditions (6.18) do not involve $g_x(t)$. Moreover, (6.18b) can be satisfied either by a fixed point $\dot g_y=\dot g_z=0$ or some complicated interdependence between $g_y(t)$ and $g_z(t)$. Regardless of what this relationship may be, or what $g_x(t)$ may be, if the condition (6.18a) is satisfied, one can prove that $g(t)$ is in the convergent region (6.11) and (6.12). This, of course, must be true if $g(t)$ is a fixed point, g_0, but it is also true for this class of dynamic EG's (6.18).

VII. MIGRATION BETWEEN LORENZ ATTRACTORS

Many of the most important and interesting complex dynamic systems have a number of attractors in their phase space (MAS) [1–11] as discussed in Ref. [12]. One of the important applications of entrainment (or near entrainment) is to make it possible to transfer a system from one attractor A_1 to another attractor A_2. If the system is in the basin of attraction BA_1 of A_1, this transfer can be achieved by first using a simple entrainment goal, such as a fixed point $g_0 \in C_1 \subset BA_1$, for a finite time, until the system and the EG is within some small distance $|x(t)-g_0|<\epsilon_e$, the entrainment error. At this time a migration goal (MG) is introduced, which goes from g_0 to the basin of attraction BA_2 of A_2. If the system follows this MG, it will enter BA_2, at which time the control can be terminated ($S=0$), and the system will tend autonomously to A_2. This achieves the desired transfer $A_1 \rightarrow A_2$.

This scenario is obviously contingent upon the satisfaction of "if the system follows this MG." If the MG is in the convergent region of the system most of the time, and dg/dt is not "too large," then the system will follow this MG. Many details of this process have yet to be worked out, but several points appear to be likely. If the MG is in a convergent region $C(\gamma)$, (2.4), and $d(\log g)/dt<\gamma$, the system should be able to follow this MG, provided that ϵ_e is sufficiently small. However, in order to move from one basin of attraction to another, the MG may leave the convergent region [12], in which case there are competing factors involved in a successful migration. The details of these considerations will be discussed in future publications (see, e.g., Ref. [24]). In this section we will simply illustrate this migration process in the Lorenz system. This will be done with two different examples involving the transfer $A_1 \rightarrow A_2$: (1) A_1 and A_2 are stable fixed points (but transient chaos); (2) A_1 is a strange attractor and A_2 is a stable fixed point.

In the first example the system is the Lorenz system with $r=17$, which has stable fixed points at

$$x_0^{\pm}=y_0^{\pm}=\pm[b(r-1)]^{1/2}, \quad z_0=(r-1) . \tag{7.1}$$

These attractors $A^{\pm}$ are the only attractors for $14<r<24$ ($\sigma=10, b=\frac{8}{13}$), but they have very convoluted basins of attraction, $BA^{\pm}$ [23,25,26]. This convoluted structure produces the phenomena of transient chaos, which is most observable for the larger values of $r<24$ (see, e.g., Ref. [27]).

Figure 13 shows the system when $r=15$, initially in the basin of attraction BA^+ of A^+. The MG in this case begins at A^+ and tends toward A^-, remaining always in the convergent region; specifically,

$$g_x(t)=g_y(t)=[b(r-1)]^{1/2}[2\exp(-t/5)-1] ,$$
$$g_z(t)=(r-1)(0.05\sin\{\pi[1-\exp(-t/5)]\}=1) . \tag{7.2}$$

When the control is initiated ($t=0$), the system is attracted toward (7.2), as can be seen from the projection on the $x=y$ plane in Fig. 13. The intersection of the convergent region's boundary with the plane $x=y$, in which (7.2) lies, is also illustrated. Once the system is "safely" in the basin of attraction BA^- of A^- (past the convoluted structure), the control is terminated ($S=0$), and the system tends to A^-.

