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## ARTICLES

# Two-dimensional models in quantum field theory: Reduction to the free-particle case

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We demonstrate that the generating functionals for two-dimensional models with two real scalar fields, one interacting with an external electromagnetic field and the other with coupling terms but without external fields, can be reduced to the case of the free-particle propagator when quasistatic solutions for this theory are used.

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## I. INTRODUCTION

Recently Bonato, Thomas, and Malbouisson [1] have established a connection between a two-dimensional scalar-field theory and a harmonic oscillator with "time-dependent" frequency. For this, they used quasistatic classical solutions for the field in the generating functional.

The motivation for this type of calculation is in the study of quantization around a nontrivial vacuum, which appears when we have nonlinear interaction terms of the fields, originating in the soliton solutions for the classical fields. This is thoroughly investigated by Rajaraman [2].

In the present work we intend to broaden the class of two-dimensional field theories that can be connected with nonrelativistic quantum problems. For this, we treat the case of real scalar fields in two situations. In the first, they interact with an external electromagnetic field and, in the second one, they are coupled to each other without any external field.

In the course of the demonstration we will see that, analogous to Ref. [1], when the classical solution of the scalar fields are static or depend only on one of the coordinates we can formally reduce the problem to that of a two-dimensional anisotropic harmonic oscillator with "time-dependent" frequencies. On the other hand, if the classical configuration is an approximate one, the equivalence will be with a two-dimensional *forced* anistropic harmonic oscillator with "time-dependent" frequencies, the external force being a measure of the "degree of exactness" of the solution. In fact, we show that they can be reduced to the free-particle case.

## II. THE MODELS

The first case that we treat here corresponds to the Lagrangian density

$$\begin{aligned}\mathcal{L}[\phi_1,\phi_2,\mathcal{A}_\mu]=&\tfrac{1}{2}[(\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2]\\&+e\mathcal{A}_\mu(\phi_1\partial^\mu\phi_2-\phi_2\partial^\mu\phi_1)\\&+(e^2/2)\mathcal{A}_\mu\mathcal{A}^\mu(\phi_1^2+\phi_2^2)\\&+V(\phi_1)+V(\phi_2)\ ,\end{aligned}\tag{1}$$

where $\mathcal{A}_\mu$ is an external electromagnetic field that depends only on the spatial coordinate. This imposition is necessary in order to guarantee that there will exist static configurations for the scalar fields, as can be seen from the Euler-Lagrange equations.

Performing the semiclassical expansion of the scalar fields around their classic static configuration $\bar{\phi}_1(x)$ and $\bar{\phi}_2(x)$,

$$\phi_i(x,t)=\bar{\phi}_i(x)+\chi_i(x,t)\ ,\quad i=1,2\ ,\tag{2}$$

we find that the Lagrangian density, up to surface terms, can be rewritten as $\mathcal{L}[\phi_1,\phi_2,\mathcal{A}_\mu]=\mathcal{L}[\bar{\phi}_1,\bar{\phi}_2,\mathcal{A}_\mu]+\mathcal{L}[\chi_1,\chi_2,\mathcal{A}_\mu]$, where

$$\begin{aligned}\mathcal{L}[\chi_i,\mathcal{A}_\mu]=&\tfrac{1}{2}[(\partial_\mu\chi_1)^2+(\partial_\mu\chi_2)^2]\\&+e\mathcal{A}_\mu(\chi_1\partial^\mu\chi_2-\chi_2\partial^\mu\chi_1)\\&+(e^2/2)\mathcal{A}_\mu\mathcal{A}^\mu(\chi_1^2+\chi_2^2)-\tfrac{1}{2}\omega_1^2(x)\chi_1^2\\&-\tfrac{1}{2}\omega_2^2(x)\chi_2^2+\epsilon_1(x)\chi_1+\epsilon_2(x)\chi_2\ ,\end{aligned}\tag{3}$$

in which

$$\omega_1^2(x)=-\left[\frac{\delta^2 V}{\delta\phi_1^2}\right]\Bigg|_{\bar{\phi}_1,\bar{\phi}_2}\ ,\tag{4a}$$

$$\omega_2^2(x)=-\left[\frac{\delta^2 V}{\delta\phi_2^2}\right]\Bigg|_{\bar{\phi}_1,\bar{\phi}_2}\ ,\tag{4b}$$

