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Stochastic dynamics of time-summating binary neural networks

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An analysis is given of the stochastic dynamics of time-summating binary neural networks. Such networks have a memory or trace of their previous output activity reaching back to some initial time. Particular attention is given to a class of networks based on a discrete time version of leaky-integrator shunting networks. The stochastic dynamics is formulated as a linear Markov process describing the evolution of densities on the infinite-dimensional space of neuronal activation states. Using certain results from the theory of linear Markov operators, due to Lasota and Mackey [*Probabilistic Properties of Deterministic Systems* (Oxford University, Oxford, 1986); Physica D **28**, 143 (1987)], conditions are derived for asymptotic stability in which the network converges to a unique limiting density. Moreover, the limiting density is shown to be a differentiable function of the parameters of the network such as the weights and decay factors. Finally, dynamical mean-field equations are derived that have periodic and chaotic solutions, implying a breaking of asymptotic stability in the thermodynamic limit.

I. INTRODUCTION

The formal neuron of Hopfield-Little [1,2] attractor networks is based on the McCullogh-Pitts [3] model of a real neuron. In this model (i) the neuron is viewed as a binary-threshold element whose output is either on or off depending on whether or not its membrane potential (local field) exceeds some threshold and (ii) the membrane potential at any given time is equal to the linear sum of inputs to the neuron at the previous time step. [We shall refer to a network of neurons with output function (i) as *binary* and use the term *standard* if they satisfy (ii).] One of the most successful analytic tools in the study of standard binary networks has been the use of statistical-mechanical techniques. For example, both linear Markov chain theory [4,5] and, in the case of large networks, spin-glass mean-field theory [6] have been applied to the analysis of the stochastic dynamics of such networks. Moreover, Gardner [7] has extended spin-glass methods to evaluate the space of interactions between the neurons and thus determine quantities such as the maximum capacity for the storage of static patterns in a Hopfield network.

However, there is growing interest in discrete-time networks that have some memory of previous input-output activity which extends beyond a single time step [8–19]. One way to achieve this is to modify (ii) so that the membrane potential is a slowly decaying function of time with decay rate $\gamma < 1$, say. Each neuron then maintains an activity trace of all previous inputs to that neuron. (We shall use the term *time summating* to denote a network of such neurons.) In this paper we use statistical-mechanical techniques to analyze the stochastic dynamics of a time-summating binary network. The application of Gardner's approach to temporal sequence storage in these networks has been developed elsewhere [17].

There are a number of reasons why time-summating networks are of interest. First, from a biological viewpoint, standard binary networks neglect the temporal aspects of the process by which real neurons integrate their inputs. For example, recent neurophysiological evidence [20] suggests that the persistence of activity on the cell membrane of certain neurons in the cortex occurs over hundreds of milliseconds. Since a single time step is of the order 1 ms, this suggests that γ could be close to unity in such cases. The dynamics of the membrane potential of a neuron can be described in some detail by so-called leaky-integrator shunting equations [21]. (More correctly, these differential equations describe the behavior of a local patch of a neuron's cell membrane [22]; however, it is possible to take account of the spatial structure of the neuron using compartmental models [23].) Bressloff and Taylor [10] have constructed a model of a time-summating binary network based upon a discrete-time approximation of these equations. An additional feature of this particular model is that the decaying membrane potential is a *nonlinear* function of the inputs to the neuron. This nonlinearity, which is distinct from the one associated with thresholding, is a consequence of shunting terms in the leaky-integrator differential equations. These terms incorporate changes of the membrane potential induced by an input from another neuron which depends on the size of the deviation of the membrane potential from some local, fixed resting potential.

Second, from a dynamical systems viewpoint, time-summating networks can display a wide range of behavior, even at the single-neuron level. (This is also true of real neurons [24].) For example, the response characteristics of a single neuron of the form considered in Ref. [10] exhibits both frequency locking and chaos [11]; the source of the chaotic behavior is the membrane potential's nonlinear dependence on inputs, which for a certain range of parameter values leads to a positive Liapunov exponent. Chaos is also observed in the single-neuron model of Aihara, Takabe, and Toyoda [12] in which the binary-threshold output function is replaced by a sigmoid function. Both of these models provide the basis for developing discrete-time versions of coupled oscillator networks [25]. The dynamics of time-summating networks with continuous-valued outputs has also been studied by Marcus and Westervelt [19].

Finally, from a computational viewpoint, time-summating networks have certain advantages over standard ones in the processing of temporal sequences. For example, the deterministic dynamics of a fully connected standard binary network with parallel update is characterized by a unique, single time step, state transition matrix [8]. Consequently, it can only store simple sequences in which a succession of states is unambiguously defined. On the other hand, the ambiguities arising from repeated patterns can be resolved by time-summating networks, provided incoming activity is held over a long enough period [13,14]. Moreover, it can be shown that the maximum storage capacity can be enhanced in certain cases [17]. (We note that an alternative approach to temporal sequence storage has been developed in Hopfield-like networks [26–28]. This involves the introduction of additional couplings between the neurons which have signal transmission delays associated with them. In the case of parallel dynamics, such neurons have a memory of previous inputs which extends over a time period equal to the number of delay lines between neuron pairs [28].) Time-summating networks have also been used in feedforward networks and allow simple on-line learning algorithms to be developed [14–16]. These deal with temporal sequences directly in the time domain so avoiding the need for mapping the temporal sequences into spatial patterns of much higher dimension.

Having indicated the importance of time-summating networks, we turn to the contents of this paper. We shall restrict our discussion of stochastic dynamics to the leaky-integrator model of Bressloff and Taylor [10], although the analysis can also be applied to related models such as the chaotic networks of Aihara, Takabe, and Toyoda [12] and the iterated-map networks of Marcus and Westervelt [19]. Moreover, we shall consider the affects of additive white noise. For zero decay rate ($\gamma=0$), the stochastic dynamics is formulated in terms of a homogeneous Markov chain defined on the discrete space of binary outputs $\{0,1\}^N$, where N is the number of neurons in the network, and shown to include the Little model [1] as a special case. If the Markov chain is irreducible then, for finite N, the long-term behavior of the network may be analyzed in terms of the Perron-Frobenius theorem [29,5]. That is, in the absence of any periodicity constraints, the system converges to a unique probability distribution on $\{0,1\}^N$, which in the case of the Little model is a Gibbs distribution [4]. On the other hand, for nonzero decay rate ($\gamma>0$), the dynamics is expressed in terms of a Markov process on the space of activation states $\mathbb{R}^N$; the corresponding dynamics on the space of binary outputs is now non-Markovian. The Markov process is generated by a linear Markov operator [30] acting on the space of probability densities on $\mathbb{R}^N$ and is analyzed using the results of Lasota and Mackey [31]. In particular, we derive conditions under which the Markov process converges to a unique limiting probability density on $\mathbb{R}^N$ (asymptotic stability). Moreover, we show that this density is a differentiable function of the network's parameters such as the decay rates and connection weights.

Asymptotic stability implies ergodicity in the sense that time averages are independent of initial conditions and may be reexpressed as ensemble averages over the limiting density. In the spin-glass approach to binary networks [6] input patterns are presented to a network by specifying the initial configuration of the neurons for fixed external inputs. Therefore, for applications such as associative memory, it is necessary to formally break asymptotic stability by taking the thermodynamic limit $N\rightarrow\infty$ in such a way that ergodicity no longer holds. Otherwise, all input patterns would generate the same average long-term behavior. In the thermodynamic limit state space decomposes into disjoint invariant subsets each characterized by its own limiting density; within each disjoint region the stochastic dynamics is still asymptotically stable. The particular subset over which ensemble averages are calculated is then determined by the initial configuration. At the macroscopic level the network converges to a fixed-point solution of dynamical mean-field equations. In the last part of this paper we discuss the thermodynamic limit for a time-summating network. In particular, we derive a dynamical mean-field equation for a homogeneous network with a noise distribution identical to that of the Little model. The mean-field equation determines the evolution of the mean output activity of the network and in the low-temperature regime exhibits periodic and chaotic solutions, implying a breaking of asymptotic stability in the thermodynamic limit.

The organization of the paper is as follows. In Sec. II we discuss standard binary networks and the Little model, emphasizing features relevant to the more general case. In Secs. III and IV, respectively, we consider the deterministic and stochastic dynamics of time-summating binary networks, based on the leaky-integrator model. In Sec. V we analyze the parameter dependence of the limiting density for asymptotically stable networks. Finally, in Sec. VI we develop the dynamical mean-field theory.

II. STANDARD BINARY NETWORKS

Consider a network of N McCullogh-Pitts binary threshold neurons [3] and denote the output of the ith neuron at each discrete time step $m=0,1,2,\ldots$ by $a_i(m)\in\{0,1\}$. The binary output indicates whether the neuron has, or has not, fired an electrochemical potential (*action potential*). The state of the network is specified by the binary vector $\underline{a}(m)=(a_1(m),\ldots,a_N(m))$. The neurons are connected by synaptic junctions of weight w_{ij}, which determine the size of the input to neuron i arising from an action potential fired by neuron j. In this simple model the *membrane potential* $V_i(m)$ of neuron i at time m is equal to the sum of all the inputs received at the previous time step,

$$V_i(m)=\sum_{j=1}^{N} w_{ij}a_j(m-1)\ , \quad i=1,\ldots,N\ . \tag{1}$$

Each neuron fires whenever its membrane potential exceeds a threshold value h_i,

$$a_i(m)=\Theta(V_i(m)-h_i)\ ,\quad i=1,\ldots,N \tag{2}$$

where $\Theta(x)=1$ if $x \geq 0$ and $\Theta(x)=0$ if $x<0$. Equations (1) and (2) determine the dynamics on the discrete space of binary outputs $\{0,1\}^N$. We shall assume throughout that the neurons are updated synchronously, since this is simpler from the dynamical systems viewpoint.