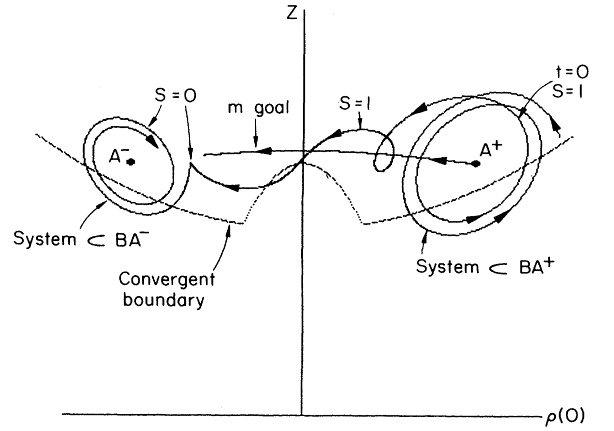


FIG. 13. The migration of the transient chaotic Lorenz system ($r=15$) from the basin of attraction BA^+ to BA^-. The migration goal (MG) is kept in the convergent region, above the convergent boundary. Once the system is in BA^-, the control can be terminated.

Several points should be noted about this process. (7.2) does not leave the convergent region even though it goes from BA^+ to BA^-. The convoluted boundary separating these two basins is not locally "unstable" in this region (it, of course, must be unstable someplace in order to separate trajectories toward A^+ and A^-). Moreover, since (7.2) does not leave C, it would be an EG if the control is maintained for all $t \geq 0$.

One of the very interesting dynamic states of the Lorenz system occurs in the approximate control-parameter range $24.1<r<24.7$ (see, e.g., Ref. [28]). In this range the system has two stable fixed points, (7.1), but it also has a strange attractor. The basins of attraction, $BA^{\pm}$, of these fixed points are cylindrical-like regions that extend to infinity, and coil around the positive-z axis in a serpentine fashion. The region outside of these topological cylinders is the basin of attraction of the strange attractor (aside from some invariant manifolds of lower dimension, and hence zero measure; see, e.g., Ref. [23]). Thus, outside of the regions $Ba^{\pm}$, the autonomous system has chaotic dynamics. Nonetheless much of these dynamics passes through the convergent regions (7.11) and (7.12), illustrated in Fig. 11.

Figure 14 illustrates the transfer of the system $r=24.1$ from the strange attractor to the stable fixed point A^-. First, the control (7.13) was initiated at an arbitrary time (labeled $t=0$ in the figure) while the system is near the strange attractor. The goal dynamics was arbitrarily

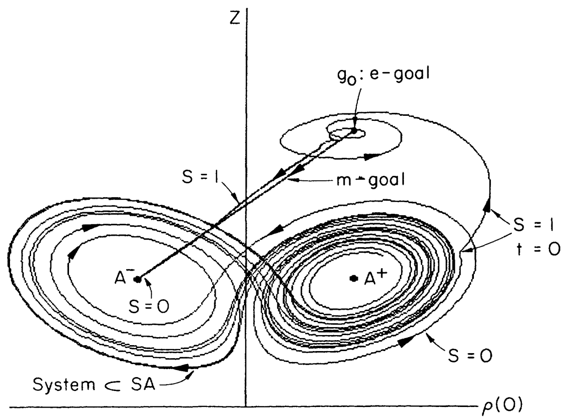


FIG. 14. Illustrates the transfer of the Lorenz dynamics ($r = 24.1$) from a strange attractor (SA) to the stable fixed point A^-. First the system is nearly entrained to $g_0 \in C$, switching on the control at any time ($t \equiv 0$). Then the MG (7.3) is used to transfer the system to the small BA^-, where the control is terminated ($S = 0$).

selected to be a fixed point well inside the convergent region

$$g_x = g_y = [b(r-1)]^{1/2}, \quad g_z = (r-1) + 50$$

labeled g_0 in the figure. This g_0 was selected in order to make this two-step control process very clear in the figure (much less exaggerated controls are possible). Once the system is nearly entrained, the control is changed to the MG,

$$g_x = g_y = [b(r-q)]^{1/2}\{2\exp[-(t-t_m)/5]-1\}, \\ g_z = (r-1) + 50\exp[-(t-t_m)/5] \quad (t \geq t_m), \tag{7.3}$$

where t_m is the arbitrary time the migration is begun. The system closely follows (7.3), and once the system is sufficiently near A^- (the distance 0.5 was used) the control can be terminated. In this particular system the basin of attraction of A^- is rather small and the attraction is very slow, so some care is required in the time when S is set to zero, and the integration methods used. In any case, this gives an example of the transfer $SA \rightarrow A^-$.

