Controlled competitive dynamics in a photorefractive ring oscillator: "Winner-takes-all" and the "voting-paradox" dynamics

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We investigate a system through which a variety of competitive dynamics can be optically realized. The interaction between the dynamical variables, in this case the intensities of resonator modes, can be controlled and even programmed. Photorefractive media are used to establish the coupling between the optical modes. We illustrate the properties of such a system through two different instances. The first is a "winner-takes-all" system in which the mutual competition between the modes leads to multiple stable fixed points in which one mode oscillates while the other ones are suppressed. The second system employs circular coupling between the modes giving rise to a dynamically recalled sequence of modes, often referred to as "voting-paradox" dynamics.

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I. INTRODUCTION

Competitive interactions play a ubiquitous role in the nonlinear sciences. In biology, for example, species competition has been used to describe predator-prey population dynamics [1]. Qualitatively similar competitive interactions have been used to model the dynamics of systems as disparate as market economics [2], computer networks [3], and multimode lasers [4]. The practical interest in understanding complex dynamics is also varied, from weather prediction to artificial self-organizing neural networks [5]. One usually has little control over the attributes of a natural system, and it is for this reason that artificial systems have a great deal of pedagogical as well as practical value: By studying the spatiotemporal behavior of a multimode laser for example, we expect to learn much about analogous natural versions and perhaps discover more than we presently know about the universal character of such systems [6].

Even in artificial systems, however, one rarely has access to the detailed interactions that take place. In the case of a laser, for example, one can easily control the number of modes, the gain, and degree of feedback; these are global parameters. It is much more difficult to establish detailed control over the dynamical parameters. Yet we know that detailed properties such as the coupling among modes of the system can have dramatic bearing on its behavior. Here we investigate a photorefractive optical system for which we have direct access to the dynamical parameters, such as the degree of coupling among the modes, of a resonator. This control allows us to synthesize and observe a variety of dynamical behaviors in a single multimode optical resonator.

Many competitive systems, including the multimode laser, can be qualitatively described by a set of coupled differential equations, known as the Lotka-Volterra equation [1]:

$$\frac{d}{dt}I_j=\alpha_j I_j-\sum_{k=1}^{N}\theta_{jk}I_kI_j\ ,\quad j=1,\ldots,N\ . \qquad (1)$$

Here I_j is the activity of the jth competing quantity. In our case I_j denotes the intensity of a resonator mode. The parameters α_j and θ_{jk} are the linear gain and saturation coefficients. For ease of discussion we will take all α's to be equal. The specific dynamics of a system described by Eq. (1) is determined by the saturation coefficient θ_{jk}. In this context, we will use the Lotka-Volterra equations to discuss two particular dynamical classes, known as "winner takes all" and "voting paradox [7,8]."

The winner-takes-all system is a multistable system in which the different modes are coupled purely competitively. This leads to a dynamics in which the modes compete with each other until one mode wins the competition and suppresses the oscillation of the remaining modes. Any mode can be the winner and the actual outcome of the competition is determined by the initial mode intensities.

The voting paradox arises from a competitive interaction leading to a contradiction in the underlying dynamics. If, in a voting process, three groups have mutually conflicting interests and group 2 has veto power over group 1, group 3 has veto power over group 2, and group 1 has veto power over group 3, then no final state can be reached. In such a case the modes will cycle through a given sequence in which each mode oscillates one after another. The mode coupling will determine the details of the dynamics such as the cycling sequence and the time spent in each mode.

In the following section we will present a further discussion of these two dynamical classes and report on an optical implementation of both types using a photorefractive ring resonator.

II. WINNER-TAKES-ALL DYNAMICS AND THE VOTING PARADOX

The first dynamical system we want to discuss is the winner-takes-all competition. Here many modes interact with each other in such a way that only one mode at a time can oscillate in a stable manner. When the system is presented with an input pattern over the different modes it decides which of the modes it will support. This decision can be based on the input intensities to the different modes, or it can be weighted through some internal bias of the modes. The system will then start to suppress the oscillation of all modes other than the chosen one. In the final state all the energy is concentrated in the only oscillating mode which constitutes a localized center of activity.

Such a dynamical system finds application, for example, in some models of self-organizing neural networks [5,7]. Here the system has to select the best match of some internal parameters to a given input. This selection has to be made in a competitive way such that the system can clearly determine which set of internal parameters is the best match, or winner, in the comparison.