The number of possible states of a binary network is finite (equal to 2^N). Therefore, in the case of synchronous dynamics, for which there is a unique transition from one state to the next (in the absence of noise), the long-term behavior is cyclic. This follows from the fact that a finite-state system must return to a state previously visited after a finite number of time steps ($\leq 2^N$). A consequence of this is that standard binary networks can only be used to store temporal sequences of patterns, corresponding to particular cycles, provided a given pattern only occurs once per cycle, e.g., the two-cycle $ABAB\ldots$ consisting of the alternating patterns $A=(a_1,\ldots,a_N)$, $B=(b_1,\ldots,b_N)$. For in order to generate a sequence of p different patterns one needs only to learn the p transitions from one pattern to its successor in the sequence. On the other hand, if a pattern occurs twice per cycle, say, then there is an ambiguity as to which is the successor of this pattern, e.g., pattern A in the sequence $AABAAB\ldots$. One approach to storing these more complex sequences involves keeping a memory of the output state of the network over a number of previous time steps [13], that is, incorporating some form of extended time summation.

Noise is usually introduced into binary networks in terms of the conditional firing probability $p(i|\underline{a})$, which determines the likelihood of neuron i firing given that the state of the network at the previous time step is $\underline{a}$. In the Little model [1] this probability has the explicit form

$$p(i|\underline{a})=\frac{1}{1+\exp\left[-\beta\left[\sum_{j=1}^{N} w_{ij}a_j-h_i\right]\right]}\ , \tag{3}$$

where β^{-1} is a *temperature* parameter. Introducing the probability $P_m(\underline{a})$ that the state of the network at time m is $\underline{a}$, we may describe the dynamical evolution of the network in terms of the homogeneous Markov chain

$$P_{m+1}(\underline{b})=\sum_{\underline{a}} Q_{\underline{b}\,\underline{a}}P_m(\underline{a})\ , \tag{4}$$

where $Q_{\underline{b}\,\underline{a}}$ is the time-independent transition probability of going from state $\underline{a}$ to state $\underline{b}$ in one time step, and which satisfies

$$Q_{\underline{b}\,\underline{a}}=\prod_{i=1}^{N}\{b_ip(i|\underline{a})+(1-b_i)[1-p(i|\underline{a})]\}\ . \tag{5}$$

Since the Markov chain generated by Eqs. (3) and (5) is irreducible (every state may be reached, after a finite number of time steps, from every other state) and assuming that N is finite, we may apply the Perron-Frobenius theorem [29].

If $\underline{Q}$ is the transition matrix of a finite irreducible Markov chain with period d then (i) the d complex roots of unity $\lambda_1=1,\lambda_2=\omega,\ldots,\lambda_d=\omega^{d-1}$, where $\omega=e^{2\pi i/d}$ are eigenvalues of $\underline{Q}$ and (ii) the remaining eigenvalues $\lambda_{d+1},\ldots,\lambda_N$ satisfy $|\lambda_j|<1$.

A Markov chain is said to have period d if, for each state $\underline{a}$, the probability of returning to $\underline{a}$ after m time steps is zero unless m is an integer multiple of d. If $\beta^{-1}>0$ then the Markov chain (4) has $d=1$ (aperiodic) so that there is a nondegenerate eigenvalue of $\underline{Q}$ satisfying $\lambda_1=1$ while all others lie inside the unit circle. By expanding the solution of Eq. (4) in terms of the generalized eigenvectors of $\underline{Q}$, it then follows that there is a unique limiting distribution $P_\infty(\underline{a})$ such that

$$\lim_{m\to\infty} P_m(\underline{a})=P_\infty(\underline{a})\ , \tag{6}$$

independently of the initial distribution, where $P_\infty(\underline{a})$ is the unique eigenvector of $\underline{Q}$ corresponding to the eigenvalue λ_1 [5]. Equation (6) implies that time averages are independent of initial conditions and may be replaced by ensemble averages over the limiting distribution $P_\infty(\underline{a})$. In other words, for any well-behaved state variable X, we have the ergodicity condition

$$\lim_{M\to\infty}\sum_{m=0}^{M}\frac{X(\underline{a}(m))}{M}=\sum_{\underline{a}} P_\infty(\underline{a})X(\underline{a})\ . \tag{7}$$

Another consequence of Eq. (6) is that the network cannot display any long-range order in time since any injection of new information only produces fluctuations about the limiting distribution and eventually disappears. Therefore, to operate the network as an associative memory it is necessary to take either the low noise limit $\beta\to\infty$, so as to include the many attractors of the deterministic system, or the thermodynamic limit $N\to\infty$. The latter leads to a breaking of the ergodicity condition (6) and is the basis of the spin-glass approach to pattern storage in Hopfield-Little networks [6] (see Sec. VI).

It is useful to reformulate the Little model in terms of a binary network with additive noise [32]. That is, we replace Eqs. (1) and (3) by the stochastic equation

$$V_i(m)=\sum_{j=1}^{N} w_{ij}a_j(m-1)+\eta_i(m-1)\ , \tag{8}$$

where η_i is an additive white-noise term. In particular, the sequence $\{\eta_i(m), m\geq 0\}$ is an independent random process with a time-independent probability density ρ_i on $\mathbb{R}$. Each density ρ_i is an L^1 function on $\mathbb{R}$ such that $\rho_i(\eta)>0$ and $\|\rho_i\|=1$ where, as usual, $\|\rho_i\|$ denotes the L^1 norm of the function ρ_i [33],

$$\|\rho_i\|=\int_{\mathbb{R}}|\rho_i(x)|dx\ . \tag{9}$$

Given a density ρ_i, then the ρ_i measure $\mu_{\rho_i}(A)$ of the subset A in $\mathbb{R}$ is defined by

$$\mu_{\rho_i}(A)=\int_A \rho_i(x)dx \tag{10}$$

and may be interpreted as the probability that, at any time step n, $\eta_i(m)\in A$. Equation (8) generates the conditional firing probability

$$p(i|\underline{a})=\int_{-\infty}^{\infty}\rho_i(\eta_i)\Theta\left[\sum_{j=1}^{N}w_{ij}a_j-h_i+\eta_i\right]d\eta_i \qquad (11)$$

and a corresponding Markov chain according to Eqs. (4) and (5). Equation (11) is equivalent to Eq. (3) on taking

$$\rho_i(\eta_i)=\frac{d}{d\eta_i}\frac{1}{1+e^{-\beta\eta_i}}\ . \qquad (12)$$

However, we are free to choose any density ρ_i. For fixed weights and thresholds, properties of the transition matrix $\underline{Q}$ such as its period and whether or not it is irreducible, then depend on the nature of ρ_i. For example, the transition matrix generated by Eq. (12) is irreducible because the nonzero support of each ρ_i is the whole real line, i.e., there is a nonzero probability of η_i taking an arbitrarily large value. However, if the support of ρ_i is restricted to the interval $[-\epsilon,\epsilon]$, $\epsilon>0$, say, then for sufficiently small ϵ the transition matrix will be reducible and will have nontrivial period. This should be contrasted with the low-noise case, $\beta\gg 1$, for a density satisfying Eq. (12), in which the Markov chain (4) is irreducible and aperiodic, but the relaxation times are large. Consequently, cyclic solutions of the underlying deterministic system may persist for long observation time [5].

The additive noise term η_i may be interpreted as a random fluctuation of the threshold h_i or, equivalently, as a random external input. Note that in real neurons a significant source of noise is associated with the quantal release of chemical transmitters at synapses [34]. This may be incorporated into Eq. (8) in the form of multiplicative white noise, by replacing each fixed weight w_{ij} by the random process $\{w_{ij}(n),n\geq 0\}$ with associated probability density ρ_{ij}. The corresponding conditional firing probability is then

$$p(i|\underline{a})=\int\prod_{j=1}^{N}\rho_{ij}(w_{ij})\rho_i(\eta_i)\times\Theta\left[\sum_{j=1}^{N}w_{ij}a_j-h_i+\eta_i\right]\prod_{j=1}^{N}dw_{ij}d\eta_i\ . \qquad (13)$$

The statistical dynamics of networks with random weights has been discussed elsewhere [32]. For simplicity we shall only consider additive white noise in this paper.

