Detection of nonclassical states using a Kerr medium

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It is possible to determine whether the input state to a Kerr medium is nonclassical by measuring the total noise of the output state. For a classical input state the total noise at the output has a minimum value that depends on the photon number and the interaction time. If the total noise is found to be less than this value, then it can be concluded that the input state was nonclassical. A certain class of generalized coherent states can be detected in this way.

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I. INTRODUCTION

Classical states of the electromagnetic field are states whose P representations are non-negative definite. Such states can be described in terms of classical stochastic fields. Nonclassical states have P representations that do not satisfy this condition. They are intrinsically quantum-mechanical fields.

Perhaps the best known nonclassical states are squeezed states [1] and states with sub-Poissonian photon statistics [2]. The first of these is detected by homodyne measurements and the second by photon counting measurements. Here we wish to discuss a method that can detect a different class of nonclassical states. Among this class are certain kinds of generalized coherent states [3–5]. This method works by measuring the total noise of a field emerging from a Kerr medium.

The total noise T of a state of a single-mode field is given by

$$T=(\Delta X_1)^2+(\Delta X_2)^2=\langle a^\dagger a\rangle-\langle a^\dagger\rangle\langle a\rangle+\tfrac{1}{2}\ , \quad (1.1)$$

where a and $a^\dagger$ are the mode annihilation and creation operators [6]. The operators X_1 and X_2 are the quadrature components of the field and are given by

$$X_1=(a^\dagger+a)/2\ , \quad X_2=i(a^\dagger-a)/2\ . \quad (1.2)$$

The total noise can be measured by averaging over the phase of the local oscillator in a homodyne measurement [7]. It assumes its minimum value of $\frac{1}{2}$ for coherent states, and it increases as a state becomes more nonclassical [7]. It should be noted that this relation goes only one way; i.e., a highly nonclassical state has a large total noise, but a state with large total noise is not necessarily nonclassical. A high-temperature thermal state, for example, has a large total noise but is, in fact, a classical state.

The basic idea of using a Kerr medium to detect a nonclassical state is the following. A coherent state incident upon a Kerr medium becomes nonclassical at the output with a consequent increase in its total noise. Because classical states are incoherent superpositions of coherent states this suggests that there might be a minimum amount of total noise that an input classical state must have as it leaves the medium. This would mean that any state that emerges with less total noise than this minimum value must have been nonclassical at the input. Therefore, measuring the total noise of a state at the output of a Kerr medium provides the possibility of determining whether a state is initially nonclassical.

Before proceeding with the development of this scheme it should be noted that a considerable amount of work has been done on the connection between Kerr media and nonclassical states. Milburn examined the Q functions of states that are produced by a Kerr medium from coherent states. He compared these to the probability distributions that one obtains by using a classical analog of the quantized Hamiltonian. Substantial differences were found between the classical and quantum behavior [8]. In a later work he considered the role of dissipation in this system [9]. Yurke and Stoler showed that for sufficiently long interaction times an initial coherent state in a Kerr medium will evolve into a state that is a linear supersition of two different coherent states, a highly nonclassical object [10]. Gerry [11] and Tanás [12] studied the conditions under which a Kerr medium will produce squeezed states. Yamamoto *et al.* found a method to produce amplitude-squeezed states by combining light emerging from a Kerr medium with light from a local oscillator in a beam splitter [13]. Finally, Agarwal has shown how nonclassical effects that arise in a Kerr medium lead to decreased fringe visibility when the light is examined in a Michelson interferometer [14].

Let us now move to a more detailed examination of the proposed detection scheme. In Sec. II the detection method itself will be developed. In Sec. III a class of states that can be shown to be nonclassical by this method will be discussed.

II. TOTAL NOISE LEVEL

The action of a Kerr medium on a single mode of frequency ω is given by the Hamiltonian

$$H=\omega a^\dagger a+\lambda(a^\dagger a)^2\ , \quad (2.1)$$

where λ is proportional to the $\chi^{(3)}$ of the medium. Note that the photon number $N=a^\dagger a$, commutes with this Hamiltonian and is, therefore, conserved.

