Modulation, signal, and quantum noise in interferometers

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(Received 15 March 1991)

We discuss the generation of quantum noise in interferometers. A heuristic model is described that enables different characteristics of the noise to be understood and calculated. In particular, the effects of modulation on signal and noise in interferometers are discussed. It is seen that the mixing in of quantum noise from different modes of the vacuum can degrade the signal-to-noise ratio of a measurement in a variety of apparently different physical situations. Michelson interferometers with both internal and external phase modulation are specifically considered, but the results are easily extendable to any type of interferometer. We also comment upon some implications for the use of squeezing in interferometers. For examples, we argue that it is possible, even in lossy narrow-band interferometers, for squeezing to reduce significantly the effective quantum-noise power.

PACS number(s): 42.50.−p, 07.60.Ly, 04.80.+z

I. INTRODUCTION

Optical interferometers are commonly used to make measurements of phase. Very high sensitivity can be achieved, for example, in laser-interferometric gravitational-wave detectors (see Thorne [1] for a review). In these instruments it is usual to employ rf phase modulation as a means of mixing the signal of interest up to frequencies at which the intensity fluctuations of the light source are low. For Michelson interferometers operating on a dark fringe, this has the additional advantage that light recycling can be used [2–4] to enhance further the sensitivity of the instrument. If maximum information is to be extracted from an interferometer, it is important that the particular modulation technique adopted does not significantly degrade the signal-to-noise ratio. Schilling [5] and Man *et al.* [6] have pointed out situations arising when "extended modulation" is used in which the ideal signal-to-noise ratio cannot be attained. With "internal" phase modulation, Schnupp [7] and Niebauer *et al.* [8] have shown that the nonstationary nature of the noise means that the signal-to-noise ratio may be worse than optimum unless the demodulation waveform is radically different from that of the modulation. In order to understand both of these results fully, we will discuss a specific model of how quantum noise enters an interferometer. We shall then argue that the physical origin of the poorer signal-to-noise ratio in these apparently disparate situations is, in fact, exactly the same: namely, the mixing in of (optical) noise from frequencies at which there is no signal.

Throughout this paper we shall confine our attention to that noise which manifests itself as fluctuations in the power of the detected light. We shall specifically ignore the back action on the interferometer resulting from radiation pressure fluctuations. This is of negligible magnitude in almost all situations of interest. Thus, while we shall describe the photon counting fluctuations as quantum noise, it must be emphasized that this is not the same as the "standard quantum limit" [9] to the sensitivity of an interferometer.

II. SIGNAL IN A MICHELSON INTERFEROMETER

We need to understand the detection of a phase change by an interferometer. Consider a sample Michelson interferometer, such as that indicated in Fig. 1. A change in the relative phase of the light in the two arms of the interferometer may be detected by conversion into a change in power at the photodetector, via interference at the beamsplitter. If E_1 and E_2 are the emerging rms field amplitudes, with a phase difference ϕ, from the arms of the interferometer, then the power I_d incident on the photodetector is

$$I_d = E_1^2 + E_2^2 + 2E_1E_2\cos\phi \ . \quad (1)$$

If we are looking for a small time-dependent signal $\delta\phi = s(t)$, it is sensible to modulate the phase difference so that

$$\phi(t) = \pi + m(t) \ , \quad (2)$$

where $m(t)$ is an arbitrary function. The modulation could, of course, just be a dc offset. However, $m(t)$ is usually chosen to be a periodic function at some high fre-

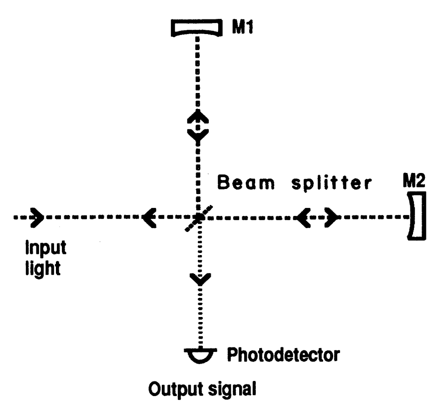

FIG. 1. The optical layout of a simple Michelson interferometer.

quency: this has the great advantage that it transfers the signal to around this frequency (where the power fluctuations of the light source may be quantum limited). This upconversion can be seen by observing the form of the power change δI_d corresponding to a signal $s(t)$:

$$\delta I_d \approx 2E_1E_2 m(t)s(t) , \tag{3}$$

where small modulation depth about the bottom of a fringe $[\sin m(t) \approx m(t)]$ has been assumed. The signal amplitude modulates the carrier formed by $m(t)$. Note that the emerging light power varies as $m^2(t)$.

The signal must now be extracted from the intensity change δI_d. The information needs to be transferred from around the modulation frequency back down to the signal frequency. This is done by coherent detection: the waveform δI_d is demodulated by multiplication with a function $d(t)$, giving a signal voltage

$$V_s(t) \approx 2E_1E_2 m(t)d(t)s(t) . \tag{4}$$

This result may also be expressed in the frequency domain:

$$\tilde{V}_s(\omega) \propto \tilde{q}(\omega) * \tilde{s}(\omega) , \tag{5}$$

where $\tilde{q}(\omega)$ is the Fourier transform of $q(t)=d(t)m(t)$ and $*$ indicates convolution. In other words, the signal that we would see on a spectrum analyzer is the convolution of the modulation-demodulation function $\tilde{q}(\omega)$ with the signal spectrum $\tilde{s}(\omega)$. For example, if $\tilde{q}(\omega)$ is a δ function then the original signal spectrum is reproduced. But we can see already an interesting feature—the *only* case in which signal *only* appears at its original frequency is when $\tilde{q}(\omega)$ is a δ function $\delta(0)$, i.e., when $d(t)m(t)$ is time independent. This is easily seen if we imagine the action of convolution being the sliding of two waveforms past each other, the convolution at each position being proportional to the overlap at that position. Any $\tilde{q}$ at a frequency ω' other than zero will shift the signal by ω'. Since a signal being diverted to spurious frequencies must, in some sense, be a waste, it would appear that all situations in which $d(t)m(t)$ is time dependent will give poorer performance than the ideal situation. We shall see that this is indeed the case [although the general form $d(t)\sin m(t)$ must be kept when the modulation depth is high—see Sec. IV]. However, for full understanding it is also necessary to consider the noise in the interferometer.