III. TIME-SUMMATING BINARY NETWORKS

The McCullogh-Pitts binary-threshold neuron is based upon a very simplified model of a real neuron. In particular, Eq. (1) neglects the temporal aspects of the process by which real neurons integrate their inputs. A more detailed account of the temporal nature of this process is given by the leaky-integrator shunting equation [21]

$$\frac{dV_i}{dt}=-\frac{V_i(t)}{\tau_i}+\sum_{j=1}^{N}A_{ij}(t)[S_{ij}-V_i(t)]\ , \qquad i=1,\ldots,N\ . \qquad (14)$$

Here $A_{ij}(t)$ is the input signal from neuron j to neuron i at time t and S_{ij} is fixed membrane reversal potential. In the absence of any inputs, the membrane potential decays to zero at the rate τ_i^{-1}. The inputs A_{ij} are taken to be positive, so that the effect of each shunting term $A_{ij}(S_{ij}-V_i)$ is for V_i to tend toward S_{ij}. Thus positive and negative S_{ij} correspond, respectively, to excitatory and inhibitory inputs. The input $A_{ij}(t)$ is usually of the form $g_{ij}f_j(V_j(t))$, where $g_{ij}>0$ is fixed and f_j corresponds to a short-term average firing rate which is assumed to be a smooth monotonic function of the membrane potential. However, this neglects the pulse-coding feature of neuronal signals. Such a feature may be incorporated into Eq. (14) by assuming that neuron i receives an impulse of size g_{ij} each time neuron j fires so that [10]

$$A_{ij}(t+t_d)=g_{ij}\sum_{n=1}^{\infty}\delta(t-T_j^n)\ , \qquad (15)$$

where T_j^n is the time at which neuron j fires for the nth occasion since $t=0$. Note that a time delay t_d has been included in Eq. (15) to account for the finite time interval between the arrival of a signal at a synapse and the resulting change in membrane potential of the target neuron. The firing times are then determined by the iterative threshold condition

$$T_j^n=\inf\{t\,|\,V_j(t)\geq h_j;t\geq T_j^{n-1}+t_R\}\ ,\quad j=1,\ldots,N\ , \qquad (16)$$

where t_R is the absolute refractory period. Equation (16) may be understood as follows. Suppose that neuron j last fired at time T. Then for $T\leq t<T+t_R$ the neuron is incapable of firing again, although $V_j(t)$ continues to evolve according to Eq. (14); for $t\geq T+t_R$ the neuron fires as soon as $V_j(t)$ is above the threshold h_j.

Imposing the initial condition that the first firing times T_j^1 are multiples of the delay t_d, and setting $t_R=t_d$, it can be shown using the threshold condition (16) that Eq. (15) may be written in the simpler form [10]

$$A_{ij}(t+t_d)=g_{ij}\sum_{m=0}^{\infty}\delta(t-mt_d)a_j(mt_d)\ , \qquad (17)$$

where $a_j(mt_d)=\Theta(V_j(mt_d)-h_j)$. In other words, $a_j(mt_d)=1$ if $T_j^n=mt_d$ for some $n\geq 1$ and $a_j(mt_d)=0$ otherwise. Equation (14) then reduces to a discrete time model of a time-summating binary network. To show this we first integrate Eq. (14), with $V_i(0)=0$ say, such that for $t\geq t_d$,

$$V_i(t)=\int_0^t\exp\left[-\int_{t'}^{t}\sum_{j=1}^{N}A_{ij}(t''-t_d)dt''\right]\times\exp\left[-\frac{t-t'}{\tau_i}\right]\times\left[\sum_{k=1}^{N}S_{ik}A_{ik}(t'-t_d)\right]dt'\ . \qquad (18)$$

Substituting Eq. (17) into Eq. (18), it follows that the dynamics of the membrane potential $V_i(t)$ is determined completely by solving Eq. (18) at the discrete times $t=mt_d$, $m\geq 0$. In particular, if t satisfies $m<t<m+1$ for some integer m, then $V_i(t)=\exp[-(t-m)/\tau_i]V_i(m)$ where

$$V_i(m)=\sum_{k=1}^{N}\sum_{r=1}^{m}\gamma_i^{r-1}w_{ik}a_k(m-r)\times\exp\left[-\sum_{j=1}^{N}\sum_{s=1}^{r}\bar{w}_{ij}a_j(m-s)\right],\quad m=1,2,\ldots \tag{19}$$

and $\gamma_i=e^{-\tau_i^{-1}}$, $w_{ij}=g_{ij}S_{ij}$, $\bar{w}_{ij}=g_{ij}\geq 0$. (We have set $t_d=1$ for simplicity.)

Equations (19) and (2) describe the dynamics of a binary network in which the membrane potential is a nonlinear function of the inputs involving an extended time summation over the previous output activity of the network. Also note that there are effectively two independent sets of weights $\{w_{ij}\}$ and $\{\bar{w}_{ij}\}$. In contrast to the standard binary-network model, Eqs. (1) and (2), the dynamics of the network in the space of binary outputs is no longer described by first-order difference equations for $\underline{a}(m)$. However, Eq. (19) may be rewritten as a first-order iterative equation for $\underline{V}(m)=(V_1(m),\ldots,V_N(m))$ given by

$$V_i(m)=F_i(\underline{V}(m-1))=\left[\gamma_i V_i(m-1)+\sum_{k=1}^{N}w_{ik}a_k(m-1)+I_i\right]\times\exp\left[-\sum_{j=1}^{N}\bar{w}_{ij}a_j(m-1)\right], \tag{20}$$

where $a_i(m)=\Theta(V_i(m)-h_i)$ and an external input I_i has been included. Note that the membrane potential of a standard binary network is discrete valued since it is given by the linear sum of binary-valued variables, Eq. (1). This implies that the dynamics is recurrent. On the other hand, the membrane potential of a time-summating network is continuous valued and generally leads to nonrecurrent behavior.

To illustrate the complex behavior of time-summating networks we briefly consider the case of a single neuron ($N=1$) with an external input I satisfying

$$V(m)=F(V(m-1))=[\gamma V(m-1)-w\Theta(V(m-1))+I]e^{-\bar{w}\Theta(V(m-1))}, \tag{21}$$

where $\gamma<1$, $\bar{w}>0$. The map F is piecewise linear with a single discontinuity at $V=0$, and has been analyzed in detail elsewhere using discontinuous circle maps [11]. In particular, if $w>0$ and $0<I<w$, then all trajectories converge to the interval $\Sigma=[V_-,V_+]$, where $V_-=(I-w)e^{-\bar{w}}$, $V_+=I$. (Other choices for I and w give trivial dynamics.) To describe the behavior on $\sum$ it is useful to introduce the average firing rate

$$\rho(V)=\lim_{M\to\infty}\sum_{n=0}^{M}\frac{\Theta(F^n(V))}{M} \tag{22}$$

(assuming the limit exists), where $\Theta(F^n(V))$ is the neuronal output $a(n)$ at time n given the initial state V. [Note that $\rho(V)$ is equivalent to the rotation number of the corresponding circle map [11]]. It can be shown that the average firing rate is independent of the initial point V, $\rho(V)=\bar{\rho}$, and that the dynamics is either periodic or quasiperiodic depending on whether $\bar{\rho}$ is a rational or irrational number. Moreover, as a function of the input I, $\bar{\rho}$ forms a "devil's" staircase in which irrational values occur for a set of Lesbegue measure zero. Such behavior was previously studied in the Nagumo-Sato model of a neuron with relative refractory period [9], which corresponds to the special case $\bar{w}=0$.

One of the interesting features of the single-neuron model described by Eq. (21) is that if the allowed range of $\bar{w}$ is extended from $\mathbb{R}^+$ to $\mathbb{R}$, with $w>0$ and $0<I<w$ as before, then chaotic dynamics on Σ can occur [11]. To indicate how this arises, consider the Liapunov exponent of Eq. (21) defined by

$$\begin{aligned}\lambda(V)&=\lim_{m\to\infty}\frac{1}{M}\ln\prod_{n=0}^{M}\left\|\frac{dF}{dV}[F^n(V)]\right\|\\&=\ln\gamma-\bar{w}\lim_{M\to\infty}\frac{1}{M}\sum_{n=0}^{M}\Theta(F^n(V))\\&=\ln\gamma-\bar{w}\rho(V)\ .\end{aligned} \tag{23}$$

If $\bar{w}>0$, then $\lambda(V)=\bar{\lambda}$, independently of V, where $\bar{\lambda}=\ln\gamma-\bar{w}\bar{\rho}<0$ and the dynamics is not chaotic. Therefore, suppose $\bar{w}<0$ with $e^{-\bar{w}}\gamma>1$. Then the dynamics on Σ is determined by the value of the quantity $\Delta=F(V_-)-F(V_+)$ [11]. If $\Delta>0$, which implies that $I<I_c=w(1-\gamma)/(1-e^{\bar{w}})$, the behavior of the neuron is similar to the case $\bar{w}>0$. In particular, $\bar{\rho}$ is well defined and increases monotonically from zero at $I=0$ to the critical value $\bar{\rho}_c=|\ln\gamma/\bar{w}|$, corresponding to zero Liapunov exponent, at $I=I_c$. On the other hand, when $\Delta<0$ or $I>I_c$, the limit in Eq. (22) does not exist and $\rho(V)$ is defined to be the set of limit points of the sequence $[\sum_{n=0}^{M}\Theta(F^n(V))/M]$. Moreover, it can be shown that the union of such sets over $V\in\sum$ is given by a closed interval $[a,b]$ where [35]

$$a=\inf_{V\in\Sigma}\lim_{M\to\infty}\inf\sum_{n=0}^{M}\frac{\Theta(F^n(V))}{M}\ ,$$

$$b=\sup_{V\in\Sigma}\lim_{M\to\infty}\sup\sum_{n=0}^{M}\frac{\Theta(F^n(V))}{M}\ .$$

The values of a and b increase monotonically with I such that $a>\bar{\rho}_c$ for $I>I_c$. It follows that the Liapunov exponent $\bar{\lambda}$ is replaced by the interval $[\lambda_a,\lambda_b]$ where $\lambda_a=\ln\gamma+|\bar{w}|a>0$ and similarly for λ_b. Since $\lambda_a>0$, the dynamics on Σ is chaotic, in the strong sense of sensitivity to initial conditions, when $I>I_c$. From the discussion of the dynamics of a single neuron, it follows that a necessary condition for Eq. (20) to have chaotic solutions when $N>1$ is that $\exp(-\sum_j\bar{w}_{ij})\gamma_i>1$ for at least one i. Then for certain mean output configurations there exists a positive Liapunov exponent. One such configuration is $a_i=1$ for all $i=1,\ldots,N$. The same inequality will also arise in our analysis of asymptotic stability in stochastic networks; see Sec. IV.

