Spatiotemporal instabilities of lasers in models reduced via center manifold techniques

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(Received 9 April 1991)

Several models of partial differential equations describing the dynamics of lasers with transverse effects are introduced by using the center manifold theorem for the elimination of irrelevant variables. By taking advantage of the different time scales associated with the relaxations of the variables, we first eliminate the polarization and later the population inversion. We show that in contrast with the plane-wave models, the use of center manifold techniques is necessary to properly describe the spatiotemporal behaviors of lasers. In particular, we characterize unsuspected Hopf bifurcations for the model obtained from the adiabatic elimination of the polarization and discuss the presence of diffusive terms induced by the interaction of radiation and matter after the elimination of the population inversion. A complex Ginzburg-Landau equation is also obtained in the small-field limit.

PACS number(s): 42.50.Tj, 42.60.−v, 42.65.−k, 05.45.+b

I. INTRODUCTION

The study of spatial effects in lasers and, more generally, in nonlinear optics is attracting growing interest. The theoretical analysis of Maxwell-Bloch equations that includes partial derivatives of the transverse coordinates [1] paved the way to the experimental observation of new instabilities leading to complex pattern formation, spatial symmetry breaking, oscillations, and eventually spatiotemporal chaos and turbulence in lasers [2]. A similar interest also developed for passive systems where hexagonal patterns have been predicted recently by two of us [3], while experiments on liquid crystals [4(a)], sodium vapor [4(b)], and photorefractive materials [4(c)] showed the formation and evolution of complex transverse patterns. Reference [5] contains a detailed list of recent references about these topics.

The theoretical and computational analysis of the models introduced in Ref. [1] is separated into two complementary branches. On one side, expansions into the modes of the empty cavity allow a clear picture of some static and dynamic behaviors of lasers in terms of characteristic resonances [6]. On the other hand, we approach here the Maxwell-Bloch equations as a system of partial differential equations in its own right and apply techniques developed in nonlinear dynamics for their theoretical and numerical analysis. The main topic of this paper is the reduction [also called adiabatic elimination (AE)] of the equations of motion describing a single-longitudinal mode laser when the relaxation time scales of the dynamical variables differ greatly, a common case in quantum optics. At difference with plane-wave models, any reduction of variables requires a correct application of the center manifold (CM) theory [7]. We show that naive eliminations of the irrelevant equations based on setting to zero the time derivative of the fast relaxing variables often lead to erroneous results.

The paper is divided as follows. In Sec. II we apply a CM technique to eliminate the polarization variable when its decay is fast in comparison with the relaxations of the electric field and population inversion. The necessity of the CM theory becomes evident when discussing the bifurcations associated with the reduced model. In Sec. III we obtain analytically the threshold of instability for Hopf and saddle-node bifurcations associated with dynamical and static pattern formation in lasers, respectively. These calculations are performed on a model which considers a flat pumping profile and almost plane mirrors. In spite of the approximations used, some of the analytical results are of general interest in the description of spatiotemporal instabilities in laser systems. What is relevant is that a naive AE would lead to a spurious instability which is instead removed when considering all the terms of the first-order expansion of the CM. Finally, we extend in Sec. IV the technique to the elimination of the population inversion when its characteristic decay time is shorter than the photon lifetime in the laser cavity. A complex Ginzburg-Landau equation in the limit of small field is then introduced and discussed.

II. THE ADIABATIC ELIMINATION OF THE POLARIZATION

We start from the Maxwell-Bloch equations for a single-longitudinal mode ring laser with transverse dependence of the fields as obtained in Ref. [1]:

$$\partial_t F = -k\left[\left\{1-i\left[\delta+a\left[\frac{\nabla^2}{4}+1-\rho^2\right]\right]\right\}F-R\right], \tag{2.1a}$$

$$\partial_t R = -(1+i\delta)R + F\Delta\ , \tag{2.1b}$$

$$\partial_t \Delta = -\gamma[\Delta-\chi(\rho)+\tfrac{1}{2}(F^*R+FR^*)]\ , \tag{2.1c}$$

where F is the complex electric field, R the complex polarization, and Δ the population inversion. δ, the atomic detuning referred to the mode pulling frequency; t, the time; k and γ, the decay rates of the field and population inversion, respectively, are all normalized to the decay rate of the polarization. Finally, ρ is the transverse radial coordinate and $\chi(\rho)$ the pump profile. These equations describe a single-longitudinal mode ring cavity with spherical mirrors. The parameter a contains the information about the geometry of the cavity. For example, in the case of the cavity shown in Fig. 1,

$$a=\frac{2}{T}\tan^{-1}\frac{L}{2\eta_0\Lambda},\quad \eta_0\equiv\frac{1}{\sqrt{2}}\left[\frac{R_C}{\Lambda}-\frac{1}{2}\right]^{1/2}, \tag{2.2}$$

where R_C is the radius of curvature of the spherical mirror, Λ is the total length of the cavity, L is the distance between the spherical and plane mirror, T is the total transmittivity of the cavity, and $(\eta_0)^{1/2}$ represents the minimum size of the beam waist. Equations (2.1) have been obtained by retaining the zeroth-order terms of a T expansion under the assumptions of $T \ll 1$, $\eta_0=O(1/T)$ and having normalized the space coordinates via

$$\rho=\left[\frac{\pi}{\lambda\Lambda\eta_0}\right]^{1/2} r\ , \tag{2.3}$$

where λ is the wavelength of the laser light [1]. This procedure remains valid when considering cavities with more than one spherical mirror. Changes affect the definition of the parameters a and η_0 only.