$$\epsilon_1(x) = -\partial_\mu\partial^\mu\bar{\phi}_1 + \left[\frac{\delta V}{\delta\phi_1}\right]\Bigg|_{\bar{\phi}_1,\bar{\phi}_2} + e^2\mathcal{A}_\mu\mathcal{A}^\mu\bar{\phi}_1 + 2e\mathcal{A}^\mu\partial_\mu\bar{\phi}_2 \,, \tag{4c}$$

$$\epsilon_2(x) = -\partial_\mu\partial^\mu\bar{\phi}_2 + \left[\frac{\delta V}{\delta\phi_2}\right]\Bigg|_{\bar{\phi}_1,\bar{\phi}_2} + e^2\mathcal{A}_\mu\mathcal{A}^\mu\bar{\phi}_2 - 2e\mathcal{A}_\mu\partial^\mu\bar{\phi}_1 \,. \tag{4d}$$

As we can see from equations (4c) and (4d) and the Euler-Lagrange equations for the Lagrangian (1), $\epsilon_i(x)$ can be thought of as being the "degree of exactness" of the static solutions $\bar{\phi}_i(x)$. Besides, the higher-order terms in the expansion fields $\chi_i(x)$ were dropped because we suppose that only first-order quantum fluctuations are important.

The generating functional of this theory in two-dimensional Euclidean space is given by

$$Z = N^{-1}\exp(-S[\bar{\phi}_1,\bar{\phi}_2,\mathcal{A}_\mu]) \times \int D\chi_1 D\chi_2 \exp\left[-\int dx\, dt\, \mathcal{L}[\chi_1,\chi_2,\mathcal{A}_\mu] + J_1(x)\chi_1 + J_2(x)\chi_2\right] , \tag{5}$$

where the normalization $N$, different from that used in Ref. [1], will be one that is usually used in quantum field theories.

$$N = \int D\phi_1 D\phi_2 \exp\left[-\int dx\, dt\, \mathcal{L}[\chi_1,\chi_2,\mathcal{A}_\mu]\right] , \tag{6}$$

and the source terms $J_i$ will depend only on one coordinate.

If we consider the simpler case in which $\omega_1^2=\omega_2^2=\omega^2$, we can achieve the goal of reducing the generating functional to free-particle propagators, by making a position dependent rotation in the scalar fields $\chi_i$:

$$\chi = \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{bmatrix}\sin\alpha(x) & \cos\alpha(x)\\ \cos\alpha(x) & -\sin\alpha(x)\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} , \tag{7}$$

where

$$\alpha(x) = e\int_x \mathcal{A}_\mu(y)dy^\mu \,. \tag{8}$$

Under this transformation the functional measure is only changed by a sign, which can be absorbed in the normalization. Substituting the transformation (7) in the Lagrangian (3) and after straightforward calculations, we obtain

$$\mathcal{L}[\psi_1,\psi_2,\mathcal{A}_\mu] = \tfrac{1}{2}[(\partial_\mu\psi_1)^2 + (\partial_\mu\psi_2)^2] - \tfrac{1}{2}\omega^2(\psi_1^2+\psi_2^2) + \tilde{\epsilon}_1\psi_1 + \tilde{\epsilon}_2\psi_2 \,, \tag{9}$$

where

$$\tilde{\epsilon}_1(x) = \epsilon_1(x)\sin\alpha - \epsilon_2(x)\cos\alpha \,, \tag{10a}$$

$$\tilde{\epsilon}_2(x) = \epsilon_1(x)\cos\alpha - \epsilon_2(x)\sin\alpha \,. \tag{10b}$$

In the second example, the Lagrangian density is given by

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu\chi_1)^2 + (\partial_\mu\chi_2)^2] - \tfrac{1}{2}[\omega^2(\chi_1^2+\chi_2^2)] + \lambda(x)\chi_1\chi_2 + \epsilon_1\chi_1 + \epsilon_2\chi_2 \,. \tag{11}$$

Note that now $V(\phi_1,\phi_2)$ has cross terms, but the restriction on the frequencies is maintained.

In this case we see that the fields $\chi_i$ stay coupled. If we make a rotation of $\pi/4$ in these fields as given below,

$$\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} , \tag{12}$$

the Lagrangian (15) decouples as

$$\mathcal{L}[\eta_1,\eta_2] = \tfrac{1}{2}(\partial_\mu\eta_1)^2 + \tfrac{1}{2}(\partial_\mu\eta_2)^2 - (\omega_-^2/2)\eta_1^2 - (\omega_+^2/2)\eta_2^2 + (1/\sqrt{2})(\epsilon_1+\epsilon_2)\eta_1 + (1/\sqrt{2})(\epsilon_1-\epsilon_2)\eta_2 \,, \tag{13}$$

where $\omega_\pm^2(x)=\omega^2(x)\pm\lambda(x)$. Thus, as in the previous example, we have reduced the problem to that of two free scalar fields.