IV. NOISE, ASYMPTOTIC PERIODICITY, AND STABILITY

We now turn to the study of noise in the time-summating binary network constructed in the preceding section. The simplest way to incorporate noise into such networks is to replace Eq. (20) by the stochastic equation

$$V_i(m)=F_i(\underline{V}(m-1))+\eta_i(m-1)\ , \tag{24}$$

where η_i is an additive white-noise term, with density ρ_i, corresponding to a random fluctuation of the membrane potential V_i. (See also Sec. VI). Deterministic dynamical systems with applied stochastic perturbations have been considered by Kifer [36], Lasota and Mackey [30], and in the physical context by Feigenbaum and Hashler [37] (see also the discussion of the structural stability of maps by Zeeman [38] and MacKay [39]). More specifically, a useful way of studying such systems is to consider the time evolution or flow of probability densities in state space. This flow is generated by picking a large number of initial states of the system and following the ensemble of trajectories associated with these states. The probability density at a particular point in state space and at a time m determines the likelihood of a trajectory of the ensemble passing through an infinitesimal neighborhood of that point at time m. Let u_m denote the probability density on the space of membrane potentials $\underline{V}(m)$. For a fixed vector $\eta=(\eta_1,\ldots,\eta_N)$, the density of the (deterministic) system (24) evolves according to the linear operator equation $u_{m+1}=K_\eta u_m$ where

$$\begin{aligned} u_{m+1}(\underline{V}')&=(K_\eta u_m)(\underline{V}')\\ &=\int_{\mathbb{R}^N}\delta(\underline{V}'-\underline{F}(\underline{V})-\eta)u_m(\underline{V})d\underline{V}\ . \end{aligned} \tag{25}$$

The Dirac δ function in Eq. (25) restricts the integral on the right-hand side to points $\underline{V}$ which satisfy $\underline{F}(\underline{V})+\eta=\underline{V}'$. On the other hand, when η is a random variable, the density of the corresponding stochastic system evolves according to the modified linear operator equation $u_{m+1}=\bar{K}u_m$ where

$$\begin{aligned} u_{m+1}(\underline{V}')&=\bar{K}u_m(\underline{V}')\\ &=\int_{\mathbb{R}^N}\rho(\eta)(K_\eta u_m)(\underline{V}')d\eta\\ &=\int_{\mathbb{R}^N}\rho(\underline{V}'-\underline{F}(\underline{V}))u_m(\underline{V})d\underline{V} \end{aligned} \tag{26}$$

and $\rho(\eta)=\prod_i\rho_i(\eta_i)$. Note that $\bar{K}$ is a linear Markov operator from the space of L^1 functions on $\mathbb{R}^N$, $L^1(\mathbb{R}^N)$, into itself. In particular, for any density u, $\bar{K}u\geq 0$ and $\|\bar{K}u\|=\|u\|$ so that $\bar{K}u$ is also a density. We shall denote the space of densities on $\mathbb{R}^N$ by D.

To analyze the asymptotic behavior of the sequence $\{u_m;m\geq 0\}$ we shall use the results of Lasota and Mackey [31]. We begin by defining a class of linear Markov operators that satisfy a spectral decomposition theorem analogous to the Perron-Frebenius theorem for irreducible Markov chains (see Sec. II). A set $\mathcal{F}\subset L^1(\mathbb{R}^N)$ is called *weakly precompact* if every sequence of functions $\{f_n\}$, $f_n\in\mathcal{F}$, contains a weakly convergent subsequence, $\{g_n\}$ say, that converges to some $\bar{f}\in L^1(\mathbb{R}^N)$. In other words,

$$\lim_{n\to\infty}\int_{\mathbb{R}^n}g_n(x)h(x)dx=\int_{\mathbb{R}^n}\bar{f}(x)h(x)dx \quad \text{for all } h\in L^1(\mathbb{R}^n)\ . \tag{27}$$

Moreover, an operator K on $L^1(\mathbb{R}^N)$ is defined to be *weakly constrictive* if there exists a weakly precompact set $\mathcal{F}\subset L^1(\mathbb{R}_N)$ such that

$$\lim_{n\to\infty}d(K^nf,\mathcal{F})=0 \quad \text{for } f\in D\ , \tag{28}$$

where $d(f,\mathcal{F})$ denotes the distance between f and the set $\mathcal{F}$ with respect to the L^1 norm [30,31]. The importance of an operator being weakly constrictive is that it then satisfies the following spectral decomposition theorem due to Komornik [40], as described in [31].

Spectral decomposition theorem. Let K be a weakly constrictive Markov operator on $\mathbb{R}^N$. Then there is an integer r, two sequences of non-negative functions $g_i\in D$ and $k_i\in L^\infty$, $i=1,\ldots,r$, and an operator R: $L^1\to L^1$ such that for all $f\in L^1$, Kf may be written as

$$Kf(x)=\sum_{i=1}^{r}\lambda_i(f)g_i(x)+Rf(x)\ , \tag{29}$$

where

$$\lambda_i(f)=\int_{\mathbb{R}^N}f(x)k_i(x)dx\ . \tag{30}$$

The functions g_i and the operator R have the following properties.

(i) $g_i(x)g_j(x)=0$ for all $i\neq j$, so that the densities g_i have disjoint supports.

(ii) For each integer i there exists a unique integer $\omega(i)$ such that $Kg_i=g_{\omega(i)}$. Moreover, $\omega(i)\neq\omega(j)$ for $j\neq i$ and thus the operator K simply permutes the functions g_i.

(iii) $\|K^nRf\|\to 0$ as $n\to\infty$ for every $f\in L^1(\mathbb{R}^N)$. It follows from Eq. (29) that K^nf may be rewritten as [31]

$$K^nf(x)=\sum_{i=1}^{r}\lambda_{\omega^{-n}(i)}(f)g_i(x)+K^{n-1}Rf(x)\ , \tag{31}$$

where $\|K^{n-1}Rf\|\to 0$ as $n\to\infty$ and $\omega^{-n}(i)$ is the nth inverse permutation of i. As the densities g_i all have disjoint support, the operation of K may be viewed as permuting the set of coefficients $\{\lambda_j,j=1,\ldots,r\}$. Hence the summation in Eq. (31) has period less than $r!$. Moreover, since $\|K^{n-1}Rf\|\to 0$ as $n\to\infty$, the sequence $\{K^nf\}$ is said to be *asymptotically periodic.* One of the interesting features of such periodicity, as demonstrated by Lasota and Mackey [31], is that for certain systems, whose limiting behavior is aperiodic in the absence of noise, the addition of noise induces asymptotic periodicity.

Consider the mapping $\underline{F}=(F_1,\ldots,F_N)$: $\mathbb{R}^N\to\mathbb{R}^N$ defined by Eq. (20), which may be written in the form

$$\underline{F}(\underline{V})=\underline{A}(\underline{V})+\underline{B}(\underline{V}) \tag{32}$$

with

$$A_i(\underline{V})=\Gamma_i(\underline{V})\gamma_i V_i\ , \quad B_i(\underline{V})=\Gamma_i(\underline{V})\sum_k w_{ik}\Theta(V_k-h_k)\ ,$$
$$\Gamma_i(\underline{V})=\exp\left[-\sum_j \tilde{w}_{ij}\Theta(V_j-h_j)\right]\ . \tag{33}$$

Furthermore, assume that (i) the density ρ generating the additive noise has finite first moment, i.e.,

$$\int_{\mathbb{R}^N}|\eta|\rho(\eta)d\eta<\infty\ , \tag{34}$$

where $|\ |$ is the norm of $\mathbb{R}^N$ and (ii) $\Gamma_i(\underline{V})\gamma_i<1$ for all $\underline{V}\in\mathbb{R}^N$ and $i=1,\ldots,N$. Then the operator $\bar{K}$ of Eq. (26) is weakly constrictive. For condition (ii) implies that

$$|\underline{F}(\underline{V})|\le|\underline{A}(\underline{V})|+|\underline{B}(\underline{V})|\le\gamma|\underline{V}|+|\underline{\xi}|\ , \tag{35}$$

where $\gamma=\max_{i,\underline{V}}\{\Gamma_i(\underline{V})\gamma_i\}<1$ and $\xi_i=\gamma\sum_k w_{ik}$. The result then follows from theorem 1 of Ref. [31], which effectively states that $\bar{K}$ is weakly constrictive if ρ satisfies Eq. (33) and $|F(\underline{V})|\le\alpha|\underline{V}|+\beta$ for some $\alpha<1$, $\beta>0$. The latter condition is satisfied for $\alpha=\gamma$, $\beta=|\underline{\xi}|$. Therefore, from the spectral decomposition theorem we deduce that the sequence of densities $\{u_n\}$ in Eq. (26) is asymptotically periodic. Note that condition (ii) is automatically satisfied if $\tilde{w}_{ij}\in\mathbb{R}^+$. However, it no longer holds when the allowed range of $\tilde{w}_{ij}$ is extended from $\mathbb{R}^+$ to $\mathbb{R}$ such that $\exp(-\sum_j\tilde{w}_{ij})\gamma_i>1$ for at least one i; the latter is a necessary condition for the occurrence of chaotic behavior in the noiseless case, as explained in Sec. III.