In the following we also analyze cavities with almost flat mirrors under the action of homogeneous pumps. Some of the assumptions introduced above and the normalization (2.3) are not appropriate for the correct treatment of this case. By considering large beam waists of order at least $1/T^2$, we observe that the parameter a scales as $1/(T\eta_0)$, the laplacian as η_0, and ρ^2 terms in (2.1a) as $1/\eta_0$. Then Eq. (2.1a) is replaced by

$$\partial_t F=-k\{[1-i(\delta+a\nabla^2)]F-R\}\ , \tag{2.4}$$

where purely numerical factors have been included into the space normalization. The physical meaning of these approximations is straightforward: $a(1-\rho^2)$ is the phase shift induced by the presence of the spherical mirror which becomes negligible once the curvature of the mirrors tends to zero. Equation (2.4) together with (2.1b) and (2.1c) under the further assumption of flat pump correspond to the model studied in Ref. [8] and will be the subject of our linear stability analysis in Sec. III.

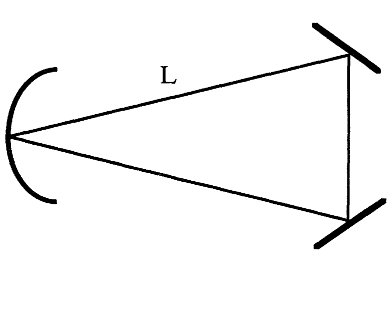

FIG. 1. Schematic representation of the ring laser cavity with a single spherical mirror of radius of curvature R_C. The total length of the ring is Λ; the distance between the curved and plane mirrors is L.

We assume that both k and γ are much smaller than unity, i.e., we restrict our analysis to lasers with a very broad homogeneous linewidth but with long spontaneous emission and photon lifetimes. Then, in agreement with Oppo and Politi [7], it is convenient to introduce

$$\Delta=h(\rho)(1+\delta^2)(1+\mu W),\quad \mu\equiv\sqrt{\gamma}\ . \tag{2.5}$$

Note that it is necessary to multiply the scaling of the population inversion by a radial function $h(\rho)$ to properly describe the behavior of the field at large distances from the beam center. In fact, even if the area around the beam center is above threshold, the tail of the intensity distribution approaches the zero state for large ρ's because of the radial shape of the pump. We therefore impose the following conditions on the function $h(\rho)$: $h(\rho)\to 1$ when $\rho\to 0$ and $h(\rho)\to\chi(\rho)/(1+\delta^2)$ for large ρ; any smooth function with these properties can serve as $h(\rho)$. Now Eqs. (2.1) become

$$\partial_t F=-k\left[\left\{1-i\left[\delta+a\left[\frac{\nabla^2}{4}+1-\rho^2\right]\right]\right\}F-R\right], \tag{2.6a}$$

$$\partial_t R=-(1+i\delta)R+h(\rho)(1+\delta^2)(F+\mu WF)\ , \tag{2.6b}$$

$$\partial_t W=-\mu\left[1+\mu W-\frac{\chi(\rho)}{h(\rho)(1+\delta^2)}+\frac{1}{2h(\rho)(1+\delta^2)}(F^*R+FR^*)\right]. \tag{2.6c}$$

At the zeroth order in the smallness parameters μ and k we obtain

$$\partial_t R=-(1+i\delta)R+h(\rho)(1+\delta^2)F\ , \tag{2.7}$$

where F acts as a constant forcing on this fast time scale. The solution of this equation is

$$R=R_0e^{-(1+i\delta)t}+h(\rho)(1-i\delta)F\ , \tag{2.8}$$

where R_0 represents the initial condition. In order to find the first-order corrections it is convenient to introduce the fluctuation of R with respect to the zeroth-order solution:

$$r\equiv R-R_0e^{-(1+i\delta)t}-h(\rho)(1-i\delta)F \tag{2.9}$$

and rewrite the partial differential equations up to the first order in μ and k:

$$\partial_t r=-(1+i\delta)r+\mu h(\rho)(1+\delta^2)WF-h(\rho)(1-i\delta)\partial_t F\ , \tag{2.10a}$$

$$\partial_t F=-k\left[\left\{1-i\left[\delta+a\left[\frac{\nabla^2}{4}+1-\rho^2\right]\right]\right\}F-r-R_0e^{-(1+i\delta)t}-h(\rho)(1-i\delta)F\right], \tag{2.10b}$$

$$\partial_t W = -\mu \left[1 + \mu W - \frac{\chi(\rho)}{h(\rho)(1+\delta^2)} + \frac{1}{2h(\rho)(1+\delta^2)} \times [Fr^* + F^* r + FR_0^* e^{-(1-i\delta)t} + F^* R_0 e^{-(1+i\delta)t} + 2h(\rho)|F|^2] \right] . \tag{2.10c}$$

