States of a dynamically driven spin. I. Quantum-mechanical model

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"Dynamical driving" means driving a system by another dynamical system, in contrast to driving by an external force (which acts on the system but is not acted upon by the system). The model consists of a localized spin driven by a polarized beam of spins. A remarkably complex variety of behaviors is possible as a result of the competition between the aligning effect of interaction with the beam and the precession due to the external magnetic field. The state of the localized spin can evolve from pure to mixed, or from mixed to pure, and its motion is characterized by an attractor. The final state of the driven spin is usually independent of its initial state, so the model provides an example of quantum-state preparation. These effects cannot be produced by a prescribed external force.

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I. INTRODUCTION

The concept of *dynamical driving* is best defined by contrasting it with driving by an *external force*. In the latter case the Hamiltonian has the form $H=H_0+V(t)$, where H_0 is the Hamiltonian of the undriven system, and $V(t)$ is the potential of a prescribed driving force. Although the behavior of a system driven by an external force can be very interesting (the periodically kicked rotor being the most extensively studied example), it has definite limitations. The motion of such a system can have no attractors. In the case of a quantum system, pure states can evolve only into pure states, and initially nearby states must always remain nearby. That is to say, if $|\psi_1(0)\rangle$ and $|\psi_2(0)\rangle$ are two initial states such that $|\langle\psi_1(0)|\psi_2(0)\rangle|=1-\epsilon$, then at any later time t we will also have $|\langle\psi_1(t)|\psi_2(t)\rangle|=1-\epsilon$. These limitations occur because the external force $V(t)$ is not a dynamical system; it acts on the object but is not acted upon by the object.

However, dynamical driving, which is the driving of an object by another dynamical system, is not subject to such restrictions. The model considered in this series of papers is shown schematically in Fig. 1. The object of interest is a localized spin (of magnitude j_1) driven by a polarized beam of spins (of magnitude j_2). Both the object (the localized spin) and the driver (the beam spins) are dynamical systems. The object is now an open system, whose motion can exhibit attractors. For the quantum system, pure states may evolve into mixed states, and mixed states may evolve into pure states. The classical system (discussed in paper II of this series) can exhibit stable attractors, limit cycles, basins of attraction, and chaos. Unlike some theories of open systems, our model involves no phenomenological dissipation [1] (friction or viscosity terms), and no stochastic environment. The complete system of object plus driver is a closed Hamiltonian system, and the great variety of behaviors that it exhibits are all consequences of conservative dynamics. The apparently irreversible motion of the state of the object towards an attractor is due to the chosen initial state of the polarized beam.

II. MODEL

The model, shown in Fig. 1, consists of a localized spin j_1, a beam of spins j_2, polarized in the y direction, and a magnetic field B in the z direction. The beam can affect the localized spin only by transferring angular momentum to it, so the interaction tends to polarize the localized spin in the y direction. The magnetic field rotates any such polarization about the z axis. The competition between these two effects is responsible for the rich variety of behaviors exhibited by the model.

The beam particles interact, one at a time, with the localized spin. While the beam particle is in the interaction region, for a duration τ, the Hamiltonian of the two interacting spins is

$$H=a(\mathbf{S}_1\cdot\mathbf{S}_2)+\mathbf{B}\cdot(\mathbf{S}_1+\mathbf{S}_2)\ . \tag{1}$$

It would be more realistic for the interaction strength to depend on the separation between the particles, and hence on time. For simplicity we take it to be constant, a, when the beam particle is in the interaction region, and zero outside of it. We also assume for simplicity that the

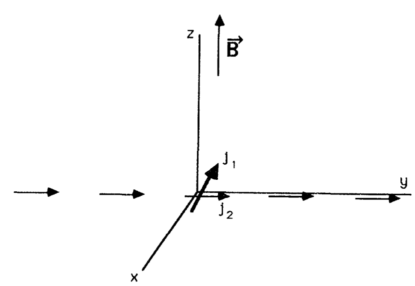


FIG. 1. The system: a localized spin j_1 is driven by a beam of spins j_2, polarized in the y direction, with an external magnetic field in the z direction.

arrival times of the beam particle are separated by exactly τ, one particle leaving the interaction region when the next one enters. The relaxation of these idealizations in order to describe a realistic experiment would not present much difficulty, but for the purposes of this paper it would only be an unnecessary complication.