A special case of the spectral decomposition theorem occurs when $r=1$, so that the summation in Eq. (29) reduces to a single term. Then the sequence $\{K^n f\}$ converges to a unique limit as $n\to\infty$ independently of f, and the operator K is said to be *asymptotically stable.* Assume that the density ρ satisfies the inequality (34) and, moreover, that it has sufficiently large nonzero support. More precisely, suppose that there exists a point $\eta_0\in\mathbb{R}^N$ and a number $r_0>(m\gamma+|\underline{\xi}|)/(1-\gamma)$, integer $m\ge0$, such that

$$\rho(\eta)>0 \text{ almost everywhere for } |\eta-\eta_0|<r_0\ . \tag{36}$$

It can then be shown that the Markov operator $\bar{K}$ defined by Eq. (26) is asymptotically stable (see theorem 2 of Ref. [31]). Denoting the unique limit of the sequence $\{u_n\}$ by u_∞, we have

$$\lim_{n\to\infty}\|u_n-u_\infty\|=0 \tag{37}$$

independently of the initial density u_0. Equation (37) implies the ergodicity condition

$$\lim_{M\to\infty}\sum_{m=0}^{M}\frac{f(\underline{V}(m))}{M}=\int_{\mathbb{R}^N}f(\underline{V})u_\infty(\underline{V})d\underline{V}\ , \tag{38}$$

where f is any integrable function [cf. Eq. (7) of Sec. II]. Note that the density used in the Little model, Eq. (12), satisfies the conditions (34) and (36).

We end this section by briefly discussing the dynamics of the network in the space of binary outputs $\{0,1\}^N$. As in Sec. II, let $P_m(\underline{a})$ be the probability that the output of the network at times m is $\underline{a}$. Then $P_m(\underline{a})$ may be obtained from the density u_m by the projection

$$P_m(\underline{a})=\int_{\mathbb{R}^N}u_m(\underline{V})\Theta_{\underline{a}}(\underline{V})d\underline{V}\ , \tag{39}$$

where

$$\Theta_{\underline{a}}(\underline{V})=\prod_{i=1}^{N}\{a_i\Theta(V_i-h_i)+(1-a_i)[1-\Theta(V_i-h_i)]\}\ . \tag{40}$$

Hence the sequence of densities $\{u_m\}$ on $\mathbb{R}^N$ induces, via Eq. (39), the sequence of probabilities $\{P_m\}$ on $\{0,1\}^N$. Furthermore, asymptotic stability of the Markov operator $\bar{K}$ implies that the latter sequence converges to a unique distribution P_∞ with

$$P_\infty=\lim_{m\to\infty}P_m(\underline{a})=\int_{\mathbb{R}^N}u_\infty(\underline{V})\Theta_{\underline{a}}(\underline{V})d\underline{V}\ . \tag{41}$$

However, the sequence $\{P_m\}$ does not, in general, satisfy a Markov chain of the form of Eq. (4), but is described, instead, by a non-Markovian process. This reflects the lack of a unique transition from $\underline{a}(m)$ to $\underline{a}(m+1)$ in the deterministic limit (see Sec. III). An exception is the case $\gamma_i=0$, $i=1,\ldots,N$, in which the projection of Eq. (26) using Eq. (39) reduces to Eq. (4), with the transition matrix $\underline{Q}$ satisfying

$$Q_{\underline{b}\,\underline{a}}=\int_{\mathbb{R}^N}\prod_{i=1}^{N}\rho_i(V_i-B_i(\underline{a}))\Theta_{\underline{b}}(\underline{V})d\underline{V}\ , \tag{42}$$

where

$$B_i(\underline{a})=\Gamma_i(\underline{a})\sum_k w_{ik}a_k\ ,$$
$$\Gamma_i(\underline{a})=\exp\left[-\sum_j\tilde{w}_{ij}a_j\right]\ . \tag{43}$$

If $\tilde{w}_{ij}=0$, $i,j=1,\ldots,N$, then Eq. (42) is identical to Eqs. (5) and (11).

V. PARAMETER DEPENDENCE OF LIMITING DISTRIBUTION

In the absence of noise, the dynamics of a time-summating binary network (Sec. III) differs greatly from that of a standard binary network (Sec. II), the latter corresponding to the limiting case $\gamma_i\to0$, $i=1,\ldots,N$. On the other hand, if additive white noise is introduced according to Eq. (24), with the density ρ having finite first moment and sufficiently large nonzero support [Eq. (36)], then the operator $\bar{K}$ of Eq. (26) is asymptotically stable and the limiting behavior of the system is described by convergence to a unique limiting distribution u_∞, irrespective of the values of γ_i (Sec. IV). In this section we consider the explicit parameter dependence of the limiting distribution u_∞, Eq. (37), and show that it is differentiable with respect to the connection weights $w_{ij},\tilde{w}_{ij}$ and decay factors γ_i. This is then used to obtain a linearized expression for P_∞ about the point in parameter space given by $\gamma_i=0$, $i=1,\ldots,N$, for fixed weights. The fact that the limiting distribution u_∞ is a differentiable function of the weights suggests that u_∞ could, in principle, be determined by some form of gradient descent learning algorithm.

Before proceeding, it is useful to rewrite the spectral decomposition theorem of Sec. IV in the following form: If K is any weakly constrictive, asymptotically stable operator on $\mathbb{R}^n$, then there exists a positive real eigenvalue $\lambda=1$ of maximum modulus, and the space $L^1(\mathbb{R}^N)$ may be decomposed into K-invariant subspaces

$$L^1(\mathbb{R}^N)=E+H \ , \tag{44}$$

where (i) E is the eigenspace of the eigenvalue λ and has dimension 1 and (ii) H has codimension 1 and $K|H$ has spectral radius less than λ. In other words, E contains a unique element f (modulo scalar multiplication), such that $K(f)=f$ and $K(h)\in H$ for all $h\in H$ with $\lim_{n\to\infty}K^n(h)=0$. The above is analogous to the Perron-Frobenius theorem for irreducible, aperiodic Markov chains (Sec. II).

To prove differentiability of the unique limiting density u_∞ we follow an argument due to MacKay [39]. First, note that the density u_∞ satisfies the fixed point equation $\bar{K}(u_\infty)=u_\infty$, where $\bar{K}$ satisfies Eq. (26). This may be written in the form (dropping the subscript ∞ for convenience)

$$\mathcal{G}(u,\alpha)(\underline{V})=u(\underline{V})-\int\rho(\underline{V}-\underline{F}(\underline{V}',\alpha))u(\underline{V}')d\underline{V}'=0 \ , \tag{45}$$

where the parameter dependence has been made explicit, with α denoting the connection weights w_{ij}, $\tilde{w}_{ij}$ and decay factors γ_i. Note that $\mathcal{G}$ may be considered as a differentiable mapping from $\Omega\times L^1(\mathbb{R}^N)\to L^1(\mathbb{R}^N)$ where Ω is the parameter space of the α's. In particular

$$\frac{\delta\mathcal{G}(u,\alpha)(\underline{V})}{\delta u(\underline{V}')}=\delta(\underline{V}-\underline{V}')-\rho(\underline{V}-\underline{F}(\underline{V}',\alpha)) \tag{46}$$

and

$$\frac{\delta\mathcal{G}(u,\alpha)(\underline{V})}{\delta\alpha}=-\int\frac{\partial\rho}{\partial\underline{V}}(\underline{V}-\underline{F}(\underline{V}',\alpha))\frac{\partial\underline{F}}{\partial\alpha}u(\underline{V}')d\underline{V}' \ . \tag{47}$$

Equations (46) and (47) may be written in the symbolic form

$$\frac{\delta\mathcal{G}}{\delta u}=1-\mathcal{Q},\quad \frac{\delta\mathcal{G}}{\delta\alpha}=-\mathcal{M} \ . \tag{48}$$

Since the operator $\bar{K}$ satisfies the spectral decomposition theorem, as described in the preceding paragraph, it follows that the tangent space to the space of densities D is H, and $1-\mathcal{Q}$ is invertible on H. Hence, applying the implicit function theorem [33] we conclude that the limiting density u locally forms a differentiable graph $u(\alpha)$ such that

$$\frac{\delta u}{\delta\alpha}=(1-\mathcal{Q})^{-1}\mathcal{M} \ . \tag{49}$$

Moreover, all its derivatives are bounded [39].