We have now cast the transverse laser equations in a form suitable for the application of the CM theory [7]. In order to make the paper self-contained, we briefly review the application of the CM theory to infinite-dimensional systems as suggested by Carr [7]. By considering the generic system of partial differential equations

$$\partial_t \mathbf{U} = \epsilon[\nabla^2 \mathbf{U} + f(\mathbf{U}, \mathbf{V}, \epsilon)] ,$$
$$\partial_t \mathbf{V} = -\alpha \mathbf{V} + \epsilon g(\mathbf{U}, \mathbf{V}, \epsilon) ,$$
$$\partial_t \epsilon = 0 ,$$

where $\mathbf{U}$ and $\mathbf{V}$ are vectors of variables, ϵ is a smallness parameter, $f(\mathbf{U},\mathbf{V},\epsilon)$ and $g(\mathbf{U},\mathbf{V},\epsilon)$ are nonlinear functions of the arguments, and α is a complex number with negative real part, Carr showed that the CM theorem can be applied so that $\mathbf{V}$ may be expressed as a function of the other independent variables of the system and effectively eliminated from the equations. The major difference here in comparison with the application of this theory to ordinary differential equations is that one has to consider $\nabla^2 \mathbf{U}$ as one of the independent variables, i.e., the perturbative expression of $\mathbf{V}$ is given by

$$\mathbf{V} = \sum_{i=0}^{\infty} \epsilon^i \mathbf{V}_i(\mathbf{U}, \mathbf{U}^*, \nabla^2 \mathbf{U}, \nabla^2 \mathbf{U}^*) ,$$

where the complex character of $\mathbf{U}$ has been made explicit. By substituting this expression in the trivial equation

$$\partial_t \mathbf{V}(\mathbf{U}, \mathbf{U}^*, \nabla^2 \mathbf{U}, \nabla^2 \mathbf{U}^*) = \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \partial_t \mathbf{U} + \frac{\partial \mathbf{V}}{\partial \mathbf{U}^*} \partial_t \mathbf{U}^* + \frac{\partial \mathbf{V}}{\partial \nabla^2 \mathbf{U}} \partial_t \nabla^2 \mathbf{U} + \frac{\partial \mathbf{V}}{\partial \nabla^2 \mathbf{U}^*} \partial_t \nabla^2 \mathbf{U}^*$$

one is then able to determine the functions $\mathbf{V}_i$ to the desired order in the perturbation expansion. A comparison of this system of partial differential equations with (2.10) yields $\mathbf{U}=(F,W)$, $\mathbf{V}=r$, and $\alpha = -(1+i\delta)$. Then, a lower-dimensional attracting surface (containing F and W and over which the long-term dynamics develop) can be determined by just applying the previous rules, i.e.,

$$r = \mu h(\rho)(1-i\delta) WF - \frac{h(\rho)}{1+\delta^2}(1-i\delta)^2 \partial_t F . \tag{2.11}$$

It is important to note that this first-order result coincides with the setting of the time derivative of the r variable equal to zero. This is, however, a peculiar consequence of the structure of our equations, as a rigorous application of the CM theory requires the use of the following equation:

$$\partial_t r(F, F^*, W, \nabla^2 F, \nabla^2 F^*) = \frac{\partial r}{\partial F} \partial_t F + \frac{\partial r}{\partial F^*} \partial_t F^* + \frac{\partial r}{\partial W} \partial_t W + \frac{\partial r}{\partial \nabla^2 F} \partial_t \nabla^2 F + \frac{\partial r}{\partial \nabla^2 F^*} \partial_t \nabla^2 F^* , \tag{2.12}$$

where r is expressed as a perturbation expansion of the given smallness parameters. One can easily check that the zeroth-order term of this expansion is identically zero and that the right-hand side of the expression (2.12) starts with second-order terms. The shape of the CM surface at orders higher than the first one greatly differs from that obtained by assuming steady-state conditions on the variable r.

We substitute the CM expression (2.11) into Eqs. (2.10) and rescale the time by $\tau = \mu t$. The negative exponential terms can then be neglected even for small τ. A further step is accomplished by inserting the lowest-order terms of Eq. (2.10b) into (2.11), so that we obtain for the field and population inversion variables:

$$\partial_\tau F = \frac{k}{\mu} \left[-\beta(\rho) F + \alpha(\rho) \left[\frac{\nabla^2}{4} + 1 - \rho^2 \right] F + \mu h(\rho)(1-i\delta) WF \right] , \tag{2.13a}$$

$$\partial_\tau W = D(\rho) - \mu W - \left[\frac{1+\mu W}{1+\delta^2} - \sigma(\rho) \right] |F|^2 + \frac{ika}{2(1+\delta^2)^2} \left[(1-i\delta)^2 F^* \frac{\nabla^2}{4} F - (1+i\delta)^2 F \frac{\nabla^2}{4} F^* \right] , \tag{2.13b}$$

where we have introduced the two complex parameters

$$\alpha(\rho) \equiv ia \left[1 - kh(\rho) \frac{(1-i\delta)^2}{1+\delta^2} \right] , \tag{2.14a}$$