The effect of interaction with one beam particle is given by the unitary time-development operator $U(\tau)=e^{-iH\tau}$. (We set $\hbar=1$.) The eigenvectors of H, and of $U(\tau)$, are the total-angular-momentum eigenvectors, which are constructed from the single-particle spin eigenvectors by means of the familiar Clebsch-Gordan coefficients,

$$|J,M\rangle=\sum_{m_1,m_2}(j_1,j_2,m_1,m_2|J,M)|j_1,m_1\rangle\otimes|j_2,m_2\rangle \ . \tag{2}$$

The corresponding eigenvalues are given by

$$H|J,M\rangle=E_{J,M}|J,M\rangle \ , \tag{3}$$

$$E_{J,M}=\tfrac{1}{2}a[J(J+1)-j_1(j_1+1)-j_2(j_2+1)]+BM \ . \tag{4}$$

The time-development operator is computed directly from its eigenvalues and eigenvectors,

$$U(\tau)=\sum_{m_1,m_2}\exp(-i\tau E_{J,M})|J,M\rangle\langle J,M| \ . \tag{5}$$

The effect of one step of the process (a beam particle in at time t, and out at time $t+\tau$) on the state operator ρ is

$$\rho(t+\tau)=U(\tau)\rho(t)U^{\dagger}(\tau) \tag{6}$$

with

$$\rho(t)=\rho_1(t)\otimes\rho_2 \ . \tag{7}$$

Here $\rho_1(t)$ is the initial state of the localized spin, and ρ_2 is the initial state of the beam particle just before interaction.

If we are interested only in the localized spin, discarding the beam particle after interaction, it is sufficient to consider only the *partial state* of the localized spin (also called the *reduced state*),

$$\rho_1(t+\tau)=\mathrm{Tr}^{(2)}\rho(t+\tau) \ , \tag{8}$$

obtained by taking a trace over the coordinates of the beam particle. Using a matrix representation, we have

$$\begin{aligned}\langle a|\rho_1(t+\tau)|b\rangle&=\sum_{\alpha,\beta}\sum_{c,\gamma}\sum_{d,\delta}\langle a\alpha|U|c\gamma\rangle\langle c|\rho_1(t)|d\rangle\\&\qquad\times\langle\gamma|\rho_2|\delta\rangle\langle d\delta|U^{\dagger}|b\beta\rangle\delta_{\alpha,\beta}\\&=\sum_{c,d}M_{ab;cd}\langle c|\rho_1(t)|d\rangle \ ,\end{aligned} \tag{9}$$

where this equation implicitly defines M.

The effect of the beam on the partial state operator for the localized spin is given by iteration of a linear mapping of the form

$$\rho_1(t+\tau)=M\rho_1(t) \ , \tag{10}$$

where the matrix M is a four-dimensional array and the vector ρ_1 is a two-dimensional array. The dynamics are determined by the eigenvalues of M,

$$M\rho_\Lambda=\Lambda\rho_\Lambda \ . \tag{11}$$

An eigenvector ρ_Λ with $\Lambda=1$ describes a steady state. An eigenvector with $|\Lambda|<1$ describes a decaying transient. ($|\Lambda|>1$ is impossible, since exponential growth would lead to violation of $\mathrm{Tr}\rho^2\leq 1$.) An oscillatory limit cycle would occur if $|\Lambda|=1$, $\Lambda\neq 1$. This case has not been observed, but there is as yet no proof whether it is possible or impossible for the model. In all generic cases (that is, excepting certain degenerate situations, to be described shortly) it has been found that there is *exactly one* eigenvalue $\Lambda=1$, all others satisfying $|\Lambda|<1$, so the system goes to a *unique* final steady state.

Since the two terms of the Hamiltonian (1) are commutative, it is permissible to treat effects of the interaction and the magnetic field separately, even though they really act simultaneously. This does not affect the mathematical analysis, but it can facilitate qualitative reasoning about the model. The effect of the magnetic field during one interaction period is to rotate all spins about the z axis through an angle $B\tau$. The separation between energy eigenvalues (4) corresponding to different values of total angular momentum J is

$$E_{J,M}-E_{J-1,M}=aJ \ . \tag{12}$$

If the state is represented in terms of the total-angular-momentum basis vectors, the effect of the interaction will be to introduce a relative phase shift of magnitude $aJ\tau$ between the J and $J-1$ components. If one of these phase shifts is a multiple of 2π, then two parts of the vector space might not be coupled by the interaction. This degenerate situation can give rise to a nonunique final state.

Because the effect of the interaction scales with J, it is convenient to introduce the parameter

$$A'=a(j_1+j_2) \ . \tag{13}$$

This is the largest of the spacings (12); it is the only spacing if the beam spin j_2 is $\frac{1}{2}$. When comparing models having different values of j_1, it is appropriate to consider equal values of A', rather than equal values of a. If we choose units such that $\tau=1$ (and $\hbar=1$), then B, a, and A' will all be measured in radians.