## III. THE EQUIVALENCE WITH THE FREE PARTICLE

In this section we will show that the harmonic-oscillator propagator can be formally reduced to that of a free particle by the use of a suitable transformation.

The above examples in fact were simplified to the treatment of a generating functional of the type

$$Z[J] = N^{-1}\int \mathcal{D}\xi \exp\left[-\int dx\, dt\, \tfrac{1}{2}(\partial_\mu\xi)^2 - [\omega^2(x)/2]\xi^2 + [\epsilon(x)+J(x)]\xi\right] , \tag{14}$$

where $N=Z(J=0)$.

Since there is a formal equivalence between the above equation and expression (2) of Ref. [1], we shall avoid unnecessary mathematical labor by using their final result:

$$Z[J]=N^{-1}\prod_{\substack{m,n\\ m\neq 0}}\left[\frac{2\pi}{E_{1n}+E_{2m}}\right]^{1/2}\int \mathcal{D}q\,\exp\left\{-\tfrac{1}{2}\int_{-L/2}^{L/2}dx\left[\left[\frac{dq}{dx}\right]^2-\omega^2(x)q^2(x)\right]\right.$$
$$\left.+\sqrt{T}\int_{-L/2}^{L/2}dx\,[\epsilon(x)+J(x)]q(x)\right\}. \tag{15}$$

This expression was obtained through an expansion of $\xi(x,t)$ in terms of eigenfunctions of the operator $\partial^2+\omega^2(x)$ with $-L/2\le x\le L/2$ and $-T/2\le t\le T/2$. On the other hand, $q(x)$ is defined by

$$q(x)=T^{-1/2}\int_{-T/2}^{T/2}\xi(x,t)dt\ .$$

$E_{1n}$ and $E_{2m}$ are the eigenvalues of the Schrödinger equation in the variables $t$ and $x$, respectively.

At this point we are able to reach our aim. In the numerator, we have the propagator of an oscillator with "time-dependent" frequency with external driving force.

Now, if we want to map this problem into that of the free particle we can use the same method applied by de Souza Dutra [3].

First we make the transformation

$$u(x)=q(x)+v(x)\ , \tag{16}$$

with the requirement

$$\frac{d^2v}{dx^2}+\omega^2(x)v=-[\epsilon(x)+J(x)]\ . \tag{17}$$

Then the exponent in Eq. (15) reduces to

$$-\frac{1}{2}\left\{\int_{-L/2}^{L/2}dx\left[\left[\frac{du}{dx}\right]^2-\omega^2(x)u^2(x)\right]\right\}+F(u,v,x)\Big|_{-L/2}^{L/2}\ , \tag{18}$$

where

$$F(u,v,x)=\tfrac{1}{2}\frac{dv}{dx}[v(x)-2u(x)]-\tfrac{1}{2}\int_x[\epsilon(y)+J(y)]dy\ . \tag{19}$$

Making use of the transformation [4]

$$r=\mu\left[\frac{du}{dx}\right]^{1/2}\sec\mu(x)\ ,\quad s=\tan\mu(x)\ , \tag{20}$$

where $\mu(x)$ is constrained by the relation

$$\left[\frac{d\mu}{dx}\right]^2\theta^2=1\ , \tag{21a}$$

with $\theta$ satisfying

$$\frac{d^2\theta}{dx^2}+\omega^2(x)\theta(x)=\theta^{-3}\ , \tag{21b}$$

it is easy to show that the exponent of Eq. (15) reduces to

$$-\frac{1}{2}\int_{-L/2}^{L/2}\left[\frac{dr}{ds}\right]^2ds+F(u,v,x)\Big|_{-L/2}^{L/2}+G(\theta,x)\Big|_{-L/2}^{L/2}\ , \tag{22}$$

where

$$G(\theta,x)=\tfrac{1}{2}u^2\left[\sin[2\mu(x)]-\frac{(d\theta/dx)\theta^{-1}}{(ds/dx)}\right]. \tag{23}$$

With this we have achieved our goal. However some observation must be made.