As discussed in the Introduction, the dynamics of many competitive systems can be qualitatively described by a set of Volterra-Lotka equations, given in Eq. (1). Here we choose these equations as a mathematical description of the system dynamics because of their simplicity and easy physical interpretation. For the winner-takes-all competition the saturation coefficients θ_{jk} are given by the matrix

$$\Theta=\begin{bmatrix} \beta & \theta & \cdots & \cdots & \theta \\ \theta & \beta & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \theta \\ \theta & \cdots & \cdots & \theta & \beta \end{bmatrix} . \tag{2}$$

The set of differential equations in Eq. (1), together with the above coefficient matrix, has many steady-state solutions, three of which are of particular interest:

$$I_j=\begin{cases} 0 \quad \text{for all } j , & (3) \\ \alpha/[\beta+(N-1)\theta] \quad \text{for all } j , & (4) \\ \alpha/\beta , \quad I_k=0 \quad \text{for all } k\neq j . & (5) \end{cases}$$

The stability of the above solutions depends on the parameters, α, β, and θ. It is easy to see that for $\alpha<0$ Eq. (3) is the only stable solution. If there is no net gain for any of the modes, all the intensities have to be zero. We therefore assume for the following that $\alpha>0$. It is also plausible that for small cross coupling, i.e., $\theta\cong 0$, the modes are virtually independent of each other. Then one expects that all modes oscillate simultaneously with equal intensity which corresponds to Eq. (4). On the other hand, if the cross-coupling coefficient is large, the strong competition will not allow a coexistence of modes. Then the stable solution will have only one mode oscillating at a time. In fact, one can show that for $\theta>\beta$ only Eq. (5) is stable, while for $\theta<\beta$ the stable solution is Eq. (4), as expected. In order to implement winner-takes-all dynamics, it is therefore necessary to make cross-coupling between modes stronger than self-saturation.

In Fig. 1 we have depicted a numerical simulation of Eqs. (1) and (2) for the case $\theta>\beta$. We have chosen the initial conditions randomly so that the intensity of one of the modes is slightly larger than the remaining ones. The system then chooses that mode as the winner and increases its intensity while decreasing the intensity for the other modes. Finally, the strongest mode has gained all the available energy and the remaining ones are shut off. Note that such a behavior of the system is also valid in the presence of noise.

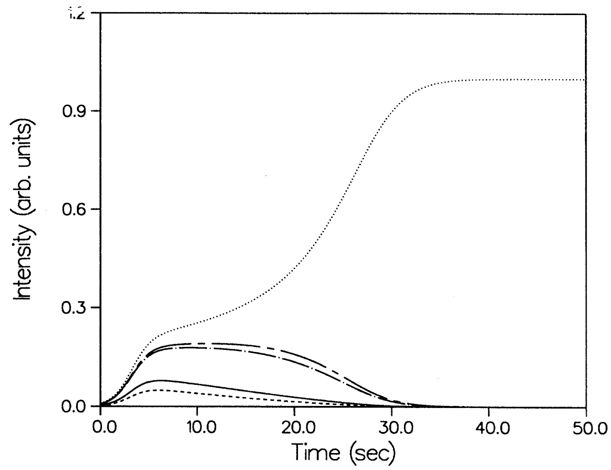

FIG. 1. Numerical simulation of the Lotka-Volterra equations in a winner-takes-all configuration. The intensities of five competing modes are plotted with respect to time. The initial intensity values are chosen randomly in the interval $[0,10^{-2}]$. The parameters in Eqs. (1) and (2) are $\alpha=1$, $\beta=1$, and $\theta=1.5$.

We have employed photorefractive two-beam coupling [9,10] to establish a physical system that exhibits winner-takes-all dynamics. An optical oscillator that uses photorefractive two-beam coupling gain in a resonator [11] is not a laser. Nevertheless, the analogies between the laser and photorefractive oscillator are strong: It has been shown that the modes of the photorefractive ring resonator are described by Lotka-Volterra equations in the weak-field case and the qualitative behavior remains similarly described in the strong-field regime [12]. In addition to providing gain for a multimode oscillator, two-beam coupling in photorefractive media can be used to establish mode coupling. The interference pattern of intersecting beams causes charge redistribution and, through the electro-optical properties of the crystal, a subsequent index of refraction grating. In materials where carrier redistribution is diffusion dominated, the phase of the index grating causes energy exchange between the two beams [10]. We can think of the coupling as gain for one of the beams and loss for the other.