III. RESULTS

The state $\rho_1(t)$ of the localized spin has always been found to converge to a steady state after interaction with sufficiently many beam particles, that is, after sufficiently many interactions of the form (10). It is neither practical nor informative to present the full steady-state density matrix, so two more useful quantities have been computed: the *normalized polarization*, $\mathbf{p}=\langle\mathbf{S}_1\rangle/j_1$; and a *purity index* for the state, $\mathrm{Tr}(\rho_1)^2$. A normalized state operator satisfies the conditions $\mathrm{Tr}\rho=1$ and $\mathrm{Tr}\rho^2\leq 1$, with $\mathrm{Tr}\rho^2=1$ if and only if ρ is a pure state. Thus the quantity $\mathrm{Tr}\rho^2$ can be regarded as a measure of the degree of purity or mixedness of the state. The normalized polarization vector **p** lies inside or on the surface of the unit sphere. One can have $|\mathbf{p}|=1$ only for a pure state, but for $j_1>\frac{1}{2}$ the state is not fully determined by **p** and there are pure states that do not satisfy $|\mathbf{p}|=1$.

The results are invariant under each of the following discrete symmetries:

(i) $B\to -B,\ S_x\to -S_x,\ S_z\to -S_z$;

(ii) $a\to -a,\ S_z\to -S_z$.

The former follows from the fact that (i) leaves invariant the Hamiltonian, the spin commutation relations, and the initial state of the beam. The transformation (ii) changes H into $-H$. This is equivalent to a time inversion, $t\to -t$, which does not affect the steady states. If the initial state of the localized spin is chosen to be invariant under one of these symmetries, then the entire evolution of the state will be invariant. The final steady state will be invariant, regardless of the initial state, so only results for positive values of a and B need be presented.

A. $j_2=\frac{1}{2}$

In this case there are only two values of total angular momentum, $J=j_1+\frac{1}{2}$ and $J=j_1-\frac{1}{2}$, and the spacing (12) between the corresponding energy eigenvalues is A'. Hence the results are periodic in A', with period 2π (using units $\hbar=1$, $\tau=1$).

Figure 2 shows the normalized polarization **p** in the final steady state for a localized spin $j_1=\frac{1}{2}$, as a function of A' and B. For $B=0$ the spin becomes fully polarized along the positive Y axis, regardless of the interaction strength. As one increases B for fixed A', the fixed point moves along a planar arc, ending somewhere on the negative Y axis for $B=\pi$. [The natural range of B is from $-\pi$ to $+\pi$, but only positive values are shown because of the symmetry (i).] For $A'=\pi$, the vector **p** lies in the plane $Z=0$, and the locus of fixed points is a semicircle of unit radius. The curve for $A'=3\pi/2$ is a mirror image in the $Z=0$ plane of the curve for $A'=\pi/2$. This is so because of the symmetry (ii) and the fact that $A'=3\pi/2$ is equivalent to $A'=-\pi/2$.

Figure 3 shows **p** at the fixed points for a localized spin $j_1=1$. For this and larger values of j_1 the curves of constant A' do not lie in a plane, as can be seen from their shadows on the sides of the box. The exception is the curve for $A'=\pi$, which lies in the plane $Z=0$ for all values of j_1, provided $j_2=\frac{1}{2}$.

It is apparent from the kink in the curve $A'=\pi$, that something peculiar happens at the larger values of B, near the negative Y axis. Figure 4 shows this also for $j_1=\frac{3}{2}$ and 2, and it persists for larger values of j_1. The phenomenon can be seen more clearly in Figs. 5–8, where the magnitude of the polarization $p=|\mathbf{p}|$ and the purity index $\mathrm{Tr}(\rho_1)^2$ are plotted as functions of B. In all cases we see that the state remains highly polarized and nearly pure for small values of B, but the polarization declines drastically for larger B values. This decline becomes more abrupt as j_1 increases. But for $j_1>1$, the state becomes partially repolarized as $B\to\pi$. For half odd-integer values of j_1 it returns at $B=\pi$ to a pure state corresponding to the eigenvector $|S_y=-\frac{1}{2}\rangle$.