III. NOISE IN A MICHELSON INTERFEROMETER

From now on let us assume that our interferometer is illuminated by a laser beam which is in a perfect coherent state. This means that the fluctuations in amplitude and phase are limited by purely quantum, rather than technical, noise. One way of modeling this situation is to imagine that the laser beam "itself" is perfectly stable and noise-free, any noise arising from this field being superposed by vacuum fluctuations [9,10]. Fluctuations in phase or photon number may be regarded as being due to the interference of the vacuum fluctuations with the laser field. While this model should not be pushed too far, it does have some nice features: it is a great aid to physical intuition; it gives some results (e.g., for single sideband or sine-wave modulation) very easily; and it allows the effect of squeezing the vacuum at different frequencies and at different places to be seen easily.

As an example of noise generation in this picture, let us consider the fluctuations in power of a monochromatic laser beam. This could be light emerging from an ideal laser, or it could be out of an interferometer with a static offset from a dark fringe. Photon number fluctuations at angular frequency ω are produced by the beating of the laser light at ω_L with noise sidebands at $\omega_L \pm \omega$ (see Fig. 2). Note that we are treating the vacuum fluctuations as being essentially classical noise. This is certainly not strictly correct. However, the only place this ever makes any difference is if we try to observe the beating of the vacuum fluctuations with themselves—the fact that $\hat{a}_k|0\rangle=0$ guarantees that this cannot be seen. With the vacuum fluctuation sidebands having rms amplitudes E_{v+}, E_{v-} and the laser beam being of rms amplitude E_0 (average power I_0), the power is

$$I=(E_0+E_{v+}+E_{v-})(E_0^*+E_{v+}^*+E_{v-}^*) , \tag{6}$$

with $*$ indicating complex configuration. For an unsqueezed vacuum, the vacuum fluctuations at the two frequencies will be of the same magnitude [10] but uncorrelated:

$$E_{v+}^2=E_{v-}^2=\tfrac{1}{4}\hbar\omega_L . \tag{7}$$

The noise spectral density is therefore

$$\delta\tilde{I}^2=8I_0E_v^2=2I_0\hbar\omega_L . \tag{8}$$

This is a standard shot-noise formula. Our picture of noise generation as the interference of "carrier" with noise sidebands makes it clear why the rms shot-noise level scales as the *amplitude* of the light.

One thing to note about the previous argument (imagining that the light is at the output of an interferometer) is that the vacuum fluctuations which produce the noise couple into a measurement in exactly the same way as does a signal: the relevant sidebands have the same fre-

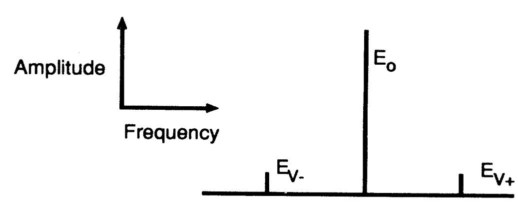

FIG. 2. An indication of the spectral composition of the field associated with a propagating laser beam. For any frequency ω away from the laser light, the effect of vacuum fluctuations is to produce uncorrelated noise sidebands, E_{v+}, E_{v-}. All of the quantum noise on the light may be regarded as being the result of the interference of these sidebands with the carrier at ω_L.

quencies and phases. There is no extra noise. So a modulation $m(t)$ which is simply a small offset from dark fringe should give the optimum signal-to-(quantum) noise ratio. Note that $m(t)d(t)$ in this case is time independent. For a monochromatic signal $s(\omega)$ and an incident laser power I, this optimum signal-to-noise ratio $(S/N)_{\rm opt}$ is given by

$$(S/N)_{\rm opt}^2=\frac{Is^2(\omega)}{2\hbar\omega_L\Delta f} , \qquad (9)$$

where Δf is the measurement bandwidth.

As Caves [9] has pointed out, this signal-to-noise may be improved if the vacuum is "squeezed," so that the noise in one phase is reduced at the expense of increasing that in the quadrature phase, the product satisfying Heisenberg's uncertainty relation. The vacuum fluctuations at *each* frequency have a magnitude that is greater in one phase than the other. Since orthogonal phases of the vacuum fluctuations produce orthogonal phases of noise (e.g., phase and amplitude) on the detected light, reduction of the appropriate phase of the vacuum fluctuations may lead to a decrease in the detected noise. Note also that the fluctuations E_{v+} and E_{v-} in the two modes that contribute to the detected noise are correlated; the squeezed vacuum is really a two-mode quantum state [11].

The place at which the most important vacuum fluctuations are generated may not be immediately obvious. Any fluctuations entering the interferometer along with the laser beam cannot exit the photodetector as long as the interferometer is exactly on a dark fringe, for they must be reflected back toward the laser in the same way as the laser light. Such fluctuations, therefore, cannot contribute to the noise in an ideal interferometer.

It might be thought that the most important source of vacuum fluctuations would be the two arms of the interferometer. But the production of fluctuations with random phase requires the presence of dissipation [12]. Just as in the classical dissipation-fluctuation theorem, the coupling-in of random noise from other modes occurs at the same rate as that of the reverse process, damping of the mode of interest. If the quality of the optics is high, and the number of reflections is not too great, so that little power is absorbed, then fluctuations generated within the arms of the interferometer will not contribute much noise. Another way of expressing this is to say that the noise generated within the arms of the interferometer has a correlation time τ_c which is comparable with the maximum light storage time τ_s of the inteferometer,

$$\tau_c \approx l/cA^2 , \qquad (10)$$

where l is the length of the interferometer and A^2 is the loss coefficient of the mirrors. If the signal is stored in the interferometer for a time τ_s', the effective noise power of the internally generated vacuum fluctuations will be

$$E_v^2 \approx \tfrac{1}{4}\hbar\omega_L(1-e^{-\tau_s'/\tau_c}) , \qquad (11)$$

Equivalently, the squeeze factor of light injected into the interferometer will decay as $e^{-\tau_s'/\tau_c}$. So, as long as the signal is stored for a time which is short compared with the correlation time, the phase uncertainty resulting from internal fluctuations will be small. For laser-interferometric gravitational-wave detectors, this will be true as long as bandwidth is not narrowed too much (see Ref. [13] for a discussion of the relation between bandwidth and signal storage time).

The most important vacuum fluctuations, as Caves [9] first pointed out, are usually those entering the interferometer from the *exit* port (but see also Sec. IV). These fluctuations can be imagined to originate at the dissipation associated with the photodetector. They then bounce off the effective mirror formed by the interferometer, return to the photodetector, and interfere with light emerging from the interferometer to produce noise. Note that this latter picture avoids worries [9,14] about the phase shift on reflection at the beamsplitter. It is clear that there will be one phase of the fluctuations (relative to the laser light) which will be detected and will produce noise. While noise in the quadrature phase results in differential radiation pressure fluctuations inside the interferometer, it does not directly produce any first order power changes at the photodetector.

