Influence of noise on the mean lifetime of chaotic transients

Marek Franaszek*
Institut für Festkörperforschung, Forschungszentrum Kernforschungsanlage Jülich, D-5170 Jülich, Germany
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Chaotic transients resulting after a boundary crisis are examined in the presence of external, additive noise. Depending on its amplitude, the mean duration of chaotic evolution may be elongated or shortened.

Chaotic transients have been found in many dissipative dynamical systems in which a boundary crisis has taken place [1–3]. In contrast to permanent chaos, transient behavior occurs only on a long, but finite, time scale, and is related to the existence of a nontrivial fractal repeller in phase space. Starting from different initial points scattered uniformly inside a neighborhood of the repeller, one can observe an ensemble of different trajectories. Their durations are distributed exponentially according to

$$M(n)=M_0 e^{-\kappa n} , \qquad (1)$$

where $M(n)$ is the number of trajectories that have not yet escaped from the given neighborhood of the repeller after n iterations, and M_0 is the number of initial points. Coefficient κ is called the escape rate and is connected with the mean lifetime $\langle n \rangle$ as $\kappa=\langle n \rangle^{-1}$. This quantity is related to other dynamical characteristics such as the generalized dimensions, entropies, or Lyapunov exponents. The above invariants, however, are not easily extracted from noisy experimental time series.

Contrary to this, the mean lifetime $\langle n \rangle$ can easily be determined in real experiments by means of the ensemble method, introduced in Ref. [3]. Investigating an ensemble of transient trajectories started from many initial points distributed uniformly in the region containing a repeller, one can simply check the duration of every single trajectory and make sufficient statistics, satisfying Eq. (1).

Since every real experiment is performed in the presence of random noise, one should be aware of its influence on the measured quantity. Therefore, in this paper we study the problem in computer simulations in which the level of noise can be kept under strict control.

Two systems are investigated: The one-dimensional (1D) logistic map

$$x_{n+1}=ax_n(1-x_n)+\sigma\eta_n , \qquad (2)$$

and the two-dimensional (2D) Hénon map

$$x_{n+1}=1-ax_n^2+y_n+\sigma\zeta_n, \quad y_{n+1}=by_n+\sigma\xi_n , \qquad (3)$$

where σ stands for amplitude of noise and η_n,ζ_n,ξ_n are numbers from a pseudorandom generator of homogeneous distribution on the interval $(-\frac{1}{2},\frac{1}{2})$. Parameters a,b are taken slightly above their critical values a^*,b^* for which a boundary crisis takes place. For the logistic map $a^*=4$ and transient trajectories escape to $-\infty$, which can be viewed as an attracting single point. In the Hénon map there are different boundary crises. We choose a particular one in which a four-piece chaotic attractor is destroyed and its basin attached to the basin of a six-piece chaotic attractor. Both attractors coexist before the crisis, which takes place for $a^*=1.080\,744\,879\ldots$, $b^*=0.3$. Beyond the crisis, transients lead to a six-piece chaotic attractor on which the system stays forever [2].

As a condition for interrupting the iteration process a geometrical criterion is used: for the 1D case iterations are stopped if $x<-0.2$, while for the 2D case we check every sixth iteration, and if both points (x_{i-6},y_{i-6}) and (x_i,y_i) are within the strip of width 3σ around the same piece of attractor, the iterations are stopped. The control parameter a in both maps is chosen close to its critical value a^*, where $\langle n \rangle$ is long, in order to ensure the existence of relatively large and clearly visible noise-induced changes of $\langle n \rangle$.