$$\beta(\rho) \equiv (1-i\delta) g(\rho) \left[1 - kh(\rho) \frac{(1-i\delta)^2}{1+\delta^2} \right] , \tag{2.14b}$$

and the three real parameters

$$g(\rho) \equiv 1 - h(\rho) , \tag{2.14c}$$

$$D(\rho) \equiv \frac{\chi(\rho)}{h(\rho)(1+\delta^2)} - 1 , \tag{2.14d}$$

$$\sigma(\rho) \equiv \frac{k}{(1+\delta^2)^2} [g(\rho)(3\delta^2 - 1) + \delta a(1-\rho^2)] , \tag{2.14e}$$

which are functions of the radial coordinate ρ. Note that $g(\rho)$, the new loss function, tends to zero for small ρ and to unity as the pump rate drops to zero at large distances from the beam center. $D(\rho)$ is the new pump parameter normalized to the on-axis laser threshold.

A first check about the correctness of the AE of the polarization is accomplished by studying the asymptotic behavior of the solutions at large distances from the beam center. In this case, Eqs. (2.13) reduce to

$$\partial_\tau F=\frac{k}{\mu}\left[-(1-i\delta)+ia\left[\frac{\nabla^2}{4}+1-\rho^2\right]\right]F\ , \qquad (2.15a)$$

$$\partial_\tau W=-\mu W-\frac{|F|^2}{1+\delta^2}\ , \qquad (2.15b)$$

which implies that the field F vanishes like a Gaussian while W relaxes smoothly to zero.

Equations (2.13) together with the expressions (2.14) are the first key result of this analysis and constitute a properly derived "rate equation" model for ring lasers with transverse effects. In spite of the complicated form of the coefficients, our model introduces great advantages for both the theoretical (as shown in Sec. III) and numerical analysis of laser systems. For example, we have removed the intrinsic stiffness of the complete model (2.1) arising from the very different relaxation scales, so that the time step of numerical computations can now be of the same order of magnitude of the characteristic oscillations of the system. Moreover, we show in the following section that AE techniques less refined than the CM theory lead to erroneous predictions of spatiotemporal instabilities.

III. SPATIOTEMPORAL INSTABILITIES

The model (2.13) represents a realistic ring cavity in presence of spherical mirrors and pumping processes which decrease with the distance from the beam center. It is, however, quite difficult to obtain analytical results from this set of equations. The radially dependent coefficient originate terms in the Fourier space which are difficult to handle. We therefore start the study of the spatial and temporal instabilities associated with the adiabatic elimination of the polarization by considering the model of almost plane mirrors and flat pump. This model is obviously less suitable to describe real lasers than (2.13). However, the instabilities detected in the approximated equations have generally served as guidelines for more realistic models. Starting from Eqs. (2.4), (2.1b), and (2.1c), considering spatially uniform pump mechanisms and repeating the same steps shown in Sec. II we obtain

$$\partial_\tau F=\frac{k}{\mu}[\alpha\nabla^2F+\mu(1-i\delta)WF]\ , \qquad (3.1a)$$

$$\partial_t W=D-\mu W-\frac{1+\mu W}{1+\delta^2}|F|^2+\frac{ika}{2(1+\delta^2)^2}[(1-i\delta)^2F^*\nabla^2F-(1+i\delta)^2F\nabla^2F^*]\ , \qquad (3.1b)$$

where the pump parameter D and the complex parameter α are now independent of the transverse coordinates. Note that the same equations can be obtained from the model (2.13) by imposing $h(\rho)=1$ for all ρ and applying the scaling considerations which yielded Eq. (2.4). The spatially homogeneous, time-independent solution $W=0$ and $|F|^2=D(1+\delta^2)$ satisfies Eqs. (3.1) and we can analyze the departure from the uniform state by studying the temporal growth of small perturbations. The linear stability analysis of the plane-wave solution allows a comparison with models based on the adiabatic elimination of the polarization by setting its time derivative equal to zero [from now on referred as standard AE (SAE)]. Recently, for example, Jakobsen *et al.* [9] reported about the existence of infinitely short-wavelength instabilities which led to temporal oscillations in a model based on SAE. One of the main results of our analysis is to show that the results based on SAE are often incorrect. They either produce spurious effect (an infinite wave-number tail for the case of $\delta<0$) or suppress important instabilities (such as in the case of $\delta>0$).