Now let us consider the general case of a Michelson interferometer with arbitrary internal modulation $m(t)$ and demodulation $d(t)$ waveforms. We saw that the signal in this case is given by Eqs. (4) and (5). Since we are modeling the noise as a signal with uncorrelated frequency components, we can rewrite (4) to give the noise voltage

$$\langle V_n^2(t)\rangle = 2I_0\hbar\omega_L\langle m^2(t)d^2(t)\rangle , \qquad (12)$$

where $\langle\ \rangle$ represents an average over the modulation period. Using the convolution theorem, we can write the equivalent expression in the frequency domain:

$$\tilde V_n^2(\omega) \propto [\tilde E_v(\omega) * \tilde q(\omega)]^2 . \qquad (13)$$

We saw earlier, from Eq. (5), that it is only when $q(t)=m(t)d(t)$ is time independent [so that $\tilde q(\omega)$ is a δ function] that the signal occurring at ω_s appears only at ω_s and not other frequencies. Equation (13) embodies the obverse message: it is only when $\tilde q(\omega)$ is a δ function that the only noise appearing at frequency ω_s at the output is that originating at the same frequency ω_s as the signal. If $m(t)d(t)$ is time dependent, then noise from other frequencies will be mixed in. (The importance of these two sides of the same coin is a recurrent theme throughout this paper.)

If the noise at different frequencies is white and uncorrelated (i.e., the vacuum is unsqueezed), so that noise contributions must be added in quadrature, then

$$[\tilde E_v(\omega) * \tilde q(\omega)]^2 = \hat E_v^2\int_0^\infty \tilde q^2(\omega)d\omega . \qquad (14)$$

We saw from Eq. (5) that the signal size is determined by the magnitude $|q(0)|$ of the component of q at zero frequency. Note that $|q(0)|$ is dimensionless. For a monochromatic signal, this means that the factor F by which the signal-to-noise ratio is reduced from its optimum value [Eq. (9)] is given by

$$F^2=\frac{|q(0)|^2}{\int_0^\infty \tilde{q}^{\,2}(\omega)d\omega} . \tag{15}$$

This relation will be a little clear, perhaps, if we take a concrete example: that of low amplitude internal sine-wave modulation at angular frequency ω_m. Consider the case of equal modulation and demodulation waveforms. The sideband structure of $m(t)$ is indicated in Fig. 3(a) (but note that phase relationships are not explicitly shown). Since the modulation amplitude is small, only the first-order sidebands are significant. The sideband amplitude has been set equal to 1 (in arbitrary units). Using the convolution theorem, it is easy to see that the corresponding structure of $\tilde{q}(\omega)$ is just that indicated in Fig. 3(b). [Remember that the value of the convolution is equal to the overlap (product) of the two functions as they slide past each other in frequency space.] It can be seen that $|q(0)|^2$ is equal to 4 in our units. Note the presence of sidebands at $\pm\omega_m$ which can mix down additional noise from these frequencies: the integral in the denominator of Eq. (15) has the value

$$\int_0^\infty \tilde{q}^{\,2}(\omega)d\omega=(1+4+1)=6 . \tag{16}$$

The integral is made particularly easy by the fact that $\tilde{q}(\omega)$ is composed of δ functions. This noise power is a factor of $\frac{3}{2}$ higher than if no additional noise had been mixed in. In other words, light which leaks out of a Michelson interferometer due to the internal sine-wave modulation, and which is demodulated with the same sine wave, has intensity fluctuations which are a factor of $\frac{3}{2}$ greater (in power) than the same power of light which leaks out due to a static offset from the dark fringe. The signal size $|q(0)|^2$ in the two cases is, however, exactly the same. So mixing in extra noise in sine-wave modulation and demodulation produces a degradation of the possible signal-to-noise ratio by a factor $F=\sqrt{\frac{2}{3}}$.

It is interesting to note what would happen if the demodulation waveform were in the quadrature phase. In this case, the sideband structure of $\tilde{q}(\omega)$ would be that shown in Fig. 3(c). There is now no contribution from the noise (or signal) from the zero-frequency peak, the only noise now coming from $\pm 2\omega_m$. The fluctuations which produce this noise are in the orthogonal phase to those detected in the demodulation phase that is most sensitive to the signal, but have the same magnitude for normal vacuum. The noise power in this quadrature is evidently only $\frac{1}{3}$ that in the signal quadrature, or $\frac{1}{2}$ that of unmodulated light. It might be thought surprising that such strongly phase-dependent noise could be produced from phase-independent vacuum fluctuations. We shall describe an experimental test of this result in Sec. V.

The above results can also be obtained without explicitly involving convolutions by considering directly the beating of the sideband field produced by the modulation with the field representing the vacuum fluctuations. In some cases this can be very simple. For example, if the modulation is single sideband, it can be seen that noise at $\sim 2\omega_m$ is detected in the same way as noise (and signal) at the signal frequency, so the signal-to-noise ratio must be a factor of $\sqrt{2}$ worse than optimum. It is also interesting to observe what happens when some of the third harmonic is added to sine-wave phase modulation. We saw that the excess noise in sine-wave modulation and demodulation is due to the mixing in of fluctuations separated from the light frequency by twice the modulation frequency. If we add some modulation which produce sidebands at $\pm 3\omega_m$, with the correct phase, then we can tend to cancel out the noise contribution from $\pm 2\omega_m$ at the expense of introducing noise from $\pm 4\omega_m$. If the third harmonic has an amplitude α compared with the first, the total noise power $\mathcal{N}$ [with the same units as the previous result (16)] is

$$\begin{aligned}\mathcal{N}&=[(1+1)^2+(1-\alpha)^2+(1-\alpha)^2+\alpha^2+\alpha^2]E_v^2\\&=[6-4\alpha(1-\alpha)]E_v^2 .\end{aligned} \tag{17}$$

So the fact that the amplitudes of the contributions from $\pm 2\omega_m$ must be added before squaring (which stems from the coherence of the modulation sidebands) enables the