VIII. CONCLUSION

A variety of controls of the dynamics of complex systems can be accomplished when the goal dynamics of these controls, $g(t) \in \mathbb{R}^n$, are largely limited to convergent regions C of the system's phase space, (2.3), and possibly dynamically contrained, as illustrated in Sec. III. These controls may involve long-time entrainment goals EG's or finite-time migration goals MG's, which may venture for short periods outside of convergent regions. In the case of EG's, the objective is to produce some desired dynamics in a region C, whereas the purpose of the MG's is to transfer the system either from one convergent region to another, or from one attractor to another.

The process of making a reliable transfer between attractors involves a two-step process. First, the system must be reliably entrained to some simply dynamics state (e.g., a fixed point, g_0), which requires some knowledge of the convergent regions in which g_0 must be located, using the Routh-Hurwitz theorem. It was also shown in several cases that the basin of entrainment of certain families of goal dynamics, $\mathrm{BE}(\{g\})$, could be analytically obtained, while others could be estimated numerically. Once this information is known, the system can be reliably entrained to this goal dynamics with only macroscopic information about the system's initial state $[x_0 \in \mathrm{BE}(\{g\})]$.

This study also illustrated that, once the system is nearly entrained, $|x(t) - g(t)| < \epsilon_e$, the control can be changed to a migration type, which moves from one basin of attraction to another. This MG control needs to be applied only for a finite time to accomplish the transfer between attractors, $A_1 \rightarrow A_2$. Such a transfer may, or may not, require such MG dynamics. As shown in Sec. IV, this is not necessary in the case of one-dimensional flows. A fixed-point goal g_0 in the convergent region of one attractor, A_1, can be found that has any other attractor A_2 within its basin of entrainment, $\mathrm{BE}(g_0) \supset A_2$, making the transfer $A_2 \rightarrow A_1$ possible using only g_0. Similarly, in the Lorenz system if $r < 14$ it is possible to use a goal $g_0 \subset \mathrm{BA}(A^+)$ to cause the system in $\mathrm{BA}(A^-)$ to transfer to $\mathrm{BA}(A^+)$, where the control can be terminated, producing the transfer $A^- \rightarrow A^+$. However, for $r < 24.7$, where two fixed points are stable, the transfer from one to the other is increasingly difficult to accomplish in a reliable fashion as r becomes larger. This is due to the convoluted structure of the basins of attraction. Thus for the transfer between these fixed points, or from the strange attractor to a fixed point ($24.1 < r < 24.7$), the use of MG's is required.

At present, not much is known about the optimal rate of speed of such transfers. This partially concerns the maintenance of "near entrainment," which involves considerations discussed in Sec. III, as well as how such controls might cause the excitation of other degrees of freedom off the inertial manifold [14,15]. The use of a variable switching function [e.g., (5.14)] illustrates one approach to this latter concern. However, such important questions will need to be studied in the future.

The present methods of control may be contrasted with other recent interesting approaches introduced by Ott, Grebogi, and Yorke [32], which have been studied experimentally by Ditto, Rauseo, and Spano [33], as well as the studies by Huberman and Lumer [34], Huberman [35], and Sinha, Ramaswamy, and Rao [36]. These approaches focus on the modification of the control parameters of the system to gain their control (parametric controls). Moreover, the changes that are made in the control parameters are dependent upon the present state of the system, which is a feedback control approach. The present method of control differs in that it uses neither of these approaches and may also open the opportunity for migration controls.

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