Three qualities of photorefractive materials are particularly remarkable. First, enormous gains are possible. Gains exceeding exp(55/cm) have been observed. Second, they are very slow by nonlinear optics standards: time constants are typically in the range 0.001–10 sec. Third, increasing the intensity of a pump beam does not increase the gain for an unsaturated signal; it merely decreases the response time of the material. The response remains nonlinear even at very low light intensities (below 1 mW/mm^2). Gain is primarily determined by the geometry of the beam interaction and by the material properties.

Our experimental optical circuit, shown in Fig. 2, is a unidirectional ring oscillator with three photorefractive barium titanate crystals. Five ring paths are formed with five optical fibers each approximately 0.75 m in length. Each fiber, in fact, carries many modes, but it is convenient to ignore the fiber's mode structure and simply refer to the collection of fiber modes as a single mode. In addition to the set of equations (1) for the mode intensity, there is a corresponding set of coupled equations governing the mode frequencies. We argue on two accounts that the latter should be ignored in light of the consequent simplicity. First, the finesse of our resonators is very low (on the order of or less than unity); therefore, gain is not a strong function of round-trip length. Second, each fiber carries so many modes ($>10^4$) that there is always some linear combination of them on or close to resonance. It is such a combination which will oscillate at any given time.

The third photorefractive crystal in Fig. 2 is pumped by an external source laser and supplies gain to all five modes of the system. For two-beam coupling gain alone it has been shown that $\theta \leq \beta$—that is, neither winner-takes-all nor the voting-paradox dynamics are possible [12]. The other two crystals allow us to program the saturation matrix. At the output of the fibers a polarizing beam splitter provides two copies of the modes. A second beam splitter divides one of these beams equally again. One of these, which we label the "resonator" beam is eventually imaged back into its respective fibers. We shall refer to the other as the "interaction" beam. A lens is used on each of these beams to produce the Fourier transform of the output of the fibers. The first photorefractive crystal is placed at the intersection of the two Fourier planes and its axis is arranged so that the interaction beams derive energy from the resonator beams. If we think of the modes as optical rays, then in the Fourier-plane junction the two sets of rays cross. We see that any of the interaction rays can deplete every resonator ray of its energy. This has the effect of increasing all saturation coefficients, including the self-saturation coefficient β. We still have, at most, $\theta=\beta$. A pair of lenses reimages both sets of beams: they intersect again, but this time in an image plane. The second photorefractive crystal is arranged to cause energy transfer from the interaction modes to the resonator modes. Here each mode intersects only its corresponding twin; hence, an interaction beam gives back whatever energy it obtained in the first crystal to its own resonator mode in the second crystal. This *reduces* the self-saturation β relative to the cross-saturation θ, such that, overall, $\theta > \beta$. The beam interactions are detailed in Fig. 3.

The performance of our system in the winner-takes-all configuration is shown in Fig. 4. Only one of the five modes is stably oscillating at any one time. The contrast ratio between the oscillating mode and the other suppressed modes is greater than 100:1. The system is optically switched between the different modes by injecting a signal into the respective mode.

The second dynamical system we want to discuss is referred to as the "voting paradox." Because of a contradiction in the underlying dynamics, systems in that class do not settle down into a steady state. Instead they perform oscillations which can be maintained indefinitely. Such a controlled, oscillatory behavior of a competitive system can, for example, be used to store and recall time-sequenced information.

We can again use the Volterra-Lotka equation (1) to give a qualitative description of such a system. Besides