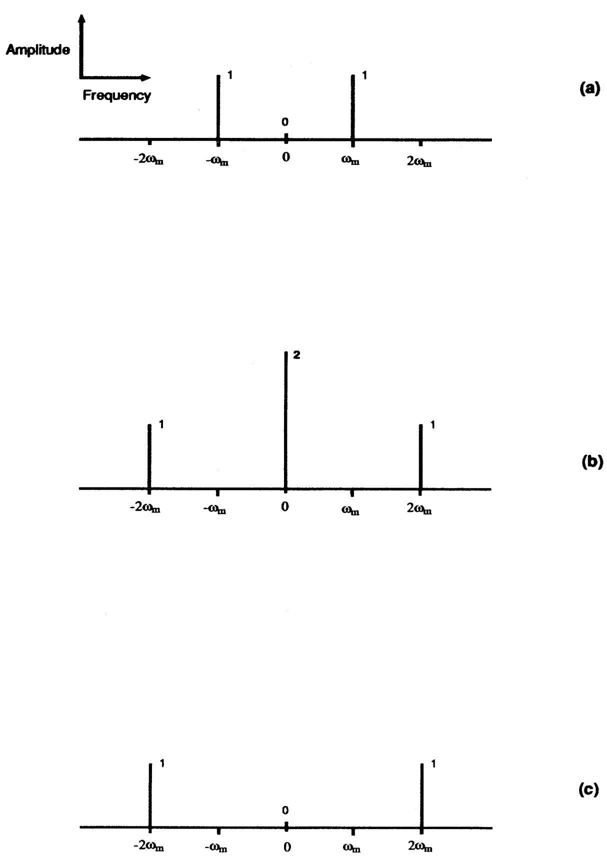

FIG. 3. The sideband structure of waveforms associated with sine-wave modulation and demodulation. (a) the spectral composition $\tilde{m}(\omega)$ with small-amplitude sine-wave phase modulation, each line being a δ function; (b) the structure of the modulation-demodulation function $\tilde{q}(\omega)=\tilde{m}(\omega)*\tilde{d}(\omega)$; (c) the structure of $\tilde{q}(\omega)$, but now with the demodulation waveform having the phase which minimizes the coupling of the signal.

overall noise to be reduced. With the optimal choice of $\alpha=\frac{1}{2}$, the addition of the third harmonic decreases the noise power of $6E_v^2$ with pure sin-wave modulation and demodulation to $5E_v^2$. This reduction in noise power by the addition of extra light power may be somewhat counter intuitive, but it is easy to understand with the picture of the noise generation process that we are using.

Similarly, the addition of some third harmonic to the demodulation waveform will help even if the modulation is a pure sine wave. This may be understood by realizing that fluctuations at $\omega_L \pm 2\omega_m$ beat with the modulation sidebands to produce noise components at both ω_m and $3\omega_m$, which can partially cancel in the demodulated signal if the demodulation waveform also contains both ω_m and $3\omega_m$.

One consequence of the picture that we have been describing is that if squeezed vacuum is to be used to reduce the equivalent phase noise of the interferometer, together with sine-wave modulation and demodulation, then it is easily seen that squeezing is important both at the laser frequency and at $\pm 2\omega_m$. It is clear that perfect squeezing at only $\omega_L \pm 2\omega_m$ would allow the "classical" ideal signal-to-noise ratio, $F=1$. However, the fluctuations at the laser frequency are the most important: they contribute $\frac{2}{3}$ of the noise power. Furthermore, the presence of any light at the original frequency, perhaps resulting from poor contrast in the interferometer, will mix in noise from $\pm\omega_m$. This will reduce the signal-to-noise ratio from its ideal value, irrespective of the modulation-demodulation combination. Squeezing at $\pm\omega_m$ will help. However, if the poor contrast is a result of aberrations, the emerging light will be in higher-order spatial modes, so the squeezing at $\pm\omega_m$ will also have to be in these high-order modes—a tough requirement. This reemphasizes the importance of ensuring that the fringe contrast is very good [14,15].

The frequency domain approach gets complicated when the modulation or demodulation waveforms produce rich sideband structure. In these situations, it is often simpler to calculate in the time domain. The expression for the signal-to-noise ratio reduction factor F is found by comparing the time average of the signal (4) with the noise (12):

$$F^2=\frac{\langle m(t)d(t)\rangle^2}{\langle m^2(t)d^2(t)\rangle} . \tag{18}$$

This relation allows the signal-to-noise ratio to be calculated for any combination of modulation and demodulation waveforms. In particular, it is clear that not only does square-wave modulation and demodulation give the optimum performance, so also does any combination with a time-dependent $m(t)d(t)$. This latter characteristic, however, is only true if (as we have assumed) light leaking out from the interferometer due to bad fringe contrast is negligible.

The expression (18) has previously been obtained by Schnupp [7] and Niebauer *et al.* [8], who emphasized that it is the nonstationary character of the noise produced by a time-dependent light power that leads to the remarkable fact that the demodulation waveform should not be the same as that of the modulation, but its inverse. We would like to stress, however, that the requirement for $d(t)m(t)$ [more accurately, $d(t)\sin m(t)$] to be time independent for optimum signal-to-noise ratio is *not* confined to situations in which the noise is nonstationary. Instead, it is a general requirement that reflects the fact that the ideal signal-to-noise ratio can only be obtained if noise is only produced by light which also produces signal. For an internally modulated interferometer with a general modulation waveform, this is the same as saying that the detected light power is time varying, so the noise is indeed nonstationary. It is also true, however, that the symmetry between $m(t)$ and $d(t)$, which leads to the conclusion that given an *arbitrary* modulation shape $m(t)$ the best $d(t)$ is $1/m(t)$, only arises when the noise is nonstationary. These points are further exemplified in the case of the stationary noise that occurs in external modulation.

IV. SIGNAL AND NOISE IN EXTERNAL MODULATION

A Michelson interferometer employing external modulation is shown in Fig. 4. The local oscillator field which converts the signal into a first-order intensity change is now not produced by differential internal phase modulation. Instead, it consists of light split off from the light in the interferometer (or from the laser) which is then interfered with the signal field at an external beamsplitter. This external reference beam is phase modulated (or frequency shifted) in order to mix the signal up to a frequency at which the laser light is quantum limited. This idea was first suggested in the context of optical interferometers by Drever [16], the motivation being to remove any phase modulators from the parts of the interferometer at which the power is very high (since they suffer from nonlinearities), or at which the losses and distortions limit the power buildup in recycled interferometers. The efficacy of external modulation has been demonstrated experimentally in a variety of situations [17,6,18]. It now forms an integral part of current proposals for laser-

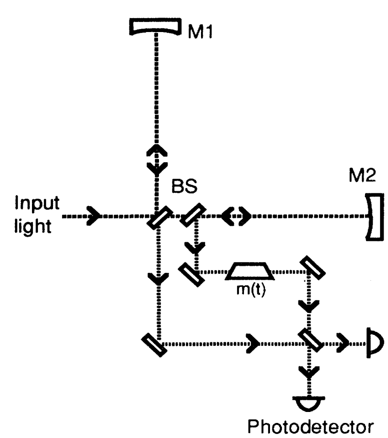

FIG. 4. A possible optical arrangement for an externally modulated Michelson interferometer.

interferometric gravitational-wave detectors [19]. This latter fact makes it especially important to know how the signal-to-noise ratio depends on the modulation and demodulation waveforms, both in shape and depth.