In order to determine how a classical state evolves under the action of this Hamiltonian we first examine the time evolution of a coherent state. From the number state expansion for the coherent state $|\alpha\rangle$ we see that

$$U(t)|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} e^{-in\omega t}e^{-in^2\lambda t}(\alpha^n/\sqrt{n!})|n\rangle \ , \tag{2.2}$$

where $U(t)=\exp(-itH)$ (we use units where $\hbar=1$). This expression can be used to find the expection value of the field amplitude in the state $U(t)|\alpha\rangle$:

$$\langle\alpha|U(t)^{-1}aU(t)|a\rangle = \alpha e^{-i(\omega+\lambda)t} \times\exp[-|\alpha|^2(1-e^{-2i\lambda t})] \ . \tag{2.3}$$

The total noise of $U(t)|\alpha\rangle$ is then

$$T=|\alpha|^2(1-e^{-2|\alpha|^2\Lambda})+\tfrac{1}{2} \ , \tag{2.4}$$

where $\Lambda=1-\cos(2\lambda t)$. Note that for $\lambda t \ll 1$ the total noise is an increasing function of the interaction time.

Let us now consider a general classical input state

$$\rho_{\text{in}}=\int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha| \ , \tag{2.5}$$

where $P(\alpha)\geq 0$. After an interaction time t the total noise of this state is

$$T=\int d^2\alpha\, P(\alpha)|\alpha|^2 - \left|\int d^2\alpha\, P(\alpha)\alpha\exp[-|\alpha|^2(1-e^{-2i\lambda t})]\right|^2+\tfrac{1}{2} \ . \tag{2.6}$$

Let us now examine in more detail the expression inside the vertical bars, which we shall call I_1, in the above equation. We have that

$$|I_1| \leq \int d^2\alpha\, P(\alpha)|\alpha\exp[-|\alpha|^2(1-e^{-2i\lambda t})]| \leq \int d^2\alpha\, P(\alpha)|\alpha|e^{-\Lambda|\alpha|^2} \ . \tag{2.7}$$

In order to proceed we note the following. Because $P(\alpha)\geq 0$, the Schwarz inequality gives us that

$$\left|\int d^2\alpha\, P(\alpha)g^*(\alpha)f(\alpha)\right|^2 \leq \left[\int d^2\alpha\, P(\alpha)|g(\alpha)|^2\right] \times\left[\int d^2\alpha\, P(\alpha)|f(\alpha)|^2\right] \tag{2.8}$$

for any functions $f(\alpha)$ and $g(\alpha)$ such that $\int d^2\alpha\, P(\alpha)|f(\alpha)|^2<\infty$ and $\int d^2\alpha\, P(\alpha)|g(\alpha)|^2<\infty$. If we choose $g(\alpha)=1$ and $f(\alpha)=|\alpha|\exp(-\Lambda|\alpha|^2)$ in the above expression we find that

$$|I_1|^2 \leq \int d^2\alpha\, P(\alpha)|\alpha|^2 e^{-2\Lambda|\alpha|^2} \ . \tag{2.9}$$

This result can be substituted into Eq. (2.6) to give us a lower bound for the total noise at the output (after an interaction time t)

$$T\geq \int d^2\alpha\, P(\alpha)|\alpha|^2(1-e^{-2\Lambda|\alpha|^2})+\tfrac{1}{2} \ . \tag{2.10}$$

Let us now define the function

$$h(\Lambda')=\int d^2\alpha\, P(\alpha)|\alpha|^2 e^{-2\Lambda'|\alpha|^2} \ , \tag{2.11}$$

which appears in Eq. (2.10). First note that

$$dh/d\Lambda' = -2\int d^2\alpha\, P(\alpha)|\alpha|^4 e^{-2\Lambda'|\alpha|^2} \ . \tag{2.12}$$