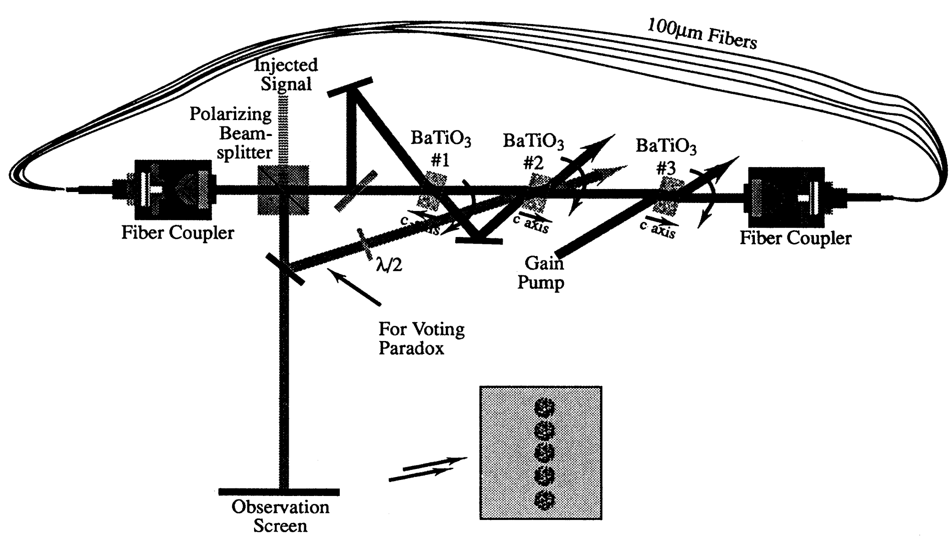

FIG. 2. Schematic of the competitive optical circuit. Five ring paths are defined by five multimode optical fibers. The polarizing beam splitter provides a beam used in the voting-paradox circuit; a half-wave plate corrects the beam polarization of the output of this beam splitter for the two-beam coupling. The gain medium pump intensity of 150 mW at a wavelength of 515 nm. The small signal gain of the gain medium (crystal No. 3) is greater than 10^3. The strong signal gains of $BaTiO_3$ crystals No. 1 and No. 2, measured with incident intensities equal to those that occur during oscillations, are 5 and 6, respectively. Round-trip passive attenuation for the system is on the order of 900 (very large).

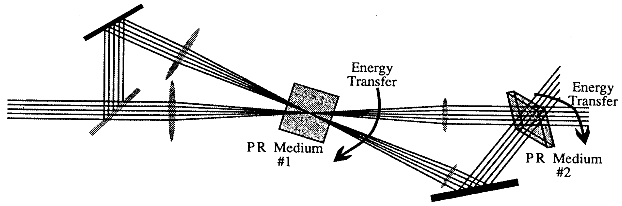

FIG. 3. Detail showing mode interaction in the two mode coupling crystals, $BaTiO_3$ No. 1 and No. 2. Crystal No. 1 is in a Fourier plane of a pair of lenses while crystal No. 2 is in an image plane. The perspective of the second interaction is distorted to clarify the fact that each twin mode couples only to its corresponding resonator mode.

the competitive coupling between the modes, as given by the matrix in Eq. (2), we require each mode to give a positive stimulus to the next mode to be recalled. Such cooperation between two modes will give rise to dynamics in which the oscillation energy will gradually move from one mode to the next. We therefore modify the saturation coefficients in Eq. (2) to the circulant matrix

$$\Theta=\begin{bmatrix} \beta & \theta & \cdots & \cdots & \theta \\ \theta & \beta & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \theta \\ \theta & \cdots & \cdots & \theta & \beta \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 & \delta \\ \delta & 0 & & & 0 \\ 0 & \delta & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & 0 & \delta & 0 \end{bmatrix} . \qquad (6)$$

Here we have assumed that the sequence will be recalled in an ascending order of the mode labels. The parameters β and θ are the same self- and cross-correlation coefficients as in Eq. (2), and δ is a positive coupling coefficient between adjacent modes.

It can be shown that the system indeed cycles through the modes if $\theta-\delta<\beta<\theta$. A numerical simulation of this case is presented in Fig. 5. An oscillating mode stimulates the oscillation of its next neighbor while suppressing all remaining ones. However, the time each mode oscillates increases with time. In an analytical solution of Eqs (1) and (6) May and Leonard [13] have shown that the period of cycling increases linearly in time. This is due to the fact that the initial value of each mode intensity decreases exponentially with time before it gets stimulated by the preceding mode. In many cases, especially in real physical systems, such behavior is unrealistic. Any kind of noise will induce fluctuations in the modes and keep their initial values around a finite average value. For a correct description of a physical system with voting-paradox dynamics, we must generalize Eqs. (1) to

$$\frac{d}{dt}I_j=\alpha I_j-\sum_{k=1}^{N}\theta_{jk}I_kI_j+f_j(t) \ , \quad j=1,\ldots,N \ . \qquad (7)$$