Let us first consider how a signal (differential phase change) $s(t)$ in the interferometer is detected in this scheme. The field leaving the interferometer and traveling towards the photodetectors is

$$E_{\rm out}=E_1e^{is(t)/2}-E_2e^{-is(t)/2}\approx iE_1s(t)\ , \tag{19}$$

where we have assumed emerging fields both equal to E_1, and a small signal. This field is then interfered, at the external beamsplitter, with the reference field. If we impose pure phase modulation $m(t)$ on this rms local oscillator field E_L, then the field incident on each of the photodetectors is

$$E_d\approx\frac{1}{\sqrt{2}}[E_Le^{im(t)}+iE_1s(t)]\ . \tag{20}$$

Note that we have chosen the phase of the local oscillator so that we maximize our later signal with simple phasing, $d(t)=m(t)$. The power at our photodetector is then

$$I_d\approx\tfrac{1}{2}[E_Le^{im(t)}+iE_1s(t)][E_Le^{-im(t)}-iE_1s(t)]\ , \tag{21}$$

which is just

$$I_d\approx\tfrac{1}{2}E_L^2+E_LE_1s(t)\sin m(t)\ , \tag{22}$$

to first order in s. If we take the component at our signal frequency and sum the outputs of the two photodetectors, we obtain the analog of Eq. (3), the intensity change due to a signal:

$$\delta I_d\approx 2E_LE_1s(t)\sin m(t)\ . \tag{23}$$

When demodulated with $d(t)$, the resulting signal voltage is

$$\langle V_s(t)\rangle\approx 2E_LE_1s(t)\langle d(t)\sin m(t)\rangle\ . \tag{24}$$

Note that $\sin m(t)$ cannot be approximated as $m(t)$, since, as we shall see, optimal modulation depths are large. With this difference, the signal is of the same form as with internal modulation.

Now the light producing the noise in external modulation has a constant power (in contrast to internal modulation). The noise is, therefore, stationary. The magnitude of the (summed) noise power is just

$$\delta\tilde{I}_d^2(\omega)=2E_L^2\hbar\omega\ . \tag{25}$$

This result can, of course, be derived explicitly using our model. Note that in the optical arrangement indicated in Fig. 4, there are *two* places from which vacuum fluctuations can reach the final photodiodes: the exit port of the interferometer and the beamsplitter used to sample the circulating light in order to produce the external reference beam. The vacuum fluctuations at each photodiode, with a sampling beamsplitter of transmission, reflection, and loss coefficients, $T_{\rm SB}^2$, $R_{\rm SB}^2$ and $A_{\rm SB}^2$, respectively ($T_{\rm SB}^2+R_{\rm Sb}^2+A_{\rm SB}^2=1$), are of magnitude

$$E_{v+}^2=E_{v-}^2=\tfrac{1}{2}(1+T_{\rm SB}^2+R_{\rm SB}^2+A_{\rm SB}^2)\tfrac{1}{4}\hbar\omega_L\ . \tag{26}$$

Since the sampling beamsplitter has a low reflectivity, almost all of the fluctuations that it couples in are those transmitted through it; but a small fraction of the fluctuations on the laser beam are also reflected onto the photodetectors. [Even more complicated, if the sampling beamsplitter is the normally unused (antireflection coated) side of the main beamsplitter, fluctuations can be coupled in from *either* side of the beamsplitter.] The multiplicity of sources for vacuum fluctuations in an externally modulated interferometer will significantly complicate the implementation of squeezing. Nevertheless, there may be some situations in which there are compensatory advantages. In particular, it is sometimes appropriate to choose a large number of reflections in order to maximally enhance the signal or to filter the output light. In this case, the losses may be so high that any squeezing is destroyed. It is then sensible to separate the signal light (which benefits from the large number of bounces) from the reference beam (which does not). This makes it possible to squeeze the reference beam, independently of the losses associated with a round trip of the interferometer. This can be seen by looking at the contributions to the noise in Eq. (26). Half of the noise power enters via the sampling beamsplitter, so (as long as $1-T_{\rm SB}^2\ll 1$) perfect squeezing of the vacuum on its "unused" side would reduce the detected noise power by a factor of nearly 2. Yet this gain in signal-to-noise ratio is independent of the losses or bandwidth of the interferometer itself.

This reduction of noise power in a lossy interferometer might be thought surprising enough, but there are some situations in which a further improvement may be possible. Long interferometers, such as the proposed gravitational-wave detectors, can have signal bandwidths much smaller than the signal frequency. These narrow-band interferometers, whether they use resonant recycling [2,4], detuned recycling [3,4], or dual recycling [4], work by ensuring that a *single* signal sideband is perfectly resonant within the optical system. The other sideband is usually off resonance and is effectively not detected. While this situation is not perfect, it still gives very good signal-to-noise ratio. Furthermore, the lack of perfection may allow some additional gain from squeezing. The detected power fluctuations (the noise) are produced by the beating of the reference beam with vacuum fluctuations at the frequencies of *both* signal sidebands. So the degree of phase dependence of the noise at both sideband frequencies is important. When squeezed light is injected into a narrow-band interferometer, the component at the frequency of the resonant sideband is coupled efficiently into the interferometer, bounces many times, and is dissipated. But the component of the squeezed light at the frequency of the nonresonant sideband does not couple into the interferometer; it is reflected back with high efficiency. One might naively argue that this means that the component of the squeezed light at the frequency of the signal sideband is destroyed, while that at the other sideband is hardly affected. This result, however, should be regarded with great caution, for it ignores the correlations between the two modes present in a squeezed state. A proper quantum-mechanical analysis is really required here. Nevertheless, the squeezed light should see losses significantly less than unity, so the naive result may well be a good approximation. This would mean that, with perfect initial squeezing, the detected noise power could be halved; or, including the gain resulting from squeezing the reference beam, the effective power of the fluctuations could be as low as $\frac{1}{4}$ of the unsqueezed value.