The linear stability analysis in the Fourier space of Eqs. (3.1) yields a cubic polynomial of the form

$$\lambda^3+A(n)\lambda^2+B(n)\lambda+C(n)=0\ , \qquad (3.2)$$

where n is the magnitude of the transverse wave vector, λ the exponents of the linear stability analysis, and

$$A(n)=\mu\left[D+1-4\frac{k^2an^2\delta}{\mu^2(1+\delta^2)}\right]\ , \qquad (3.3a)$$

$$B(n)=\frac{k^2a^2n^4}{\mu^2}+2kD-\frac{k^2an^2\delta}{1+\delta^2}[4+D(1+\delta^2)]\ , \qquad (3.3b)$$

$$C(n)=\frac{k^2an^2}{\mu}[an^2(D+1)-2D\delta]\ . \qquad (3.3c)$$

We then separate two cases depending on the sign of the detuning δ. For atomic frequencies smaller that the cavity frequency (negative δ) all the coefficients in (3.3) are positive and the instabilities occur only via a Hopf mechanism, whenever the inequality

$$A(n)B(n)-C(n)<0 \qquad (3.4)$$

is satisfied. Simple algebra shows that there are no Hopf instabilities if either

$$\mu\ll k,\quad k\ll\mu^2,\quad \text{or}\ |\delta|<3\mu(D+1)\left[\frac{3}{D}\right]^{1/2}\ . \qquad (3.5)$$

Otherwise, for any value of the parameters there is an interval of values of an^2 between

$$an^2_{\min}\cong\frac{\mu^2(D+1)}{k|\delta|},\quad an^2_{\max}\cong\frac{\mu}{k}\left[\frac{D(1+\delta^2)}{2}\right]^{1/2} \qquad (3.6)$$

such that the corresponding wavelengths are unstable to small perturbations. The most unstable wave number is given by

$$an_0^2\cong\frac{\mu}{k}\left[\frac{D(1+\delta^2)}{6}\right]^{1/2}\ , \qquad (3.7a)$$

which, in terms of unscaled variables, yields

$$n_r^2=\frac{4\pi}{\lambda\Lambda\eta_0}n_0^2=\frac{4\pi}{c\lambda}\left[\frac{\gamma_\parallel\gamma_\perp D(1+\delta^2)}{6}\right]^{1/2}\ , \qquad (3.7b)$$

where c is the speed of light, and $\gamma_\parallel$ and $\gamma_\perp$ are the unscaled decay rates of the population inversion and polarization, respectively. The corresponding Hopf frequency

$$\Omega \cong \left[\frac{D(1+\delta^2)}{6} + 2kD \right]^{1/2} , \tag{3.8}$$

associated with the spatiotemporal instabilities, may be orders of magnitude larger than the frequency of the relaxation oscillations [the second term in large parentheses in Eq. (3.8)]. For example, for the single-longitudinal mode Nd:YAG microchip laser [10] (where YAG denotes yttrium aluminum garnet), the inclusion of transverse effects may result in the generation of ultrashort pulses by using the same modulation techniques developed for nanosecond pulses. More importantly, the presence of unstable wave numbers at low pump values allows one to easily excite temporal oscillations on a spatial scale of the order of millimeters. Figure 2(a) shows the results of the inequality (3.4) for $k=\mu=10^{-3}$, $\delta=-0.2$, and $D=3$ and 5, i.e., within ranges easily accessible to experiments for both gas and solid-state lasers. The short-dashed lines correspond to the SAE case showing the short-wavelength catastrophe, which is removed by a proper application of the CM theory. In order to confirm that this effect is not a spurious consequence of the AE procedure, we have numerically evaluated the largest real part of the eigenvalues of the linearization of the full five equation model of Lugiato, Oldano, and Narducci [8] [corresponding to Eqs. (2.4), (2.1b), and (2.1c); the revised coefficients of the fifth-order polynomial are presented in the Appendix] and compared it with the results of both Eqs. (3.3) and the SAE model. The agreement between our theory and the equations without AE is so close that they cannot be distinguished within the resolution of the graphics of Fig. 3(a), which simultane-

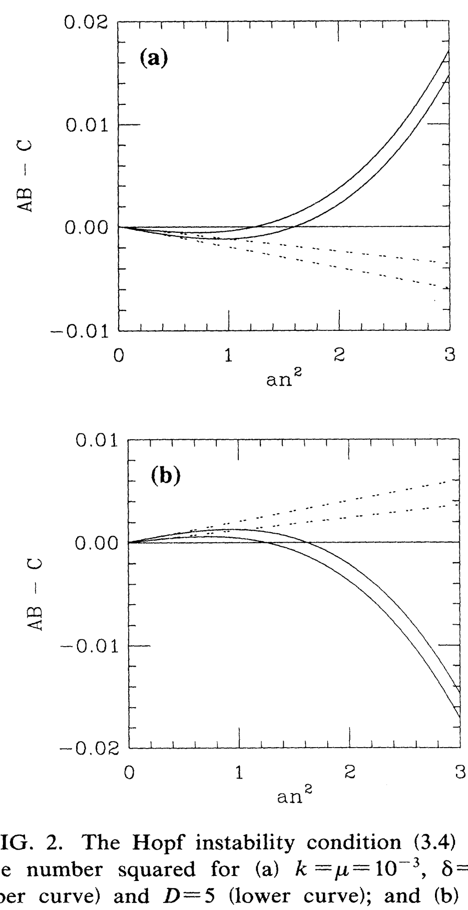

FIG. 2. The Hopf instability condition (3.4) vs the scaled wave number squared for (a) $k=\mu=10^{-3}$, $\delta=-0.2$, $D=3$ (upper curve) and $D=5$ (lower curve); and (b) $k=\mu=10^{-3}$, $\delta=0.2$, $D=3$ (lower curve) and $D=5$ (upper curve). The short-dashed lines correspond to the SAE case.