This unexpected return to a pure state has a simple explanation. The total-angular-momentum states $|J,M\rangle$ have the following form for $M=0$:

$$|j_1+\tfrac{1}{2},0\rangle=(|j_1,\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|j_1,-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle)/\sqrt{2}\ ,$$

$$|j_1-\tfrac{1}{2},0\rangle=(|j_1,\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle-|j_1,-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle)/\sqrt{2}\ .$$

(Here we use eigenvectors of the y component of angular momentum.) At the fixed point, the state of the localized spin and a newly arrived beam particle will be

$$|j_1,-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle=(|j_1+\tfrac{1}{2},0\rangle-|j_1-\tfrac{1}{2},0\rangle)/\sqrt{2}\ . \tag{14}$$

The effect of the interaction is to shift the relative phase of these two terms by $A'\tau=\pi$, changing the state into

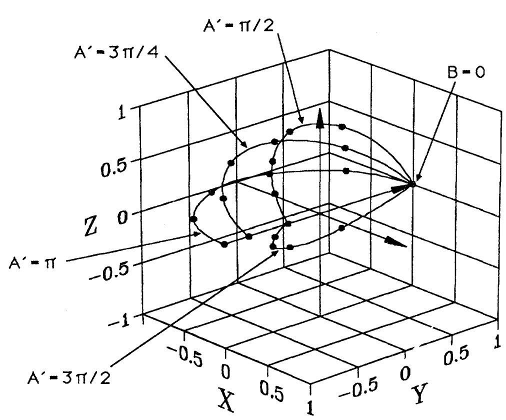


FIG. 2. Loci of the fixed-point polarization for $j_1=\frac{1}{2}$, $j_2=\frac{1}{2}$. Along each curve of constant A', B varies from 0 to π in steps of 0.2π.

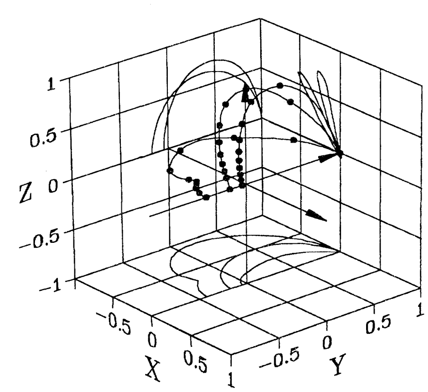


FIG. 3. Loci of the fixed-point polarization for $j_1=1$, $j_2=\frac{1}{2}$. The curves, from highest to lowest, correspond to $A'=\pi/4$, $\pi/2$, and π. From right to left along each curve, B varies from 0 to π in steps of 0.1π. (Shadows on the sides of the box help to visualize the three-dimensional shapes of the curves.)

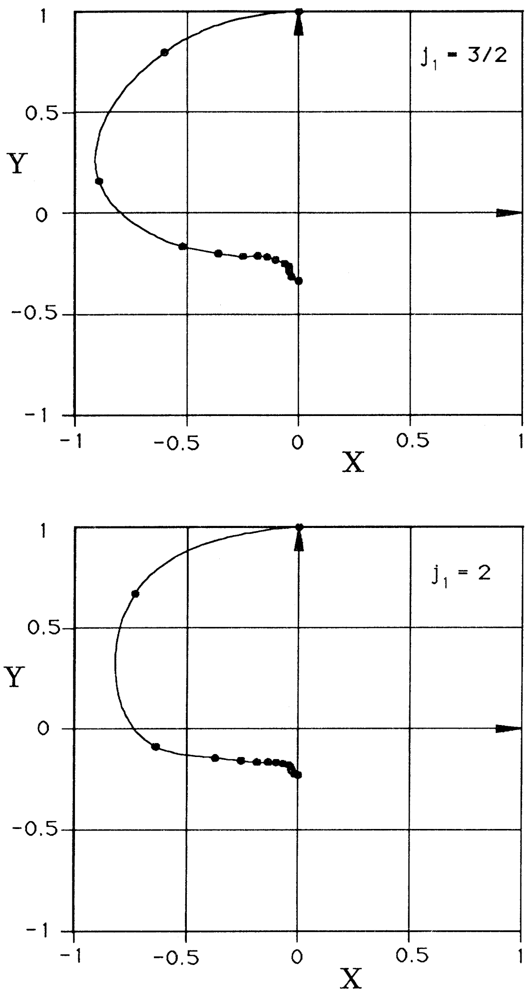


FIG. 4. Loci of fixed-point polarization for $j_2=\frac{1}{2}$ and $A'=\pi$, for $j_1=\frac{3}{2}$ and 2. From top to bottom along each curve, B varies from 0 to π in steps of 0.1π.

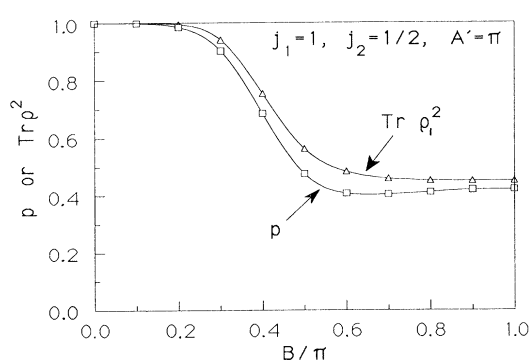


FIG. 5. Magnitude of the normalized polarization p and state purity index Trρ_1^2 as functions of B, for $j_1=1$, $j_2=\frac{1}{2}$, $A'=\pi$.