We also have, using Eq. (2.8) with $f(\alpha)=e^{-\Lambda'|\alpha|^2}$ and $g(\alpha)=|\alpha|^2 e^{-\Lambda'|\alpha|^2}$, that

$$h(\Lambda')\leq\left[\int d^2\alpha\, P(\alpha)|\alpha|^4 e^{-2\Lambda'|\alpha|^2}\right]^{1/2} \times\left[\int d^2\alpha\, P(\alpha)e^{-2\Lambda'|\alpha|^2}\right]^{1/2} \leq\left[\int d^2\alpha\, P(\alpha)|\alpha|^4 e^{-2\Lambda'|\alpha|^2}\right]^{1/2} \ , \tag{2.13}$$

because the second factor on the first line is less than 1. Putting Eqs. (2.12) and (2.13) together gives

$$dh/d\Lambda' \leq -2h(\Lambda')^2 \ , \tag{2.14}$$

which can also be expressed as

$$\frac{d}{d\Lambda'}\left[\frac{1}{h}\right]\geq 2 \ . \tag{2.15}$$

If we now integrate both sides from 0 to Λ and note that $h(0)=\langle N\rangle$ we have that

$$\frac{1}{h(\Lambda)}-\frac{1}{h(0)}\geq 2\Lambda$$

or

$$h(\Lambda)\leq\langle N\rangle/(2\Lambda\langle N\rangle+1) \ , \tag{2.16}$$

Substituting this back into our expression for the total noise, Eq. (2.10), gives

$$T\geq\langle N\rangle\left[1-\frac{1}{2\Lambda\langle N\rangle+1}\right]+\tfrac{1}{2}\equiv T_{\min}(\langle N\rangle,\Lambda) \ . \tag{2.17}$$

Therefore, any classical input state with mean photon number $\langle N\rangle$ will have a total noise greater than the expression on the right-hand side of Eq. (2.17) at the output of the Kerr medium. This means that if the total noise of an output state is less than $T_{\min}(\langle N\rangle,\Lambda)$, then the input state was nonclassical.

Consider now the expression for $T_{\min}(\langle N\rangle,\Lambda)$. We shall assume that $\lambda t\ll 1$, which will generally be true. We then have that $\Lambda\cong 2(\lambda t)^2$. It is then clear that $T_{\min}$ will vary most as a function of t when $(\lambda t)^2\langle N\rangle$ is of order 1. It ranges from a value of $\frac{1}{2}$ for $t=0$ to $\langle N\rangle+\frac{1}{2}$ for $(\lambda t)^2\langle N\rangle\gg 1$.

Let us now examine whether $T_{\min}$ can be made to differ significantly from $\frac{1}{2}$ in a real system. The system to be considered will be the traveling-wave fiber arrangement discussed by Levenson *et al.* [15]. Ideally one would like to make $\Lambda\langle N\rangle$ of order 1, but for a 100-m fiber with a $\chi^{(3)}$ of 10^{-14} esu and a laser power of 1 W this is not possible. On the other hand, the first term on the right-hand side of Eq. (2.17) will be of order 1 under these conditions. In particular, if $\Lambda\langle N\rangle \ll 1$, then this term is approximately given by $(\lambda t\langle N\rangle)^2$. If λ is expressed in terms of $\chi^{(3)}$, $\langle N\rangle$ in terms of the field amplitude E, the frequency ω, and the quantization volume, and if the interaction time is taken to be the length of the fiber L over c, then

$$\lambda t\langle N\rangle = (\omega L/c)\chi^{(3)}|E|^2 . \tag{2.18}$$

The condition that $\lambda t\langle N\rangle$ be of order 1 or greater is, then, just that $(\omega L/c)\,\chi^{(3)}|E|^2>1$. This is the condition which Levenson *et al.* found to be necessary for the generation of squeezing in a fiber, and it is achievable [15]. Therefore, it should be possible to make $T_{\min}$ differ appreciably from $\frac{1}{2}$.

Note that using Eq. (2.17) to determine whether a state is nonclassical involves two measurements. One first needs to determine the mean photon number $\langle N\rangle$. This can be done either before or after the light has passed through the Kerr medium because the interaction of Eq. (2.1) does not affect the photon statistics. One then measures the total noise at the output of the Kerr medium and compares the result to $T_{\min}(\langle N\rangle,\Lambda)$. If the result is less than $T_{\min}$, then the input state is nonclassical.