If we use the usual normalization given by Eq. (6) and apply the same procedure as for the numerator, we obtain the ratio between Eq. (18) and

$$-\frac{1}{2}\left\{\int_{-L/2}^{+L/2}dx\left[\left[\frac{du}{dx}\right]^2-\omega^2(x)u^2(x)\right]\right\}+\mathcal{F}(m,v',x)\Big|_{-L/2}^{+L/2}\ , \tag{24}$$

where

$$\mathcal{F}(u,v',x)=\frac{1}{2}\frac{dv'}{dx}[v'(x)-2u(x)]-\tfrac{1}{2}\int dy\,\epsilon(y)v'(y)\ , \tag{25}$$

with $v'(x)$ satisfying

$$\left[\frac{dv'(x)}{dx}\right]^2+\omega^2(x)v'=-\epsilon(x)\ . \tag{26}$$

Then this ratio permits us to cancel the exponentials that contains oscillators with time-dependent frequencies in the variables $u(x)$, that appear in the numerator as well as in the denominator. In other words, we have

$$Z[J]=\exp\left[F(u,v,x)\Big|_{-L/2}^{+L/2}\right]\times\{\exp[\mathcal{F}(u,v',x)]|_{-L/2}^{+L/2}\}^{-1}\ . \tag{27}$$

Now we do not need to make the transformation (20) that leads us to the free-particle propagator. That will be necessary only if we use the same normalization of Bonato, Thomaz, and Malbouisson. In this case we have

$$N=\prod_{\substack{m,n\\ m\neq 0}}\left[\frac{2\pi}{\epsilon_{1n}+\epsilon_{2m}}\right]^{1/2}\times\mathcal{D}q\,\exp\left[-\tfrac{1}{2}\int_{-L/2}^{L/2}dx\left[\frac{dq}{dx}\right]^2\right], \tag{28}$$

where $\epsilon_{1n}=(2\pi n/T)^2$ and $\epsilon_{2m}=(2\pi m/L)^2$; $m,n\in\mathbb{Z}$, being $\epsilon_{1n}=E_{1n}$ and $\epsilon_{2n}\neq E_{2n}$. Therefore we keep

$$Z[J]=\prod_{\substack{m,n\\ m\neq 0}}\left[\frac{2\pi}{E_{1n}+E_{2m}}\right]^{1/2} K\int \mathcal{D}r(s)\exp\left[-\tfrac{1}{2}\int_{-L/2}^{L/2}ds\left[\frac{dr}{ds}\right]^2\right]$$
$$\times\exp\left[F(u,v,x)\Big|_{-L/2}^{+L/2}-G(\theta,x)\Big|_{-L/2}^{+L/2}\right]$$
$$\times\left\{\prod_{\substack{m,n\\ m\neq 0}}\left[\frac{2\pi}{\epsilon_{1n}+\epsilon_{2m}}\right]^{1/2}\int \mathcal{D}q\,\exp\left[-\tfrac{1}{2}\int_{-L/2}^{L/2}dx\left[\frac{dq}{dx}\right]^2\right]\right\}^{-1}. \tag{29}$$

The factor $K$ arises due to the change in the functional measure when we make the transformation (20) and it is given by

$$K=\frac{\partial r}{\partial u}\bigg|_{-L/2}\frac{\partial r}{\partial u}\bigg|_{+L/2}. \tag{30}$$

## IV. CONCLUSIONS

In this work we have formally reduced the problem of a complex scalar field interacting with an external electromagnetic field and another one with self-interaction but without an electromagnetic field, to that of a quantum-mechanical one. In order to apply this approach in a particular case, one has to pay the price of obtaining the classical quasistatic configurations exactly and to solve the equation of a driven harmonic oscillator with time-dependent frequency.

The first model that we have considered here is special, in the sense that the form of the potential is restricted by the imposition $\omega_1^2=\omega_2^2=\omega^2$. However, the general case remains to be solved.

In general, the equations that we must solve are not easy, but there are solutions for some particular cases. For example, we can quote that treated by Bonato, Thomaz, and Malbouisson [5] that has appeared recently in the literature. In this paper they work with the potential $V(\phi)=g(\phi^2-a^2)^2$ in the presence of a constant source $J$, obtaining the static bounce solution.

Finally, it is necessary to say that the generating functional that appears in this work is not the usual one. The generating functional presented here is for the time-averaged correlation functions, in contrast to that used in Ref. [1] which is a generating functional for the time- and space-averaged correlation functions.

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