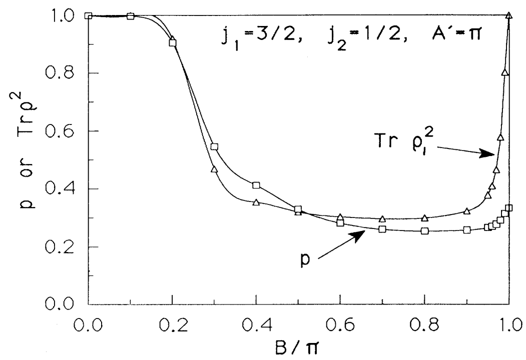


FIG. 6. Similar to Fig. 5 for $j_1=\frac{3}{2}$.

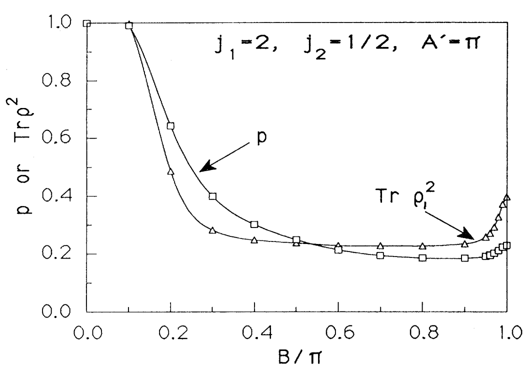


FIG. 7. Similar to Fig. 5 for $j_1=2$.

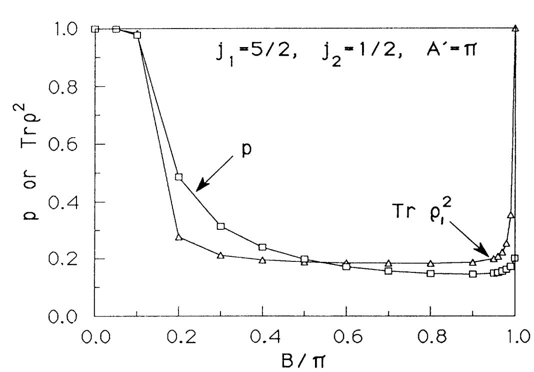


FIG. 8. Similar to Fig. 5 for $j_1=\frac{5}{2}$.

$|j_1,\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle$. The magnetic field then rotates the spin through the angle $B\tau=\pi$, so that the localized spin state returns to the fixed point, $|j_1,-\frac{1}{2}\rangle$. No such pure-state fixed point is available if j_1 is an integer, but the numerical results show that the system makes its "best effort" to achieve a similar polarization.

B. $j_1=j_2=1$

In this case the interaction Hamiltonian has three energy levels (corresponding to $J=0,1,2$), with separations of a and $2a=A'$. Figure 9 shows loci of the fixed-point polarization as a function of B for fixed A'. The results in the upper figure are qualitatively similar to those in Fig. 3, except that now $A'=\pi$ has no special symmetry. The loci in the lower figure are somewhat different, in that they cross from the half space $X<0$ to $X>0$.

For $A'=2\pi$, $B=0$ we find the first example of a nonunique final state, with the eigenvalue $\Lambda=1$ in Eq. (11) being fourfold degenerate. This can be understood in terms of angular momentum coupling. The usual fixed point for $B=0$ is the pure state that is fully polarized in the y direction, which we denote as $|1\rangle\otimes|1\rangle$. (The factors are eigenvectors of the y component of spin for the localized and beam particles, respectively.) This is an eigenvector of total angular momentum, and so is a stationary state. Suppose now that the localized spin was initially in a state corresponding to the zero eigenvalue of S_y,