It might be thought that it would be a good idea to stop the coupling of vacuum fluctuations transmitted through the sampling beamsplitter by placing a high reflectivity mirror on its normally unused side. However, this only helps a little. It does stop transmitted fluctuations entering (and, in principle, allows more efficient use of the light), but it also reflects back fluctuations generated at the photodetectors. So the total noise would be unchanged. In this case squeezed light would have to be injected in two places rather than three.

Our earlier comments about the required frequency spectrum for the squeezing and the effect of poor contrast are also valid for an externally modulated interferometer.

Returning now to the case of unsqueezed vacuum, we can write the noise voltage in the externally modulated system as

$$\langle V_n^2(t)\rangle = 2E_L^2\hbar\omega\langle d^2(t)\rangle \ , \tag{27}$$

which means that the signal-to-noise reduction factor in external modulation is

$$F^2 = \frac{\langle d(t)\sin m(t)\rangle^2}{\langle d^2(t)\rangle} \ . \tag{28}$$

In contrast to the corresponding expression (18) for internal modulation, this does not embody a symmetry between $m(t)$ and $d(t)$. This is largely a result of the noise being independent of modulation. It can also be seen that optimization of the signal-to-noise ratio requires a high degree of modulation. For a static phase offset, it is easy to see that the signal is maximum if $m=\pi/2$ rad, so that the signal and local oscillator fields have the same phase. If a periodic modulation is used, the same maximum signal is obtained if the modulation is a square wave, chopping between $\pm\pi/2$ rads. That such a square-wave modulation, with square-wave demodulation, is indeed optimum is confirmed explicitly by evaluation of Eq. (28), giving $F=1$. Note that the ideal signal-to-noise ratio is obtained when $d(t)\sin m(t)$ is time independent: this is a requirement in *all* situations. However, it is clear that the time independence of $d(t)\sin m(t)$ is a necessary but by no means sufficient condition for obtaining the ideal signal-to-noise ratio. Any modulation which spends a significant time at a phase other than $\pm\pi/2$ rads will not give ideal performance.

Signal-to-noise ratio reduction factors F for several cases of different modulation and demodulation waveforms of interest are given in Table I. Some of the these results have previously been obtained by Schilling [5].

The traditional modulation-demodulation combination has been sine wave–sine wave. This is significantly easier to apply in practice than, say, square wave–square wave, for two reasons. First, application of only a single frequency to the phase modulator allows a tuned circuit to be used to increase the voltage across the modulator, facilitating the achievement of the required modulation depth. Second, detection of only a narrow frequency range around the modulation frequency allows the use of tuned detectors, with corresponding better noise performance. The examples shown in Table I, in which some third or fifth harmonic is added to the modulation waveform, are attempts at approximating a square wave without requiring infinite bandwidth. Note that adding

TABLE I. Maximum signal-to-noise ratio reduction factors F_{max} for different modulation $m(t)$ and demodulation $d(t)$ waveforms with external modulation. The corresponding optimum modulation depth Φ_{opt} is also shown.

$m(t)$	$d(t)$	F_{max}	Φ_{opt} (rad)
$\sin\omega t$	$\sin\omega t$	0.82	1.8
	$\sin(\sin\omega t)$	0.85	1.92
	Square wave	0.79	2
Square wave	Square wave	1.0	$\pi/2$
	$\sin\omega t$	0.9	
$\Phi_1\sin\omega t + \Phi_3\sin 3\omega t$	$\sin\omega t$	0.888	$\Phi_1=1.9$, $\Phi_3=0.45$
	$\sin m(t)$	0.9	$\Phi_1=1.8$, $\Phi_3=0.8$
$\Phi_1\sin\omega t + \Phi_3\sin 3\omega t + \Phi_5\sin 5\omega t$	$\sin\omega t$	0.893	$\Phi_1=1.9$, $\Phi_3=0.5$, $\Phi_5=0.2$
	$\sin m(t)$	0.946	$\Phi_1=1.9$, $\Phi_3=0.5$, $\Phi_5=0.36$
Single sideband	$\sin\omega t$	$1/\sqrt{2}$	

the higher harmonics improves the attainable signal-to-noise ratio even with sine-wave demodulation. The mechanism for this is not reduction of the detected noise, as it is for internal modulation, but increase of the signal. The maximum signal is obtained when the first-order sidebands J_1 are maximized. An increase in J_1 by the addition of higher harmonics is only possible with large modulation depths: it relies on $\sin[\Phi_1\sin\omega_m t+\Phi_3\sin3\omega_m t]$ having a larger component of $\sin\omega_m t$ than does $\sin(\Phi_1'\sin\omega_m t)$. However, the reason that increase of the signal is important is that there is noise being produced by light which does not contribute to the signal. By increasing J_1, for example, we are reducing the fraction of the light which generates noise but not signal. Once more we see the opposite sides of the same coin: maximization of the signal reduces the contribution to the noise from fluctuations at frequencies at which there is no signal.

We saw earlier [Eq. (28)] that the signal-to-noise ratio reduction factor F in external modulation is given by

$$F^2=\frac{\left[\frac{1}{2\pi}\int_0^{2\pi}d(\phi)\sin m(\phi)d\phi\right]^2}{\frac{1}{2\pi}\int_0^{2\pi}d^2(\phi)d\phi}\equiv\frac{u^2}{v} . \tag{29}$$

Here we have used $\phi=\omega_m t$ and written out the averages explicitly. We would like to know, given a modulation waveform $m(t)$, what is the demodulation waveform $d(t)$ that gives the highest signal-to-noise ratio, the maximum value of F. If we differentiate Eq. (29) with respect to $d(t)$, we find the condition for F to be a maximum:

$$u\frac{\partial v}{\partial d}-2v\frac{\partial u}{\partial d}=0 . \tag{30}$$

We also have

$$\frac{\partial v}{\partial d}=\frac{1}{2\pi}\int_0^{2\pi}2d(\phi)d\phi \tag{31}$$

and

$$\frac{\partial u}{\partial d}=\frac{1}{2\pi}\int_0^{2\pi}\left[\sin m(\phi)+d(\phi)\frac{\partial\sin m(\phi)}{\partial d}\right]d\phi . \tag{32}$$

Using these relations it is simple to confirm that our guess that

$$d_{\text{opt}}(\phi)=\pm\sin m(\phi) , \tag{33}$$

which gives the demodulation the same sideband structure as the modulation, is a solution of Eq. (30), and is, therefore, the optimum demodulation waveform. Thus, once again we see that although we want $d(t)\sin m(t)$ to be time independent in order to obtain the ideal signal-to-noise ratio, it is *not* true that if we are given the modulation waveform $m(t)$ then the optimum demodulation waveform is $1/\sin m(t)$.