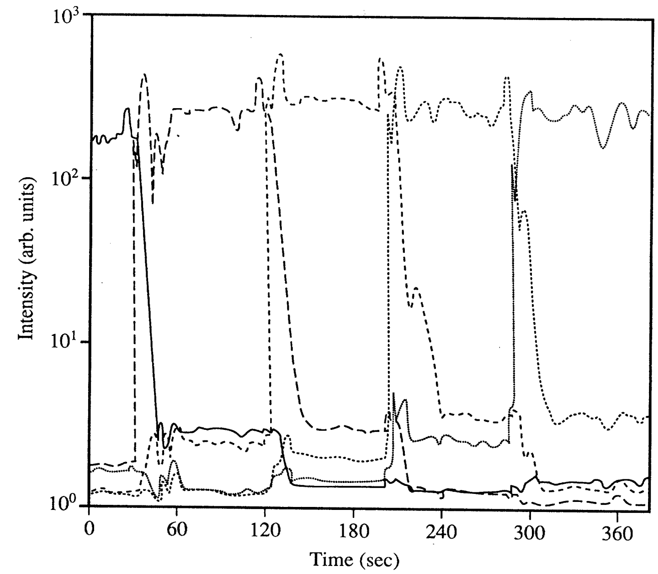

FIG. 4. Oscillation intensities of the five modes for winner-takes-all system. Mode identities are signified by various densities of dashed lines. In each case, light is injected in the direction of a given mode to turn it on. Once the oscillation of the mode is established the injected signal is terminated. After a time, the injected signal is again presented, and then moved to another mode. This accounts for the peaks seen at the beginning and end of the oscillation of each mode. Other variations are due to fluctuations in the resonator environment.

The matrix (θ_{jk}) again specifies the coupling between the modes and is given by Eq.(6). The functions $f_j(t)$ are time-dependent noise forces with

$$\bar{f}_j=\langle f_j(t)\rangle=\epsilon \ , \qquad (8)$$

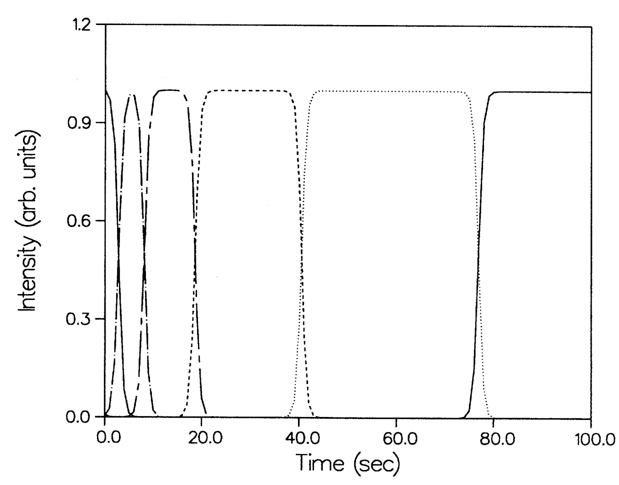

FIG. 5. Numerical simulation of the Lotka-Volterra equations in a voting-paradox configuration [Eqs. (1) and (6)]. The intensity of mode 1 (solid curve) at time $t=0$ is chosen to be 1, while the remaining intensities are randomly chosen from the interval $[0,10^{-2}]$. The parameters $\alpha=\beta=\delta=4$ and $\theta=6$ and were chosen such that the oscillations occur on a similar time scale as in the experiment. Note that the oscillation time for each mode increases with time.

$$\langle [f_j(t)-\bar{f}_j][f_i(t')-\bar{f}_i] \rangle = \tfrac{1}{3}\epsilon^2\delta_{ij}\delta(t-t') \ . \quad (9)$$

The parameter ϵ determines the strength of the fluctuations. The mean value for the noise in Eq. (8) is not zero because $f_j(t)$ is a noise function for the intensity. We have further assumed that the fluctuations are equally distributed in the interval $[0,2\epsilon]$ (Ref. [14]). This leads to the coefficient $\frac{1}{3}$ in the correlation function in Eq. (9). A numerical simulation of the equations of motion for the mode intensities as given by Eqs. (7)–(9) is shown in Fig. 6 in the case of five modes. In the presence of noise the system cycles through the five modes with a constant period. The period is mainly determined by the noise strength ϵ. It is interesting that a finite noise level substantially changes the system's dynamics without affecting its stability.

The voting-paradox dynamics is again experimentally achieved through a photorefractive ring resonator. Because of the similarities between winner-takes-all and voting-paradox dynamics, as illustrated by Eqs. (2) and (6), only a few modifications to the winner-takes-all system are necessary (see Fig. 2). In addition to the competitive interaction between the resonator and interaction beams, we now couple each mode cooperatively to its neighbor. This is achieved by coupling the s-polarized part of the beams, obtained from the first polarizing beam-splitter cube, into a second set of fibers. This allows us to randomly change the order of the modes and thus control the sequence of recalled events. Here we have chosen a cyclic permutation of the five modes. The light from these fibers is then passed through a second polarizing beam-splitter cube. The p component of the beam is coupled into the second photorefractive crystal, which was oriented to amplify the signal beam. Recall that this crystal is in an image plane of the five modes.