Figure 1 shows the results obtained for the logistic map with fixed parameter a and three distinct values of σ. The middle line with dots corresponds to the noiseless case $\sigma=0$; the slope of this line is equal to the escape rate $\kappa(0)$. Two facts should be noted: (i) the exponential distribution of lifetimes, Eq. (1), is excellently fulfilled for a purely deterministic as well as a noisy repeller; (ii) ac-

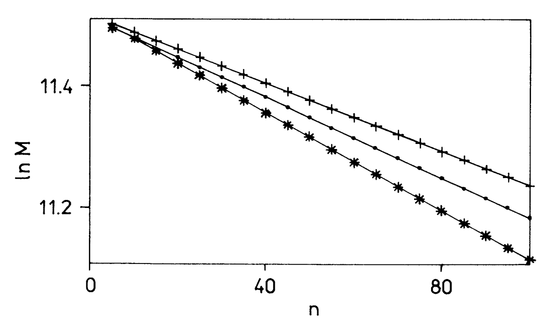

FIG. 1. Distribution of lifetimes $\ln M(n)$ vs iteration number n for the logistic map at fixed control parameter $a=4.0001$; number of initial points $M_0=10^6$. The slope is equal to the escape rate κ. Values of noise amplitude σ are as follows (from top to bottom): 10^{-4}, 0, 5×10^{-4}.

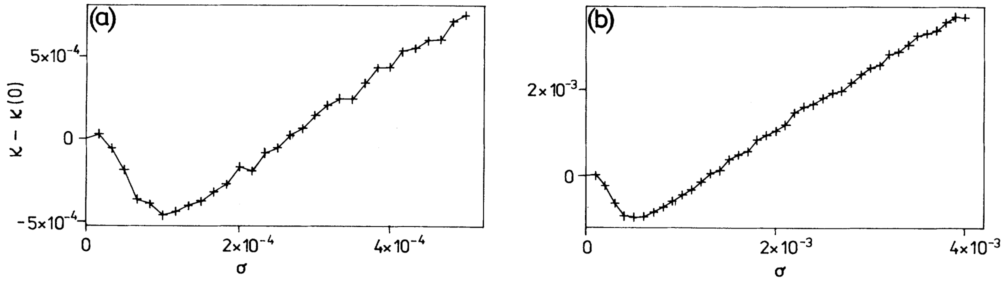

FIG. 2. Dependence of escape rate κ on noise amplitude σ for the logistic map with fixed control parameter: (a) $a=4.0001$; (b) $a=4.0005$. The difference $\kappa(\sigma)-\kappa(0)$ is plotted on the vertical axis, where $\kappa(0)$ is the escape rate in the absence of noise. Notice the shift in the position of the minimum with change of parameter a.

cording to its amplitude, noise modifies the mean lifetime in such a way that one can observe both longer and shorter mean durations of transients. Thus, for fixed system parameters one can expect a nonmonotonic dependence of the escape rate κ on noise amplitude σ. Indeed, systematic investigations confirm this, as shown in Fig. 2(a). On the vertical axis the difference $\kappa(\sigma)-\kappa(0)$ is plotted, and therefore all points below zero correspond to the elongated mean lifetime while those above zero refer to shorter transients. The longest mean duration (minimum on the plot) is about 20% greater than the noiseless one. The plot $\kappa(\sigma)-\kappa(0)$ has a similar shape for other values of the control parameter a [see Fig. 2(b)]. Again the longest transients are about 20% longer than the noiseless ones for a given value of a. Runs with other values of a seem to suggest that the critical value of the noise amplitude σ^*, for which the longest transient occurs, depends on the a parameter of the map roughly as $\sigma^*\approx(a-a^*)$. The same investigations were also performed for a 2D system. As an example, we demonstrate the dependence $\kappa(\sigma)-\kappa(0)$ vs σ for fixed a in Fig. 3. Again, in some range of σ, noise has a tendency to elongate chaotic transients, and the longest $\langle n \rangle$ is about 18% longer than the noiseless one. It is a little surprising that the maximum relative elongation $[\langle n(\sigma^*)\rangle-\langle n(0)\rangle]/\langle n(0)\rangle$ is oscillating in a narrow range around the same constant value (≈ 0.2) in both 1D and 2D systems and for different values of the control parameter a. It should be added that for amplitudes σ larger than those shown in the figures, there is already a monotonic decrease of mean lifetime $\langle n \rangle$ for both maps.