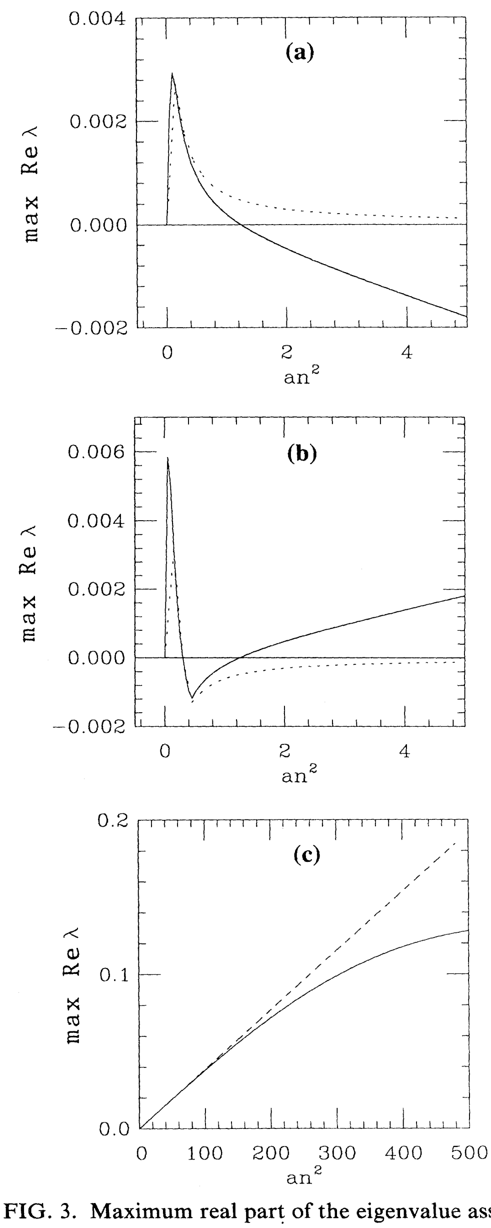

FIG. 3. Maximum real part of the eigenvalue associated with the linear stability analysis of the five-equation model (Ref. [8] and the Appendix) (solid line), model (3.1) (dashed lines, sometimes indistinguishable from the solid lines), and the model after SAE procedures (short-dashed lines), vs the scaled wave number squared. The parameters are (a) $k=\mu=10^{-3}$, $D=3$, and $\delta=-0.2$ or (b) and (c) $\delta=0.2$.

ously demonstrates the deficiencies of the SAE.

An unsuspected Hopf instability affects the spatiotemporal evolution of the system (3.1) for positive detuning δ. By similar calculations to those previously described, we obtain the well-known saddle-node instability [8] for

$$an^2 < \frac{2D\delta}{D+1} \tag{3.9}$$

corresponding to change of sign of $C(n)$. This long-wavelength instability coexists, however, with a Hopf bifurcation obtained via Eq. (3.4),

$$an^2 > \frac{\mu}{k}\left[\frac{D(1+\delta^2)}{2}\right]^{1/2}, \tag{3.10}$$

which is a short-wavelength instability [see Fig. 2(b)]. Again, we have checked that this feature of our model (3.1) is not an artifact of the AE procedure. In Fig. 3(b) and 3(c) we show the largest real part of the eigenvalues in a comparison between the full five equations model [8], Eqs. (3.1), and the model obtained from SAE. The first important observation concerns the fact that the SAE wrongly suppresses the Hopf instability, which is a genuine feature of the original system of equations, showing again its unreliability. The second comment regards the excellent agreement of the CM results with the original model, which starts to break up only at the limit of the approximations, i.e., when the laplacian of the field cannot be treated in a perturbative way any more [see Fig. 3(c)]. For cases where short distances are relevant to the dynamics, one has to include the spatial derivatives at the zeroth-order in the perturbation expansion, i.e., to know the solution of a partial differential equation exactly. The third feature is the unexpected character of the Hopf instability for positive detunings. As Fig. 2(c) clearly shows, a correct description of the spatial and temporal dynamics involves both long wavelengths via the saddle-node mechanism and oscillations on extremely short scales inducing serious limitations on realistic computer simulations. At present, the physical origin of the latter instability is not completely clear. By using the unscaled formalism of (3.7b) for the case of a Nd:YAG laser, we found that the spatial scale of the oscillations is comparable with the laser wavelength (i.e., of the order of micrometers). This may be due to the unphysical assumptions of infinitely extended plane mirrors and pump, and absence of transverse boundary conditions. For this reason, our present efforts focus on the numerical integration of realistic models such as (2.13) which contain physical hypotheses on the transverse dependence of parameters. We are numerically testing the mechanisms of instability under the conditions specified by the inequality (3.10).

In conclusion, we have shown that the presence of the transverse Laplacian term in the original equations requires the application of the CM theory even for the case of very large decay rate of the polarization. Note that this case is well known to be "regular" in the plane-wave approximation where the system (3.1) reduces to a set of ordinary differential equations. The use of CM theory following the outline of the present calculations is then appropriate whenever partial differential equations for laser dynamics are analyzed.