One of the demodulation waveforms used in Table I is $d(t)=\sin m(t)$. However, in contrast to the case of internal modulation, this optimum demodulation waveform is now not guaranteed to give the ideal signal-to-noise ratio, $F=1$. For example, sine wave modulation gives $F=0.85$ with $d(t)=\sin m(t)$ [or $F=0.822$ with $d(t)=m(t)$]. Note also that square-wave modulation with sine-wave demodulation can give a greater signal-to-noise ratio, $F=0.9$. This emphasizes the importance of getting the *modulation* waveform right.

We have seen that failure to attain the ideal signal–to–quantum-noise ratio using a particular modulation scheme (whether internal or external) can always be ascribed to the mixing in of quantum noise from frequencies at which there is no signal. It is probably better to restate this and say that the noise originates in *modes* in which there is no signal. This generalized statement then includes even the case of the demodulation waveform having the incorrect phase.

V. MEASUREMENT OF PHASE-DEPENDENT NOISE

Our model of how noise is generated in an interferometer has led us to make some rather surprising predictions concerning modulation and signal-to-noise ratio in interferometers. We saw earlier that the noise power of light emerging from an ideal Michelson interferometer, with sine-wave modulation and demodulation, was predicted to vary between $\frac{3}{2}$ times the standard shot-noise value (for the signal phase) to only $\frac{1}{2}$ of the standard level (for the quadrature phase). We decided to test this experimentally.

The optical system was adapted from that used for dual recycling and external modulation experiments [18]. The layout of the system used is shown in Fig. 5. Note that an electro-optic phase modulator was inserted into one of the arms of the interferometer. The 10-MHz modulation was used both to keep the Michelson interferometer on a dark fringe and to produce modulated

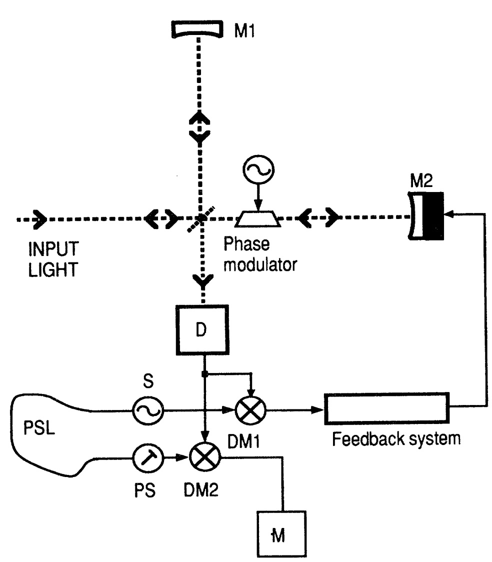

FIG. 5. The experimental arrangement for the observation of phase-dependent power fluctuations. *D*, photodiode; *S*, modulation source; DM, demodulator; PS, phase shifter; PSL, cable phase shifter; *M*, amplifier and monitor.

light for noise measurements. The phase modulator limited the effective contrast of the interferometer to a give a minimum output power of 0.040±0.002 of the input power. (The typical input power was a few mW and the output power a few hundred μW.) The modulation index was chosen to produce an output power of three times this at an interference minimum (to 2% accuracy).

The signal from the photodiode was band pass filtered by the tuned photodiode amplifier. The tuned circuit here had a Q of ~ 25, so there could be no contribution from the demodulation of any frequencies apart from those close to the modulation frequency. This means that the shape of the waveform (provided it has its fundamental component at the modulation frequency) used for the demodulation is not important as any shape will have the same effect as a sine wave.

We wished to compare the photon shot-noise level, in the case where the demodulation was in phase with the modulation, to that with the demodulation in the quadrature phase. So that we could be confident of the results, considerable care was taken to ensure that the gain of the rf electronics did not change between these two measurements. The phase of the demodulation waveform was varied either by altering the length of one of the cables or by using an active phase shifter. Various different combinations of the two methods were used so that we could be confident that the gain of the signal channel was not changing: the observed noise levels were identical to within 0.1 dB. The phase of the demodulating waveform was measured by monitoring the size of the signal due to a 6-kHz arm length modulation at the output. The minimum of this signal corresponded to the quadrature phase and the maximum to the in-phase case. The phase was known to $\sim 3°$.

The signal from the photodiode was measured on a rf spectrum analyzer (before the demodulation) and compared to noise due to torch light producing the same photocurrent. Apart from excess noise at low frequency the two results were the same within the accuracy of the instrument used (~ 0.5 dB). This measurement confirmed that the laser power was shot-noise limited at the measurement frequency.

The signal from the photodiode was demodulated to provide separate control and measurement signals. With a particular phasing, the demodulated (measurement) signal was amplified using a low-noise amplifier and spectrum analyzed in the region of 99 to 100 kHz. This frequency range was chosen since there was no sign of any noise in excess of photon shot noise. Several hundred averages were used to reduce the measurement uncertainty. The laser power supply was adjusted to maintain the photocurrent to within 2% of the initial value throughout the measurement period (~ 10 min). Measurements were also made with torch light producing the same photocurrent, and with unmodulated light from the laser (attenuated with a polarizing beamsplitter).

The phase of the demodulation was then changed and the above measurements were repeated. There was no discernible difference between the two results produced by the torch light or by the light directly from the laser (within 0.15 dB). The noise levels produced by the modulated light did, however, change. This can be seen in Fig. 6: the noise in the signal phase is, indeed, higher. The results shown must be corrected to allow for the presence of electronic noise in the measurement system. This was done by subtracting the noise power observed with no light reaching the photodiode. The results of this are shown in Table II. In order to calculate the expected change in noise levels attributable to the modulation effects it was necessary to allow for the noise power due to the one third of the detected power which was unmodulated. The predictions for this situation are shown in the table for comparison. Also shown are results from when the whole experiment was repeated with a different light power and with a 3-dB attenuator placed on the signal input to the mixer. This changed the signal level on the photodiode slightly. It can be seen that these results were very similar to those from the earlier experiment: the observations are in good agreement with the predictions.

So the demodulated photon noise is, indeed, higher for one phase than the other. Furthermore, the difference in noise level is in good agreement with the predictions. This gives us confidence that our model is a reasonable representation of the mechanism of noise generation in an interferometer.