A straightforward corollary of the above result is that the limiting density is a continuous function of the weights w_{ij} and scaling factors γ_i. In particular, assuming that the weights are fixed, to a first approximation

$$u(\underline{V},\gamma)=u(\underline{V},0)+\sum_i\frac{\delta u}{\delta\gamma_i}(\underline{V},0)\gamma_i \ , \tag{50}$$

where $\gamma=(\gamma_1,\ldots,\gamma_N)$. Multiplying both sides of Eq. (50) by $\Theta_{\underline{b}}(\underline{V})$ and integrating over $\underline{V}$ we obtain, using Eq. (41),

$$P(\underline{b},\gamma)=P(\underline{b},0)+\sum_i\frac{\partial P}{\partial\gamma_i}(\underline{b},0)\gamma_i \ . \tag{51}$$

The derivative $\partial P/\partial\gamma_i$ may be calculated directly from the limiting probability distribution $P(\underline{b},0)$. For using Eq. (47) and the identity

$$\rho(\underline{V}'-\underline{F}(\underline{V},\gamma)) \equiv\sum_{\underline{a}}\left[\prod_i\rho_i(V_i'-\Gamma_i(\underline{a})\gamma_iV_i-B_i(\underline{a}))\right]\Theta_{\underline{a}}(\underline{V}) \ , \tag{52}$$

we have

$$\frac{\delta\mathcal{G}(u,0)(\underline{V})}{\delta\gamma_i}=\sum_{\underline{a}}\frac{\partial\rho}{\partial\underline{V}}(\underline{V}-\underline{B}(\underline{a})) \times\int\Theta_{\underline{a}}(\underline{V}')\frac{\partial\underline{F}}{\partial\gamma_i}(\underline{V}',0)u(\underline{V}',0)d\underline{V}' \ , \tag{53}$$

where $\partial\underline{F}/\partial\gamma_i=V_i$ and Γ_i and B_i are defined in Eq. (43). Moreover, from Eqs. (45) and (52),

$$u(\underline{V},0)=\sum_{\underline{a}}\rho(\underline{V}-B(\underline{a}))P(\underline{a},0) \ . \tag{54}$$

Finally, combining Eqs. (49), (53), and (54) we obtain

$$\frac{\partial P}{\partial\gamma_i}(\underline{b},0)=\sum_{\underline{a}}\left[\sum_i(1-\underline{Q})^{-1}\underline{M}^{(i)}\underline{\tilde{M}}^{(i)}\right]_{\underline{b}\,\underline{a}}P(\underline{a},0) \ , \tag{55}$$

where $\underline{Q}$ is the transition matrix (42), and

$$\begin{aligned}M^{(i)}_{\underline{b}\,\underline{a}}&=\int\frac{\partial\rho}{\partial V_i}(\underline{V}-\underline{B}(\underline{a}))\Theta_{\underline{b}}(\underline{V})d\underline{V} \ ,\\ \tilde{M}^{(i)}_{\underline{b}\,\underline{a}}&=\int\rho(\underline{V}-\underline{B}(\underline{a}))V_i\Theta_{\underline{b}}(\underline{V})d\underline{V} \ .\end{aligned} \tag{56}$$

Thus, to first order in γ, $P(\underline{b},\gamma)$ may be calculated explicitly if $P(\underline{b},0)$ is known. The latter is particularly simple when ρ satisfies Eq. (12) and $w_{ij}=w_{ji}$, $\tilde{w}_{ij}=0$. For then $P(\underline{b},0)$ is given by the limiting distribution of the Little-Hopfield model, which is a Gibbs distribution [4].

VI. MEAN-FIELD THEORY

So far we have considered stochastic dynamics from the viewpoint of linear Markov theory. In this section we turn to the issue of the macroscopic behavior of time-summating networks in the thermodynamic limit $N\to\infty$. For simplicity we shall consider a homogeneous network described by Eqs. (2) and (20) with $\tilde{w}_{ij}=0$, $w_{ij}=-w/N$, $I_i=I$, $\gamma_i=\gamma$ for all $i,j=1,\ldots,N$. Moreover, it is convenient to introduce the noise term η_i via Eq. (2) rather than Eq. (20), with η_i now interpreted as a random threshold. That is,

$$V_i(m+1)=F_i^{\eta}(\underline{V}(m))$$
$$=\gamma V_i(m)-\frac{w}{N}\sum_{j=1}^{N}\Theta(V_j(m)+\eta_j(m))+I\ , \tag{57}$$

so that Eq. (26) becomes

$$u_{m+1}(\underline{V}')=\int_{\mathbb{R}^N}\rho(\eta)\delta\left[\underline{V}'-\left[\gamma\underline{V}-\frac{w}{N}\sum_{i=1}^{N}\Theta(V_i+\eta_i)+I\right]\right]u_m(\underline{V})d\underline{V}\,d\eta$$
$$=\int_{\mathbb{R}^N}\frac{1}{\gamma^N}\rho\left[\frac{\underline{V}'-\underline{F}(\underline{V})}{\gamma}\right]\times u_m\left[\underline{V}-\frac{\underline{V}'-\underline{F}(\underline{V})}{\gamma}\right]d\underline{V}\ , \tag{58}$$

where as usual $\rho(\eta)=\prod_{i=1}^{N}\rho_i(\eta_i)$. We see that ρ in Eq. (58) is equivalent to Eq. (26) provided the densities are rescaled such that $\rho_i'(x)=\gamma^{-1}\rho_i(x/\gamma)$. In the following we shall assume that the density ρ_i satisfies Eq. (12), in which case this corresponds to a simple rescaling of the temperature. [Note that for nonlinear threshold activity, i.e., $\tilde{w}_{ij}\neq 0$, there is an extra nontrivial Jacobian in Eq. (58).] Although there is no longer a straightforward equivalence between random threshold noise and random fluctuations of the membrane potentials, the similarity between (58) and (26) suggests that the analysis of the preceding sections can be extended to the former case.

To derive mean-field equations for a homogeneous network we proceed along the lines of Ref. [32]. That is, consider a fixed vector $\underline{V}$ and define the associated random variables $V_i'=F_i^{\eta}(\underline{V})$, with mean and variance

$$\bar{V}_i'=\langle F_i^{\eta}(\underline{V})\rangle_\rho\ ,$$
$$(\Delta V_i')^2=\langle[F_i^{\eta}(\underline{V})-\bar{V}_i'][F_i^{\eta}(\underline{V})-\bar{V}_i']\rangle_\rho\ . \tag{59}$$

Here $\langle\ \rangle_\rho$ denotes averaging with respect to the random thresholds. From Eqs. (12) and (57)

$$\bar{V}_i'=\gamma V_i-\frac{w}{N}\sum_{j=1}^{N}\frac{1}{1+e^{-\beta V_j}}+I \tag{60}$$

and

$$(\Delta V_i')^2=\frac{w^2}{N^2}\sum_{j=1}^{N}\left[\frac{1}{1+e^{-\beta V_j}}-\left[\frac{1}{1+e^{-\beta V_j}}\right]^2\right]\ , \tag{61}$$

where we have used the fact that each η_i, $i=1,\ldots,N$, is independently distributed. Each term in the summations of Eqs. (60) and (61) is finite. Therefore, in the thermodynamic limit $N\rightarrow\infty$, fluctuations vanish since $(\Delta V_i')/\bar{V}_i'\rightarrow 0$. In other words, in a given trial the probability that $V_i'=\bar{V}_i'$, where $\bar{V}_i'$ satisfies Eq. (60), approaches unity. Such a result holds if $\underline{V}$ is replaced by the dynamical vector $\underline{V}(m)$ satisfying Eq. (57), since the random thresholds are independently generated at each time step. Then $\bar{V}_i'=V_i(m+1)$ and we obtain for large N the dynamical mean-field equations

$$V_i(m+1)=\gamma V_i(m)-\frac{w}{N}\sum_{j=1}^{N}\frac{1}{1+e^{-\beta V_j(m)}}+I\ . \tag{62}$$

Similarly, the mean activity of the network, defined by $M(m)=\sum_{i=1}^{N}a_i(m)/N$, satisfies

$$M(m)=\frac{1}{N}\sum_{j=1}^{N}\frac{1}{1+e^{-\beta V_j(m)}}\ . \tag{63}$$

The mean output activity $M(m)$ determines the degree of spatial ordering of the network at time m, with $M=1,0$ corresponding to an ordered macroscopic state and $M=0.5$ a disordered one. Since $V_i(m+1)-V_j(m+1)=\gamma[V_i(m)-V_j(m)]$, $\gamma<1$ for all $i\neq j$, it follows that the long-term macroscopic behavior of the network is effectively governed by the single mean-field equation

$$X(m+1)=F_\beta(X(m))=\gamma X(m)-w\frac{1}{1+e^{-\beta X(m)}}+I\ , \tag{64}$$

where $X(m)$ is the mean membrane potential of the network, $X(m)=\sum_{i=1}^{N}V_i(m)/N$. The mean output activity is now

$$M(m)=\frac{1}{1+e^{-\beta X(m)}}$$
$$=\frac{1}{1+\exp\left[-\beta\sum_{r=1}^{m}\gamma^{r-1}[I-wM(m-r)]\right]}\ . \tag{65}$$

We note that the mean-field equation for the mean output activity, Eq. (65), can also be derived using a functional approach similar to that developed for standard binary networks [10].