IV. THE ADIABATIC ELIMINATION OF THE POPULATION INVERSION

For completeness, we now proceed to eliminate the population inversion variable W by taking advantage of the form of Eqs. (2.13), in the case of lasers with long photon lifetime, i.e.,

$$\frac{k}{\mu^2}=\frac{k}{\gamma} \ll 1 . \tag{4.1}$$

Note that in agreement with Oppo and Politi [11] one cannot consider k/μ as the smallness parameter because such scaling would originate a singular perturbation expansion (see also the end of this section for further comments on this point). The zeroth-order equation for the population inversion can be solved exactly by regarding the field F as a constant forcing:

$$W=\frac{D(\rho)(1+\delta^2)-|F|^2}{\mu(1+\delta^2+|F|^2)} . \tag{4.2}$$

As usual, the first-order corrections are studied by introducing the fluctuation of W from the value given by its long-term solution

$$u=W-\frac{D(\rho)(1+\delta^2)-|F|^2}{\mu(1+\delta^2+|F|^2)} \tag{4.3}$$

and by writing its corresponding differential equation

$$\partial_\tau u=\partial_\tau W+\partial_\tau|F|^2\left[\frac{(1+\delta^2)[D(\rho)+1]}{\mu(1+\delta^2+|F|^2)^2}\right] . \tag{4.4}$$

After some manipulations one obtains

$$\begin{aligned}\partial_\tau u=&-\mu\left[\frac{1+\delta^2+|F|^2}{1+\delta^2}\right]u+\sigma(\rho)|F|^2\\&+\frac{ika}{2(1+\delta^2)^2}\left[(1-i\delta)^2F^*\frac{\nabla^2}{4}F\right.\\&\qquad\left.-(1+i\delta)^2F\frac{\nabla^2}{4}F^*\right]\\&+\partial_\tau|F|^2\left[\frac{(1+\delta^2)[D(\rho)+1]}{\mu(1+\delta^2+|F|^2)^2}\right],\end{aligned} \tag{4.5a}$$

$$\begin{aligned}\partial_\tau F=\frac{k}{\mu}&\left[-\beta(\rho)F+\alpha(\rho)\left[\frac{\nabla^2}{4}+1-\rho^2\right]F\right.\\&+\frac{h(\rho)(1-i\delta)[D(\rho)(1+\delta^2)-|F|^2]}{1+\delta^2+|F|^2}F\\&\left.+\mu h(\rho)(1-i\delta)uF\right] .\end{aligned} \tag{4.5b}$$

It may seem that this form of the equations is not suitable for the CM theory because of the $|F|^2$ dependence of the decay rate of the variable u. However, as $|F|^2$ is always positive, its effect is to accelerate the relaxation of the variable u to its long-term value justifying better and better the application of the CM method. More important, the cutoff of large wave numbers n^2 necessary when considering the Laplacian terms of Eq. (4.5b) to be small is now shifted to sizes of the order of $\mu^2/(ka)$ due to the characteristic decay time of the variable u.

Considering terms up to the first order in k/μ^2, evaluating the time derivative of $|F|^2$ at the zeroth order in k and renormalizing the time by $\tau'=(k/\mu)\tau=kt$ we obtain the single equation

$$\partial_{\tau}F=-\beta(\rho)F+\alpha(\rho)\left[\frac{\nabla^2}{4}+1-\rho^2\right]F+\eta(\rho,|F|^2)F+\zeta(\rho,|F|^2)F\left[F^*\frac{\nabla^2}{4}F-F\frac{\nabla^2}{4}F^*\right], \tag{4.6}$$

where we have defined the following complex quantities:

$$\eta(\rho,|F|^2)\equiv\frac{h(\rho)(1-i\delta)}{1+\delta^2+|F|^2}\left\{D(\rho)(1+\delta^2)-|F|^2\left[1-\sigma(\rho)(1+\delta^2)+\frac{2k(1+\delta^2)^2[D(\rho)+1]}{\mu^2(1+\delta^2+|F|^2)^2}\left[g(\rho)-\frac{h(\rho)[D(\rho)(1+\delta^2)-|F|^2]}{1+\delta^2+|F|^2}\right]\right]\right\}, \tag{4.7a}$$

$$\zeta(\rho,|F|^2)\equiv\frac{ikah(\rho)(1-i\delta)(1+\delta^2)^2[D(\rho)+1]}{\mu^2(1+\delta^2+|F|^2)^3}. \tag{4.7b}$$

Note that both η and ζ yield k/μ^2 terms which are obviously larger than k terms because of the smallness of μ. This is an *a posteriori* proof that Eq. (4.6) is valid in the limit (4.1) while Eqs. (2.13) have to be used in any other case in agreement with Ref. [11].

A linear stability analysis of stationary states of (4.6) is not available because of the explicit dependence of the parameters on the transverse coordinates. Again, we analyze the case of lasers with flat pump and almost plane mirror cavities. By repeating the calculations shown in Sec. III, we have investigated the stability of the uniform solution to small perturbations of a certain spatial size. The only instability found in Sec. III which survives the limit (4.1) is the saddle-node bifurcation (3.9) showing that the mechanism related to the onset of spatiotemporal oscillations is strictly related to the slow dynamics of the population inversion. These results are correct up to magnitudes of the wave vector of order $\mu^2/(ka)$. Shorter wavelengths require the use of more sophisticated CM techniques.