VI. CONCLUSION

We have described a model of how quantum noise is generated in an interferometer. This way of looking at the situation enables considerable physical insight to be gained. It provides a unifying conceptual framework in which apparently very different phenomena can be seen to have common origins. Thus, the production of strong-

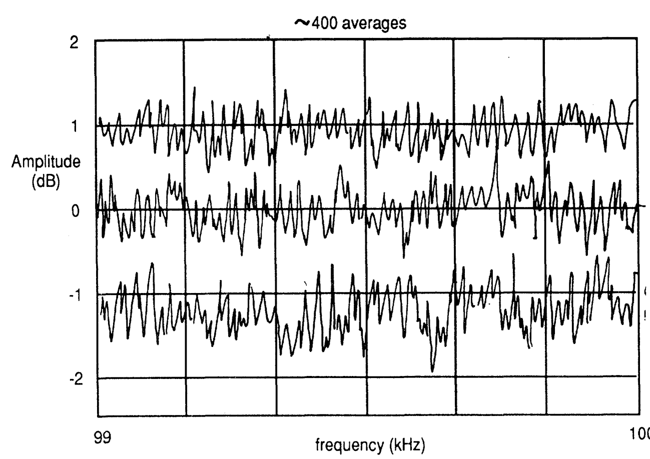

FIG. 6. The noise levels observed on the spectrum analyzer. The middle trace was produced by torch light (equivalent to unmodulated laser light). Modulated laser light with the demodulation waveform in the signal phase produced the upper trace, while changing the phase of $d(t)$ by 90° resulted in the lower trace.

TABLE II. A summary of the results of the experiment to investigate the effects of modulation on measurement noise. All of the results are normalized so that the noise produced by torch light is 0 dB. The first column of figures shows the noise level predicted with one-third of the light unmodulated. The second column shows the measured noise levels and in the final column the electronic noise power has been subtracted and the results renormalized to torch light. The two sets of results shown (1 and 2) were obtained for the two slightly different signal levels, as described in the text.

	Predicted noise (dB)	Measured noise (dB)	Corrected noise (dB)
In phase 1	1.25	1.1 ± 0.2	1.2 ± 0.2
Torchlight 1	0	0	0
Quadrature 1	-1.76	-1.4 ± 0.2	-1.6 ± 0.2
Electronic 1		-10 ± 1	
In phase 2	1.25	0.9 ± 0.2	1.2 ± 0.2
Torchlight 2	0	0	0
Quadrature 2	-1.76	-1.2 ± 0.2	-1.8 ± 0.2
Electronic 2		-5.1 ± 0.5	

ly phase-dependent white noise in demodulated laser light; the reduction in signal-to-noise ratio in various modulation schemes but with the possibility of recovery by adding other sidebands; the somewhat different characteristics of internal and external phase modulation; the effects and requirements of squeezing—all can be understood to be the consequence of the beating of the laser light with vacuum fluctuations. Furthermore, we have demonstrated experimentally that the phase-dependent noise does exist, as predicted.

Along the way we have analyzed the efficiency of different types of modulation, with particular emphasis on external modulation: square-wave modulation is best, but the signal-to-noise ratio is only degraded by a small factor in most other modulation schemes. Much of this degradation can be avoided by the addition of higher harmonics to the modulation waveform. We have seen that a necessary, but not sufficient, condition for obtaining the ideal signal-to-noise ratio in *all* situations is that the modulation $m(t)$ and demodulation $d(t)$ waveforms must be such that $d(t)\sin m(t)$ is time independent. If this condition is not met, noise will be mixed into the detection process from frequencies at which there is no signal, spoiling the signal-to-noise ratio. Furthermore, degradation of signal to (quantum) noise can *always* be traced to mixing in of noise from modes in which there is no signal.

We have seen that modulation schemes usually mix in quantum noise from several different optical modes. If squeezing the vacuum is to give its full potential benefit, then the squeezing must cover these modes. Square-wave modulation, for example, requires a greater squeezing bandwidth than does sine-wave modulation. The relative importance of different frequencies, of different spatial modes and of different places in which fluctuations can enter is made clear by our model of noise generation. Thus, we have argued that perfect squeezing of an externally modulated, optimally narrow-banded interferometer in which only a single sideband is resonant may reduce the detected quantum noise power by a factor as high as 4.

We hope that these results will help in the design of interferometer for sensitive measurements.

ACKNOWLEDGMENTS

We would like to acknowledge interesting conversations with Jim Hough, Euan Morrison, Gavin Newton, Tim Niebauer, Roland Schilling, Lise Schnupp, and Harry Ward. We are also grateful for the support of the University of Glasgow, the Science and Engineering Research Council and (for B.J.M.) the Royal Society.

[1] K. S. Thorne, in *300 Years of Gravitation,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
[2] R. W. P. Drever, in *Gravitational Radiation,* edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983).
[3] J.-Y. Vinet, B. J. Meers, C. N. Man, and A. Brillet, Phys. Rev. D **38**, 433 (1988).
[4] B. J. Meers, Phys. Rev. D **38**, 2317 (1988).
[5] R. Schilling (unpublished).
[6] C. N. Man, D. Shoemaker, M. Pham Tu, and D. Dewey, Phys. Lett. A **148**, 8 (1990).
[7] L. Schnupp (unpublished).
[8] T. M. Niebauer, R. Schilling, K. Danzmann, A. Rüdiger, and W. Winkler, Phys. Rev. A **43**, 5022 (1991).
[9] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[10] H.-A. Bachor and P. J. Manson, J. Mod. Opt. **37**, 1727 (1990).
[11] C. M. Caves and B. Schumaker, Phys. Rev A **31**, 3068 (1985).
[12] J. Gea-Banacloche, Phys. Rev. A **35**, 2518 (1987).
[13] B. J. Meers, Phys. Lett. A **142**, 465 (1989).
[14] J. Gea-Banacloche and G. Leuchs, J. Mod. Opt. **36**, 1277 (1989).
[15] B. J. Meers, and K. A. Strain Phys. Rev. D **43**, 3117 (1991).
[16] R. W. P. Drever (private communication).

[17] B. J. Meers (unpublished).
[18] K. A. Strain and B. J. Meers, Phys. Rev. Lett. **66**, 1391 (1991).
[19] J. Hough, B. J. Meers, G. P. Newton, N. A. Robertson, H. Ward, G. Leuchs, T. M. Niebauer, A. Rüdiger, R. Schilling, L. Schnupp, H. Walther, W. Winkler, B. F. Schutz, J. Ehlers, P. Kafka, G. Schäfer, M. W. Hamilton, I. Schütz, H. Welling, J. R. J. Bennet, I. F. Corbett, B. W. H. Edwards, R. J. S. Greenhalgh, and V. Kose, Max-Planck-Institut für Quantenoptik Report No. MPQ 147, Garching, Germany, 1989 (unpublished).