III. EXAMPLE OF STATES THAT CAN BE DETECTED BY THIS METHOD

We now want to examine briefly one kind of nonclassical state that can be detected by this scheme. A more exhaustive analysis of the nonclassical states that can be detected in this way will appear in a future publication.

Passage through a Kerr medium affects correlations between the amplitude fluctuations and the phase fluctuations of a field. This suggests that input states whose output total noise is less than $T_{\min}$ should possess amplitude-phase correlations. In particular, suppose we start with a coherent state and then, in some fashion create correlations between its amplitude and its phase. We now pass this state through our Kerr medium detector, which "undoes" these correlations, leaving us with a coherent state again at the output. A coherent state has a total noise of $\frac{1}{2}$, which is clearly less than $T_{\min}$. Such a nonclassical state can be detected by this scheme.

This leads us to consider what kinds of amplitude-phase correlations can be undone by a Kerr medium. One answer is those created by another Kerr medium. The constant λ appearing in Eq. (2.1) can be of either sign. Suppose we have two Kerr media, the first with $\lambda=-\lambda_1<0$ and the second with $\lambda=\lambda_2>0$. The first serves to prepare the state, the second to detect it. We first inject a coherent state $|\alpha\rangle$ into the first medium where it interacts for a time t_1. The resulting state is

$$|\Psi\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} e^{-in\omega t_1} e^{in^2\lambda_1 t_1}(\alpha^n/\sqrt{n!}|n\rangle . \tag{3.1}$$

This is an example of a generalized coherent state [3–5]. This nonclassical state now serves as the input to the second Kerr medium, which is the detector. The interaction time there is t_2 and upon emerging the field has a total noise given by

$$T=|\alpha|^2(1-e^{-2|\alpha|^2\mu})+\tfrac{1}{2} , \tag{3.2}$$

where $\mu=1-\cos[2(\lambda_2 t_2-\lambda_1 t_1)]$. For the input state $|\Psi\rangle$ we find that $T_{\min}$ is given by

$$T_{\min}=|\alpha|^2[1-1/(2\Lambda|\alpha|^2+1)]+\tfrac{1}{2} , \tag{3.3}$$

where in this case $\Lambda=1-\cos(2\lambda_2 t_2)$. By measuring the total noise at the output of the second medium we shall be able to assert that $|\Psi\rangle$ is nonclassical if $T<T_{\min}$. This will be true if

$$e^{-2|\alpha|^2\mu}>1/(2\Lambda|\alpha|^2+1) . \tag{3.4}$$

If $\lambda_1 t_1$ and $\lambda_2 t_2$ are much smaller than 1 this becomes

$$\exp[-4|\alpha|^2(\lambda_2 t_2-\lambda_1 t_1)^2]>1/[(2\lambda_2 t_2|\alpha|)^2+1] . \tag{3.5}$$

This condition is clearly satisfied when $\lambda_2 t_2=\lambda_1 t_1$ and for some range of $\lambda_2 t_2$ about $\lambda_1 t_1$. Therefore, generalized coherent states of the form given by $|\Psi\rangle$ can be detected by using this method.

IV. CONCLUSION

It has been shown that a detector consisting of a Kerr medium and a homodyne detector can be used to detect nonclassical states. The homodyne detector is used to measure the total noise of the field state at the output of the medium. If the total noise is below a certain level, then one can conclude that the input state is nonclassical. A certain class of generalized coherent state, which can be produced by a second Kerr medium, can be detected in this way. A more thorough study of the properties of the states that can be detected by this scheme will appear in a subsequent publication.

Finally, it has been shown that it is possible for $T_{\min}$ to differ appreciably from its minimum value of $\frac{1}{2}$ in a traveling-wave fiber system. It is possible that even higher values can be obtained in a pulse or cavity arrangement. To determine whether this is actually the case will require further investigation.

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