$$|0\rangle\otimes|1\rangle=(|2,1\rangle-|1,1\rangle)/\sqrt{2}\ . \qquad (15)$$

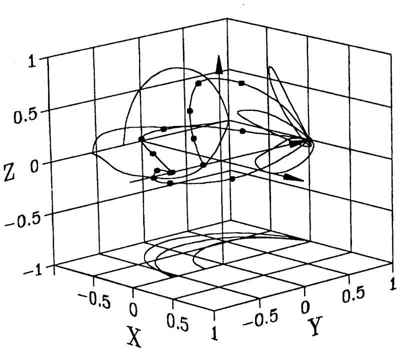
The right-hand side is its expansion in terms of total-angular-momentum vectors, using the y basis. The effect of the interaction is to shift the relative phase of the two terms on the right by A' (that is, by $A'\tau$, with $\tau=1$). But for $A'=2\pi$ there is no effect. Thus the S_y eigenvectors $|1\rangle$ and $|0\rangle$ for the localized spin both correspond to fixed points in this case. The four linearly independent components of the density matrix ρ_1 that correspond to $\Lambda=1$ in Eq. (11), and so can be steady states, are the pure states: $|1\rangle\langle 1|$, $|0\rangle\langle 0|$, $\frac{1}{2}(|1\rangle+|0\rangle)(\langle 1|+\langle 0|)$, and $\frac{1}{2}(|1\rangle-|0\rangle)(\langle 1|-\langle 0|)$. Of course any mixture of these four is also a steady state.

In the case shown in Fig. 9, the initial density matrix of the localized spin was a multiple of the unit matrix. The final state for $B=0$ is $\rho_1=\frac{1}{2}(|1\rangle\langle 1|+|0\rangle\langle 0|)$, yielding $\langle S_y\rangle=0.5$. But $\langle S_y\rangle$ could take any value between 0 and 1, depending on the initial state.

The degeneracy of the eigenvalue $\Lambda=1$ in Eq. (11) is broken by taking either $B\neq0$ or A' not equal to a multiple of 2π. The lower part of Fig. 9 compares $A'=2\pi$ with $A'=1.9\pi$. The curves are close for all but the lowest values of B, for which they differ very greatly. This indicates the instability of the results that arise from the degeneracy at $A'=2\pi$, $B=0$.

For $B=\pi$, $A'=2\pi$ the unique final state is pure. But, unlike the results for $j_2=\frac{1}{2}$, it is unpolarized and corresponds to the spin eigenvector $|S_y=0\rangle$.

Instead of the fully polarized beam state, $|S_y=1\rangle$, one can alternatively prepare the beam particles in the spin eigenstate $|S_y=0\rangle$. In this case the polarization lies entirely along the z axis (direction of B), and its variation with B is shown in Fig. 10 for $A'=\pi$. This beam state does not replicate itself on the localized spin when $B=0$, unlike the polarized beam state. Although the polarization is zero for $B=0$, $\pi/2$ and π, the final state is not rotationally symmetric in these cases. (See Ref. [2] for a classification of the pure and mixed states of a spin-1 particle.) It is a mixed state for $B=0$ and π. For $B=\pi/2$ it is an eigenvector of the x component of spin, $|S_x=0\rangle$.



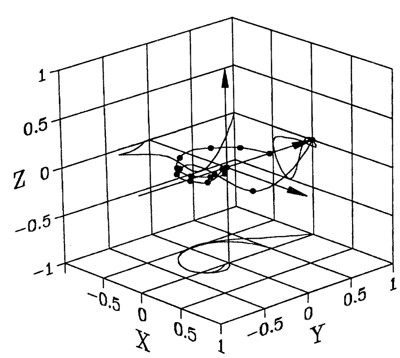


FIG. 9. Loci of the fixed-point polarization for $j_1=1$, $j_2=1$. The curves, from highest to lowest, correspond to $A'=\pi/2$, π, $3\pi/2$ (top figure), and $A'=2\pi$, 1.9π (bottom figure), with B increasing from 0 to π along each.

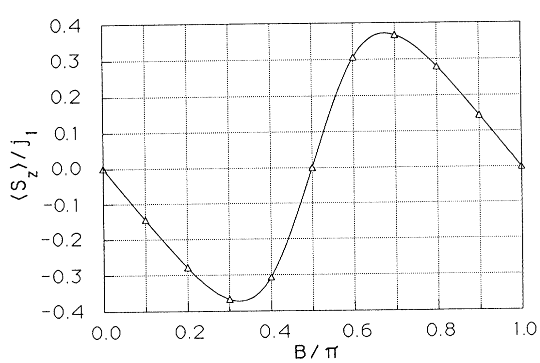


FIG. 10. Normalized polarization vs B, for $A'=\pi$, $j_1=1$, $j_2=1$, but beam particle state $|S_y=0\rangle$.

C. $j_1=j_2=\frac{3}{2}$

In this case the interaction Hamiltonian has four energy levels (with $J=0,1,2,3$), the spacings between them being a, $2a$, and $3a=A'$. For $3a=2\pi$ the final state is not unique, with the $\Lambda=1$ eigenvalue in Eq. (11) being fourfold degenerate, for reasons essentially the same as in the previous case. For $2a=2\pi$ the final state is, nevertheless, unique. Thus the vanishing (mod2π) of a phase shift between two J eigenvectors is not a sufficient condition for degeneracy of $\Lambda=1$, although it seems to be a necessary condition. Of course if $a=2\pi$ then all phase shifts would be multiples of 2π, and this case would be equivalent to no interaction at all.