The form of the Eq. (4.6) is valid for any values of the laser intensity and of the detuning δ. However, a physical interpretation is difficult because of the complexity of the terms. We then introduce a revised version of the Ginzburg-Landau equations by taking the limit of Eq. (4.6) for small intensities, i.e., $|F|^2\to 0$,

$$\partial_{\tau}F=A(\rho)F+B(\rho)F|F|^2+\alpha(\rho)\left[\frac{\nabla^2}{4}+1-\rho^2\right]F+\zeta(\rho)F\left[F^*\frac{\nabla^2}{4}F-F\frac{\nabla^2}{4}F^*\right], \tag{4.8}$$

where

$$A(\rho)\equiv(1-i\delta)\left[h(\rho)D(\rho)-g(\rho)+\frac{kg(\rho)h(\rho)(1-i\delta)^2}{1+\delta^2}\right], \tag{4.9a}$$

$$B(\rho)\equiv-h(\rho)(1-i\delta)[D(\rho)+1]\times\left[\frac{1}{1+\delta^2}+\frac{2k}{\mu^2}[g(\rho)-h(\rho)D(\rho)]\right], \tag{4.9b}$$

$$\alpha(\rho)\equiv ia\left[1-kh(\rho)\frac{(1-i\delta)^2}{1+\delta^2}\right], \tag{4.9c}$$

and

$$\zeta(\rho)=\frac{ikah(\rho)(1-i\delta)[D(\rho)+1]}{\mu^2(1+\delta^2)}. \tag{4.9d}$$

A first difference between (4.8) and an equivalent equation obtained by SAE procedures is represented by the "diffusive" term due to the real part of the coefficient (4.9c). Such a term changes its sign with the detuning originating the unusual process of "antidiffusion" for positive δ. We mention here that the integration of Eq. (4.8) for negative δ has revealed the existence of new and unexpected solutions [12] where it is possible to study the interaction of optical vortices [13]. Other peculiar features of Eq. (4.8) are related to the presence of nonlinear laplacian terms of the form $|F|^2\nabla^2F$ and to the explicit dependence of the coefficients on the transverse coordinates, a case not previously considered in the analysis of Ginzburg-Landau equations. Our present studies are focusing on the effects of these terms on the process of pattern formation in lasers [12].

V. CONCLUSIONS

The adiabatic elimination of irrelevant variables has been shown to be very sensitive to the method used for the perturbation expansions in the case of partial differential equations which describe laser dynamics. The center manifold theory supplies a solid mathematical framework within which the fast variables as well as the characteristic scalings of the long-term dynamics are properly determined. More specifically, the laser equations, which have been already shown to take advantage of these techniques in the temporal domain [7], *require* the application of the CM theory *whenever* spatial effects are taken into account. For example, we have proved here that some instabilities determined by using standard techniques do not have any counterpart in the complete set of the original equations. A more refined application of the CM is now under investigation in order to treat the equations at large wave numbers. In this case one has to solve partial differential equations exactly even at the zeroth order of the perturbation expansion. This fact makes the application of these techniques extremely difficult in the case where no analytical solutions are explicitly known. The interaction of numerical methods and computerized algebra can, however, overcome this difficulty leading to reduced set of partial differential equations easy to integrate. This will be the subject of future communications.

ACKNOWLEDGMENTS

Useful discussions with J. R. Tredicce, L. A. Lugiato, L. M. Narducci, J. V. Moloney, L. Gil, A. Kent, and A. Politi are gratefully acknowledged. This work has been partially supported by the CNRS (Nice, France), by the SERC Grants Nos. GR/F 75087 and GR/G 12665, by a Twinning project within the SCIENCE program of the European Community, and by the University of Strathclyde through the faculty funds.

APPENDIX

The coefficients a_i of the fifth-order characteristic polynomial

$$\sum_{i=0}^{5} a_i \lambda^i$$

of the linear stability analysis of the homogeneous solution of Eqs. (2.4), (2.1b), and (2.1c) corresponding to the case of almost plane mirrors and flat pump are

$$a_5 = 1 \ ,$$

$$a_4 = 2(1+k)+\mu^2 \ ,$$

$$a_3 = 2\mu^2(1+I_0)+(1+k^2)(1+\delta^2) + 2k(1-\delta^2)+k^2 an^2(an^2-2\delta) \ ,$$

$$a_2 = \mu^2[(1+k)(1+\delta^2)+2k(1-\delta^2)+I_0(3k+1)] + k^2 an^2(2+\mu^2)(an^2-2\delta) \ ,$$

$$a_1 = 2k\mu^2 I_0(1+k)+k^2 an^2\{an^2[1+\delta^2+\mu^2(2+I_0)] - \mu^2\delta(4+I_0)\} \ ,$$

$$a_0 = k^2\mu^2 an^2[an^2(1+\delta^2+I_0)-2\delta I_0] \ ,$$

where

$$I_0 = D(1+\delta^2)$$

is the field intensity of the stationary solution.

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