Equation (64) is identical in form to the single-neuron model of Aihara, Takabe, and Toyoda [12] and can have both periodic and chaotic solutions. We shall assume that β, w, and γ are fixed and discuss the dynamics of F_β as a function of the global external input I. Without loss of generality we take I and w to be positive with $I<w$. Suppose that $\kappa\equiv[w\beta/2\gamma)-1]>1$ so that the function F_β has two critical points at $\pm X^*$, where $\beta X^*=\ln[\kappa\pm\sqrt{(\kappa^2-1)}]$ (Fig. 1). There is also a unique fixed point, denoted $X=X_0$, which lies in the interval $[-X^*,X^*]$. Since β, γ, and w are fixed, changing the value of the external input I simply shifts the graph in Fig. 1 vertically. For $\beta\gg 1$ (low-temperature regime), there exists an interval $\Omega=[w_-,w_+]$, with $0<w_-<w_+<w$, such that for all $I\in\Omega$ the fixed point is unstable. That is, the slope of the graph at the fixed point X_0 has modulus greater than unity. All trajectories then converge to the closed interval $\sum=[F_\beta(-X^*),F_\beta(X^*)]$ on which the dynamics is either periodic or chaotic. The possibility of chaotic dynamics arises from the fact that for $\beta\gg 1$ the negative gradient branch of the graph of F_β has an average slope of modulus greater than unity, which can lead to a positive Liapunov exponent [12]. On the other hand, if $I\notin\Omega$, then the fixed point is stable and all trajectories converge to X_0. In Fig. 2 the bifurcation diagram of $M(m)$, satisfying Eq. (65), is plotted as a functon of I with $\beta^{-1}=0.04$, $w=1.0$, and $\gamma=0.5$. In Fig. 3 the bifurcation diagram of $M(m)$ is plotted as a function

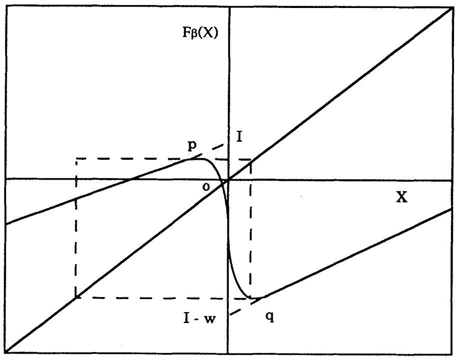

FIG. 1. Graph of the map F_β for $\beta=0.04$. The critical points $-X^*$, X^*, and the unique fixed point X_0 are indicated by p, q, and o, respectively. The dotted rectangular region denotes the restriction of F_β to $\sum$.

of the temperature $T=\beta^{-1}$ for fixed external input $I=0.09$. Note that in the zero-temperature limit the sigmoid function reduces to a threshold function and chaotic dynamics cannot occur for any value of I. Moreover, there exists a range of values of $I\in[0,w]$ over which chaos cannot occur at any temperature. This is illustrated in Fig. 4 where the critical temperature T_c, defined as the lowest temperature at which the Liapunov exponent is positive, is plotted as a functon of I for $w=1.0$, $\gamma=0.5$. From Fig. 4 we find that a finite critical temperature does not exist when $I\in[0.5-\Delta,0.5+\Delta]$ where $\Delta\approx0.16$. The behavior of F_β on $\sum$ may be understood further by observing that if the points $F_\beta(-X^*)$ and $F_\beta(X^*)$ are identified as points of a circle, then F_β restricted to $\sum$ is equivalent to a nonmonotonic circle map [41] with a single discontinuity at the identified point (see also Ref. [11]).

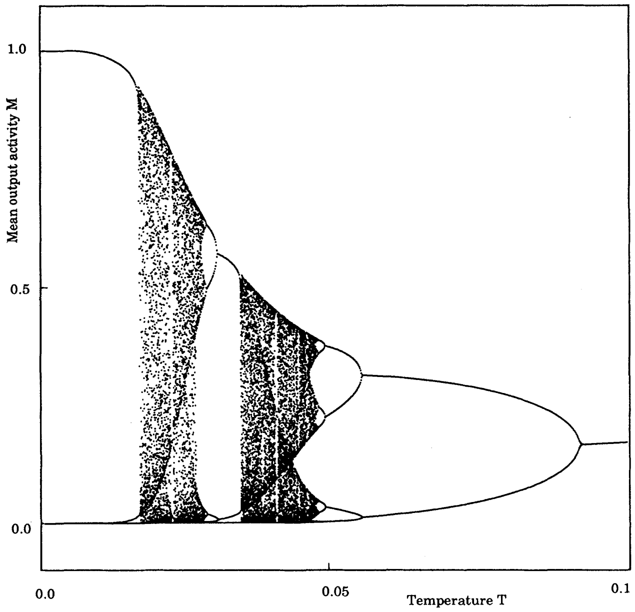

FIG. 3. Bifurcation diagram for the mean output activity as a function of temperature $T=\beta^{-1}$ for fixed external input $I=0.09$.

We now interpret the solutions to Eqs. (64) and (65) in terms of the behavior of the underlying neural network. First, the existence of periodic and chaotic solutions implies that in the limit $N\rightarrow\infty$ the network exhibits macroscopic behavior in which asymptotic stability is broken. For if the network were asymptotically stable, then Eq. (38) would hold, implying that the ensemble average of $M(V)$, say, converges to the fixed point

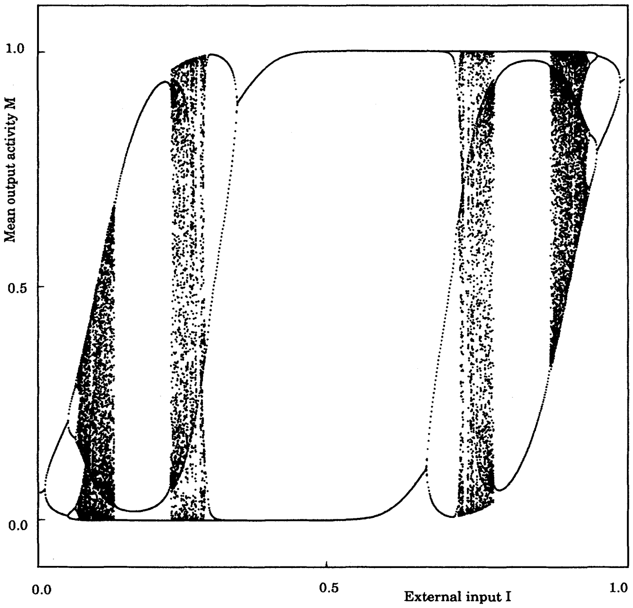

FIG. 2. Bifurcation diagram of the mean output activity $M(m)$ as a function of the external input I for $w=1.0$, $\gamma=0.5$, $\beta^{-1}=0.04$.

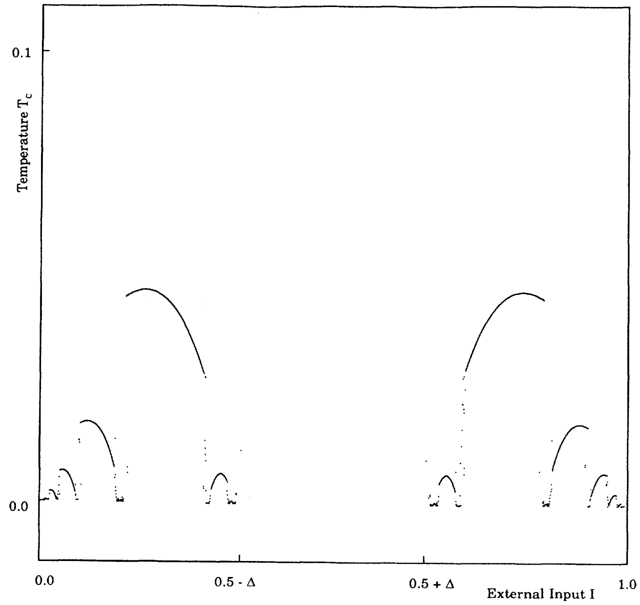

FIG. 4. Critical temperature T_c for onset of chaos as a function of the external input I.

$$M_\infty = \int_{\mathbb{R}} M(\underline{V}) u_\infty(\underline{V}) d\underline{V} \ . \tag{66}$$

(It is assumed throughout that the thermodynamic limit is taken before any long-time limits.) It is important to note that the macroscopic behavior of the homogeneous network, Eq. (57), is independent of initial conditions so that in this particular case the breaking of asymptotic stability does not lead to the breaking of ergodicity. A second consequence of Eqs. (64) and (65) is that for certain values of the external input the dynamics is phase locked in the sense that the firing patterns of (almost) all the neurons are synchronized. This only occurs when the network dynamics is restricted to ordered macroscopic states, i.e., $M(m) \approx 0,1$ for all $m \geq 0$. Such states correspond to cases in which the membrane potentials of the neurons have relatively large modulus so that fluctuations can be ignored in the low-temperature regime. (On the other hand, phase locking cannot arise for disordered macroscopic states, irrespective of whether the dynamics is periodic or chaotic, since the particular neurons which are active at any given time is random.) For example, when $I \in [0.4, 0.6]$ the network is bistable, with the mean output activity oscillating between the ordered states 0 and 1. On the other hand, when $I \approx 0.1$ the dynamics of $M(m)$ is chaotic and includes both ordered and disordered macroscopic states so that there is no phase locking.