D. $B=\pi$

We have already seen that interesting effects occur in the neighborhood of $B=\pi$, making this case worthy of more detailed study. For $B=\pi$ the final state polarization lies on the y axis. Figure 11 shows the value of $\langle S_y \rangle$ produced by a $j_2=\frac{1}{2}$ beam for three different interaction strengths and several values of j_1. It is apparent that, for fixed A', $\langle S_y \rangle$ rapidly approaches a constant limit as $j_1 \to \infty$. This fact has a simple explanation for $A'=\pi$ and half odd-integer values of j_1 (see Sec. III A), but this more general result was not anticipated. A similar $j_1 \to \infty$ limit seems to exist for other values of j_2, but it is approached more slowly than for $j_2=\frac{1}{2}$. The relation of these limits to the classical limit will be considered in paper II.

We saw in Figs. 6–8, for $j_2=\frac{1}{2}$, that the polarization and the purity index of the localized spin state increase as we approach the parameter values $B=\pi$, $A'=\pi$. As j_1 increases, this peak in the (B, A') parameter space sharpens as a function of B, as can be seen from a comparison of Figs. 6–8 and 12. However, it is not nearly so sharp as a function of A' as is shown in Fig. 13 for $j_1=5$. A similar peak occurs for $j_2=1$, its dependence on B and A' being shown in Figs. 14 and 15. (For $j_2>\frac{1}{2}$ there is no symmetry about $A'=\pi$, and the peak occurs at $A'\approx 1.1\pi$ for $j_2=1$.) As j_1 increases this peak becomes localized within a diminishing range of B, but it is quite robust as a function of the interaction strength A'.

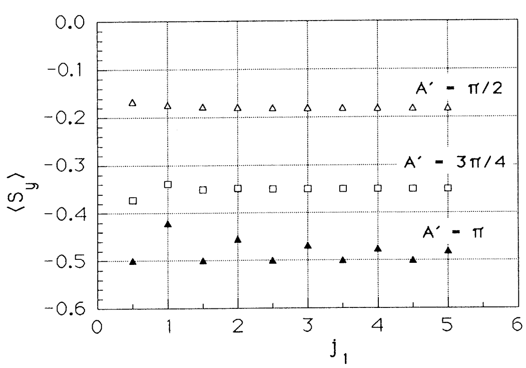


FIG. 11. Showing the limiting behavior of $\langle S_y \rangle$ at large j_1 values, for $B=\pi$ and $j_2=\frac{1}{2}$.

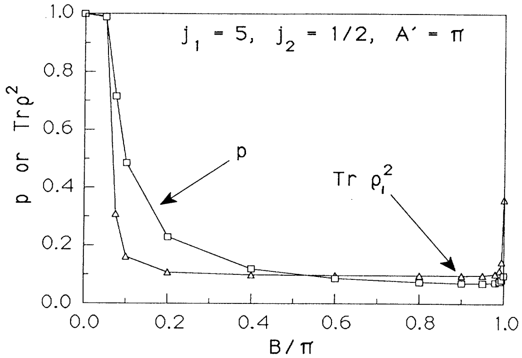


FIG. 12. Magnitude of the normalized polarization p and state purity index $\mathrm{Tr}\rho_1^2$ as functions of B, for $j_1=5$, $j_2=\frac{1}{2}$, $A'=\pi$.

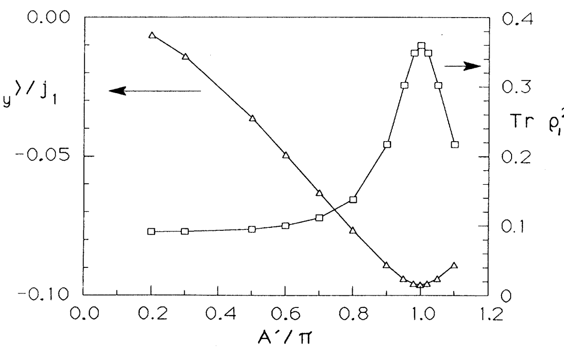


FIG. 13. Magnitude of the normalized polarization p and state purity index $\mathrm{Tr}\rho_1^2$, as functions of A', for $j_1=5$, $j_2=\frac{1}{2}$, $B=\pi$.