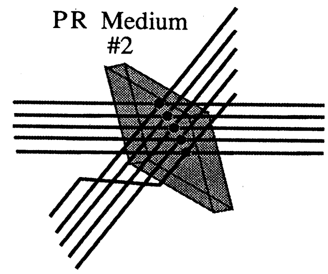

FIG. 7. Detail showing additional mode interaction for voting-paradox dynamics. Each mode is coupled to its upward neighbor, with the last mode looped back to the bottom. This ordering is established with the help of an auxiliary set of five fibers not shown in Fig. 2.

Therefore, each mode only couples into one of the resonator modes. Figure 7 details this interaction.

A time record of the voting-paradox experiment is shown in Fig. 8. The intensity of all five modes is recorded as the system makes two complete cycles through the set of modes. Note that the dynamics occurs on time scales of seconds and a complete cycle takes about 40 sec. The slight overall decrease of mode intensities during the cycle is due to some external disturbance of the system. It is not characteristic of the experiment as other time windows show a different drift of the intensities. A comparison of Fig. 8 to the theoretical simulation in Fig. 6 demonstrates a good agreement between the experiment and its qualitative description of Lotka-Volterra equations.

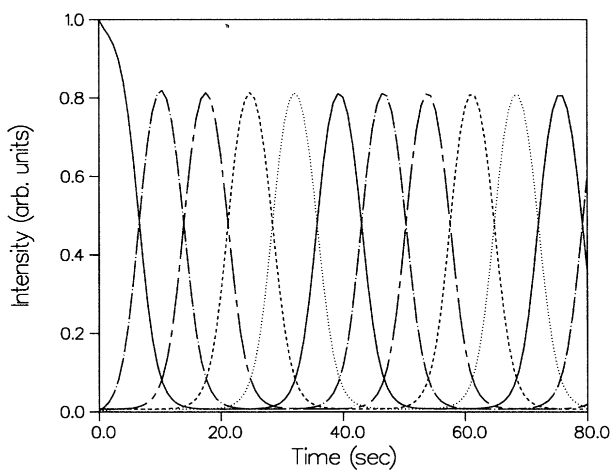

FIG. 6. Numerical simulation of the Lotka-Volterra equations in a voting-paradox configuration with noise added to the dynamics [Eqs. (7) and (6)]. The initial mode intensities are the same as in Fig. 5. The parameters are chosen such that the simulations most closely matches the experimental results: $\alpha=\beta=\delta=1.4$, $\theta=2.1$, and $\epsilon=0.01$. Note that the cycling time is constant in time.

III. SUMMARY

We have used a multimode resonator to demonstrate two competitive dynamical systems: winner-takes-all dynamics and the voting paradox. The competitive and cooperative interactions between the resonator beams is

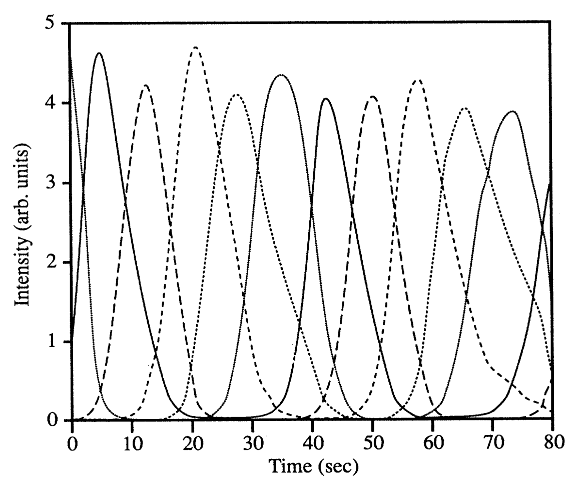

FIG. 8. Voting-paradox dynamics shows time record of the mode intensities. The modes continuously cycle through the set of five. The slight downward trend in the peak mode intensities is coincidental: it is probably due to a slow drift in the system parameters.

achieved through photorefractive beam coupling. This mechanism is well suited to our purposes because it allows us to control the strength of the interactions and hence to program the dynamical behavior of the overall system. We believe it is possible, with minor modifications to our present system, to obtain any dynamics qualitatively described by Lotka-Volterra equations.

ACKNOWLEDGMENTS

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