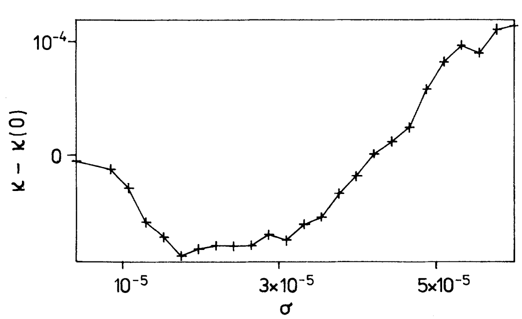

FIG. 3. The same dependence as in Fig. 2 but for the Hénon map with fixed parameters $a=1.080\,750$ and $b=0.3$.

The above results suggest a general tendency for small noise amplitude to stabilize chaotic transients, and just the opposite effect for large noise amplitude. Here, the term "small" or "large" has its sense for fixed parameters of the system and one should be aware that at other values of the system parameters the same noise may have a different influence on the same physical system. In other words, instead of a pair of independent parameters a and σ one can use a single dimensionless parameter $\rho=\sigma/(a-a^*)$. Its value indicates whether the observed mean lifetime is longer or shorter in comparison with the noiseless one. In the systems investigated here, we found a critical value ρ_0 above which $\kappa(\sigma)-\kappa(0)$ is always positive. It equals about 2.5 and is roughly the same for both the 1D and 2D cases.

It should be stressed that we investigated the effect of external noise on the mean lifetime $\langle n \rangle$, which is the averaged, statistical quantity. Of course, a single transient trajectory can behave in exactly the opposite way but similar investigations performed with very small statistics for other dynamical systems give qualitatively the same results [4].

Finally, let us make one more general comment. The escape rate κ is related to other dynamical characteristics. For the simplest 1D case [3] we have

$$\kappa=(1-D)\Lambda\ , \tag{4}$$

where Λ is the Lyapunov exponent and D stands for the information dimension with respect to the natural invariant measure on the repeller. From the study of permanent chaos it is well known that noise destroys details of fractal geometry below a length scale comparable with the amplitude σ [5]. The same also holds for repellers. Thus, the hierarchical structure survives for a length scale greater than σ and in this range of length one can estimate dimension D correctly. On the other hand, there is no such possibility in determining the value of the escape rate. As illustrated, Eq. (1) is excellently fulfilled for the noisy repeller in a reasonable range of n but with the value of κ being noise dependent. The calculation of Λ from experimental time series is probably the most problematic and least confidential procedure [6], but some authors [7] found numerically a sensitivity of the greatest Lyapunov exponent on the level of external noise. Analytic investigations of fully developed chaotic maps [8] reveal nonmonotonic dependence of $\Lambda(\sigma)$ that appears to have a shape just opposite that of $\kappa(\sigma)$, i.e., the Lyapunov exponent Λ increases for small σ and decreases for larger σ. One can also expect a similar dependence of $\Lambda(\sigma)$ for slightly larger values of the control parameter where transient chaos exists. Changes of the Lyapunov exponent due to noise may thus be compensated by changes of the escape rate κ. Consequently, Eq. (5) may also be correct in the case of noisy repellers.

Summarizing, we can conclude once again that external noise can either stabilize or damage transient chaos. The particular kind of reaction depends on neither the value of the noise amplitude σ nor the system parameter a independently, but on the dimensionless parameter $\rho=\sigma/(a-a^*)$. For $\rho<\rho_0$ noise always makes transient chaos more persistent, while for $\rho>\rho_0$ the effect is just the opposite. The critical value ρ_0 found in numerical investigations is nearly the same for 1D and 2D systems.

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*On leave from: Higher Educational School, Podchorążych 2 Kraków, Poland.

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