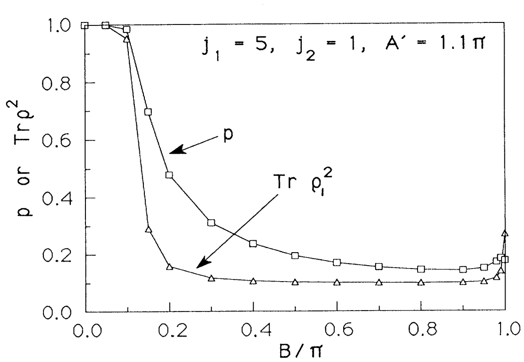


FIG. 14. Magnitude of the normalized polarization p and state purity index $\mathrm{Tr}\rho_1^2$ as functions of B, for $j_1=5$, $j_2=1$, $A'=1.1\pi$.

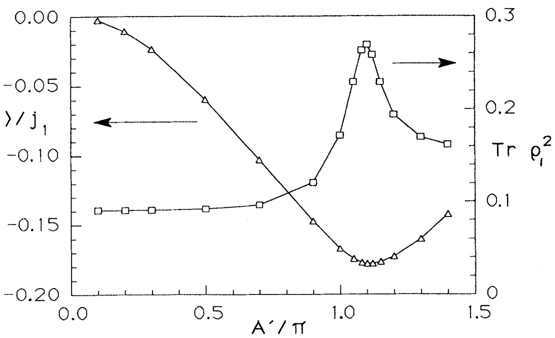


FIG. 15. Magnitude of the normalized polarization p and state purity index $\mathrm{Tr}\rho_1^2$ as functions of A', for $j_1=5$, $j_2=1$, $B=\pi$.

IV. DISCUSSION AND CONCLUSIONS

We have seen that a very simple model can give rise to an interesting and complex variety of behaviors, which derive from competition between the aligning effect of interaction with the polarized beam and the rotating effect of the external magnetic field. Depending upon the values of the parameters, the final state of the driven spin may be either a pure state or a mixed state. The transition between these two cases can be quite abrupt, although it is always continuous. In most cases the final state is independent of the initial state of the driven spin, and so the model provides an example of a quantum-state preparation procedure.

The driven spin being an open system, it may evolve dynamically from a pure state to a mixed state, or from a mixed state to a pure state. There is no undirectionality about the pure-to-mixed or mixed-to-pure evolution. For example, if there is no magnetic field, the effect of the beam is to drive the localized spin into a pure state, polarized in the beam direction. If the localized spin were initially polarized, but in a different direction, the evolution of its state would be from pure to mixed to pure. These phenomena are possible because of dynamic driving, that is, driving by another dynamical system, rather than by a prescribed external force.

Slosser, Meystre, and Braunstein [3] have treated another example of a dynamically driven quantum system. Their model consists of a harmonic oscillator driven by a beam of two-level atoms. There is considerable similarity between their model and ours: the states of a two-level atom are isomorphic to those of a spin-$\frac{1}{2}$ particle, and both a harmonic oscillator and a spin-j particle in a magnetic field have uniformly spaced energy levels. However, the interactions in the two models are different, and there is no one-to-one correspondence between the results of the two models. Slosser, Meystre, and Braunstein note that the dynamics of the model are governed by the spectrum of eigenvalues Λ in an equation analogous to our (11). They find that their eigenvalues tend to cluster into two groups: one close to unity, and one close to zero. This happens in our model for some parameter values, but it is not a general, or even typical, behavior. More commonly the eigenvalues for our model are distributed more or less uniformly over a wide range. Slosser, Meystre, and Braunstein have also found a case with an eigenvalue $\Lambda=-1$, corresponding to an oscillatory limit cycle. No such case has been discovered for our model.

A very useful feature of our model is that both its quantum and classical versions can be treated with equal ease, and hence the quantum-to-classical limit can be studied. That will be done in paper II of this series.

Although we have not studied the problem of experimentally realizing this model, two possibilities suggest themselves. A polarized atomic beam could be used to drive the spin of an atom held in a trap. Alternatively, one could work in the rest frame of the beam by shooting the "localized" particle parallel to the surface of a polarized substance, which would serve as the "beam."

[1] J. C. Camparo and R. P. Frueholz, Phys. Rev. A **43**, 338 (1991) have found a fractal attractor in the motion of a two-level atom driven by an external field. They emphasize that the inclusion of phenomenological relaxation terms is necessary for the existence of the attractor.

[2] L. E. Ballentine, *Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1990), Prob. 8-2 (pp. 169, 468, and 469).

[3] J. J. Slosser, P. Meystre, and S. L. Braunstein, Phys. Rev. Lett. **63**, 934 (1989); J. J. Slosser and P. Meystre, Phys. Rev. A **41**, 3867 (1990).