VII. DISCUSSION

In this paper we have analyzed the stochastic dynamics of time-summating neural networks with noisy external inputs, based on the discrete time leaky-integrator model of Bressloff and Taylor [10]. Moreover, we have shown how nontrivial periodic and chaotic dynamics can be generated at the microscopic and macroscopic levels. A number of interesting issues are raised by this work.

(i) In Sec. VI we derived dynamical mean-field equations describing the response characteristics of a homogeneous time-summating binary network as a function of an external input I, in terms of the mean output activity $M(m)$, Eq. (63). One possible way to extend such an analysis is to take the connection weights w_{ij} in Eq. (20) to be of the Hebbian form [42] $w \sum_{\mu=1}^{P} \xi_i^\mu \xi_j^\mu / N$, for a set of P random pattern vectors ξ^μ, $\xi_i^\mu = \pm 1$, and to consider external inputs $\sum_{\mu=1}^{P} I^\mu \xi^\mu$. If the outputs of the neurons are expressed in terms of the "spin representation" $S_i = 2\Theta(V_i(m) + \eta_i(m)) - 1 = \pm 1$, then following the analysis of Sec. VI, we can derive the mean-field equations (for large m and finite P, as $N \rightarrow \infty$)

$$M^\mu(m) = \left\langle\!\!\left\langle \xi^\mu \tanh \left[\beta \left(\sum_{r=1}^{m} \gamma^{r-1} \sum_{\nu=1}^{P} \xi^\nu [I^\nu - w M^\nu(m-r)] \right) \right] \right\rangle\!\!\right\rangle \ . \tag{67}$$

$M^\mu = \sum_{i=1}^{N} \xi_i^\mu S_i / N$, and $\ll \gg$ denotes self-averaging over the random pattern vectors [6]. However, the analysis of Eq. (67) in nontrivial since (i) the average long-term behavior of the network will depend on the initial conditions, implying that ergodicity as well as asymptotic stability has been broken; (ii) the various attractors of the dynamical mean-field equations will generally be periodic orbits or chaotic (rather than fixed points, as in the case of the Little-Hopfield model with symmetric weights); and (iii) it is necessary to perform the self-averaging over the space of pattern vectors. Moreover, although functional techniques along the lines of Ref. [32] can be used to derive dynamical mean-field equations when the number of patterns P increases with the size of the network, $P = \alpha N$, the resulting self-consistency equations for the autocorrelation and response functions appear intractable. These issues need to be confronted before an analysis of the pattern storage capacity of stochastic time-summating networks with Hebbian learning can be carried out. (See also Ref. [28] for a discussion of stochastic time-delay networks.)

(ii) The Hebbian form for the connection weights is based on a learning rule in which each w_{ij} is modified according to the output activities on either side of the connection (ij) and does not depend on the particular values of the internal membrane potentials V_i, V_j. Since the dynamical variables of time-summating networks are the internal activation states of the neurons rather than the binary outputs, it would be of interest to develop alternative learning rules which have an explicit dependence on the V_i (especially of the target neuron).

(iii) Instead of considering the relationship between the macroscopic behavior of a network and the underlying microscopic behavior of the individual neurons, it is possible to view the network itself as a single formal unit along similar lines to Amari [43] and Anninos *et al.* [44]. One may then analyze the dynamics of networks of such units or "netlets" with the important variable of each unit being the mean activity, say.

[1] W. A. Little, Math. Biosci. **19**, 101 (1974).

[2] J. J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554 (1982).

[3] W. S. McCullogh and W. Pitts, Bull. Math. Biophys. **3**, 115 (1943).

[4] P. Peretto, Biol. Cybern. **50**, 51 (1984).

[5] J. W. Clark, Phys. Rep. **158**, 92 (1988).

[6] D. J. Amit, H. Gutfreund, and H. Sompolinsky, Phys. Rev. A **32**, 1007 (1985); Phys. Rev. Lett. **55**, 1530 (1985); Ann. Phys. (N.Y.) **173**, 30 (1987); D. J. Amit, *Modelling Brain Function* (Cambridge University Press, Cambridge, 1989).

[3] W. S. McCullogh and W. Pitts, Bull. Math. Biophys. **3**, 115 (1943).

[4] P. Peretto, Biol. Cybern. **50**, 51 (1984).

[5] J. W. Clark, Phys. Rep. **158**, 92 (1988).

[6] D. J. Amit, H. Gutfreund, and H. Sompolinsky, Phys. Rev. A **32**, 1007 (1985); Phys. Rev. Lett. **55**, 1530 (1985); Ann. Phys. (N.Y.) **173**, 30 (1987); D. J. Amit, *Modelling Brain Function* (Cambridge University Press, Cambridge, 1989).

[7] E. Gardner, J. Phys. A **21**, 257 (1988).

[8] E. R. Caianiello, J. Theor. Biol. **1**, 204 (1961).

[9] J. Nagumo and S. Sato, Kybernetik **10**, 155 (1972).

[10] P. C. Bressloff and J. G. Taylor, in *Proceedings of the International Neural Network Conference, Paris, 1990* (Kluwer Academic, Dordrecht, 1990); P. C. Bressloff and J. G. Taylor (unpublished).

[11] P. C. Bressloff and J. Stark, Phys. Lett. A **150**, 187 (1990); P. C. Bressloff, in *Proceedings of the Neural Computing Meeting, London, 1990,* edited by J. G. Taylor and C. L. T. Mannion (Springer-Verlag, New York, 1991).

[12] K. Aihara, T. Takabe, and M. Toyoda, Phys. Lett. A **144**, 333 (1990).

[13] I. Guyon, L. Personnaz, J. P. Nadel, and G. Dreyfus, Phys. Rev. A **38**, 6365 (1988).

[14] J. G. Taylor and M. Reiss (unpublished).

[15] W. S. Stornetta, T. Hogg, and B. A. Huberman, in *Neural Information Processing Systems,* edited by D. Z. Anderson (American Institute of Physics, New York, 1988); H.-U. Bauer and T. Geisel, Int. J. Neural Syst. **1**, 187 (1989).

[16] M. C. Mozer, Complex Syst. **3**, 349 (1989).

[17] J. G. Taylor (unpublished); P. C. Bressloff and J. G. Taylor (unpublished).

[18] L. F. Abbott, J. Phys. A **23**, 3835 (1990).

[19] C. M. Marcus and R. M. Westervelt, Phys. Rev. A **42**, 2410 (1990).

[20] G. Major, A. Larkman, and J. Jack, Physiol. Soc. **23**, 1 (1990).

[21] S. Grossberg, Neural Networks **1**, 17 (1988).

[22] J. Jack, D. Noble, and R. Tsein, *Electric Current Flow in Excitable Cells* (Clarendon, Oxford, 1975).

[23] W. Rall, *Neural Theory and Modelling,* edited by R. F. Reiss (Standord University Press, Stanford, 1964).

[24] N. Takahashi, Y. Hanyu, T. Musha, R. Kubo, and G. Matsumoto, Physica D **43**, 318 (1990).

[25] Z. Li and J. J. Hopfield, Biol. Cybern. **61**, 379 (1989); A. Atiya and P. Baldi, Int. J. Neural Syst. **1**, 103 (1989).

[26] D. Kleinfeld, Proc. Natl. Acad. Sci. U.S.A. **83**, 9469 (1986); H. Sompolinsky and I. Kanter, Phys. Rev. Lett. **57**, 2861 (1986).

[27] R. Kuhn, J. L. van Hemmen, and U. Reidel, J. Phys. A **22**, 3123 (1989).

[28] A. V. M. Herz, Z. Li, and J. L. van Hemmen, Phys. Rev. Lett. **66**, 1370 (1991).

[29] G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (Oxford University Press, Oxford, 1986).

[30] A. Lasota and M. C. Mackey, *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, 1985).

[31] A. Lasota and M. C. Mackey, Physica D **28**, 143 (1987).

[32] P. C. Bressloff, in *New Developments in Neural Computing,* edited by J. G. Taylor and C. L. T. Mannion (Hilger, Bristol, 1989); P. C. Bressloff and J. G. Taylor, Phys. Rev. A **41**, 1126 (1990).

[33] W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, Kogakusha, 1976).

[34] B. Katz, *The Release of Neural Transmitter Substances* (Thomas, Springfield, 1969).

[35] M. Misiurewicz, Ergod. Theor. Dynam. Sys. **6**, 117 (1986).

[36] Y. Kifer, *Ergodic Theory of Random Transformations* (Birkhauser, Boston, 1986).

[37] M. J. Feigenbaum and B. Hashler Phys. Rev. Lett. **49**, 605 (1982).

[38] E. C. Zeeman, Nonlinearity **1**, 115 (1988).

[39] R. S. MacKay (unpublished).

[40] J. Komornik, Tohoku Math. J. **38**, (1986).

[41] R. S. Mackay and C. Tresser, Physica D **19**, 206 (1986).

[42] D. O. Hebb, *The Organisation of Behavior* (Wiley, New York, 1949).

[43] S-I Amari, Trans. Syst. Man Cybern. **2**, 643 (1972).

[44] P. A. Anninos, B. Beek, T. J. Csermely, E. M. Harth, and G. Pertile, J. Theor. Biol. **26**, 121 (1970).