Dynamics of a soliton in a generalized Zakharov system with dissipation

Hichem Hadouaj, Boris A. Malomed,* and Gérard A. Maugin
Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, Tour 66, 4 place Jussieu, 75252 Paris CEDEX 05, France
(Received 19 February 1991)

A generalized Zakharov system (describing interaction of dispersive and nondispersive waves in one dimension), with direct self-interaction of the dispersive waves and weak dissipation in the dispersive subsystem, is considered. Evolution of a one-soliton state under the action of weak dissipation is analyzed. It is proved analytically that three different scenarios of evolution are possible: adiabatic (slow) transformation of a moving subsonic soliton into a stable quiescent one, complete adiabatic decay of a transsonic soliton with small amplitude, and the appearance of a transsonic one with a large amplitude into a critical state, from which a further adiabatic evolution is not possible (it corresponds to a local minimum in the dependence of the soliton's momentum on its velocity). In the latter case, numerical investigation of the further evolution of the soliton is performed. It is demonstrated that, in a general case, it abruptly splits into a stable quiescent soliton, the slowly decaying small-amplitude transsonic one, and a pair of left- and right-traveling acoustic pulses slowly fading under the action of weak dissipation.

I. INTRODUCTION

In this work, we will analyze the dynamics of a solitary pulse (soliton) governed by the generalized Zakharov system (ZS)

$$iu_t + u_{xx} + 2\lambda|u|^2u + 2nu = 0 , \tag{1.1}$$

$$n_{tt} - c^2 n_{xx} = -\mu(|u|^2)_{xx} + \gamma n_{txx} , \tag{1.2}$$

where the wave fields $u(x,t)$ and $n(x,t)$ are complex and real, respectively, and the parameters λ, c, μ, and γ are all real. The system of Eqs. (1.1) and (1.2) with $\lambda=0$ and $\gamma=0$ has been first derived by Zakharov in Ref. 1 to describe the interaction between Langmuir (dispersive) and ion acoustic (approximately nondispersive) waves in a plasma. Later, it has become commonly accepted that the ZS is a general model to govern interaction of dispersive and nondispersive waves in one dimension. In this sense, it is as universal as the nonlinear Schrödinger (NS) equation, which governs evolution of an envelope of weakly nonlinear dispersive waves [2]. The ZS has found a number of applications in various physical problems, such as interaction of intramolecular vibrations giving rise to Davydov solitons with acoustic disturbances [3], interaction of high-frequency and low-frequency gravity disturbances in an atmosphere [4], and so on. There are a few review papers surveying the dynamics of nonlinear waves governed by the ZS [5].

In all the works mentioned, the ZS appeared with $\lambda=0$ [see Eq. (1.1)], i.e., no direct self-interaction of the dispersive waves was presumed. In this work, we will deal with the generalized system ($\lambda\neq0$). The analysis of this system was stimulated by the recent work [6] of two of the present authors where propagation of nonlinear shear surface waves was considered in a model of a semi-infinite elastic body [7] covered by a thin "lid." It has been demonstrated that, among three branches of the surface waves in this model, one is dispersive and two are not, and propagation of the dispersive waves is governed by the NS equation provided their coupling to the nondispersive branches is ignored. In the same approximation, the nondispersive waves obey d'Alembert equations (with different sound velocities). It the coupling between the dispersive branch and nondispersive ones is taken into account, we arrive at a straightforward generalization of the ZS including two nondispersive components n_1 and n_2 (proof of this shall be given elsewhere). The equation for the dispersive component of the generalized ZS contains two different nonlinear terms: the direct self-interaction $\lambda|u|^2u$, like the single NS equation, and the nonlinear coupling to the nondispersive components, which can be written in the form $(n_1+n_2)u$. In this work, we deal with the generalized ZS based on Eqs. (1.1) and (1.2), i.e., for the sake of simplicity only one nondispersive component is retained. Note that the ZS in the form of Eqs. (1.1) and (1.2) can also be obtained in another way: take the ZS with two nondispersive components and with no direct self-interaction of the dispersive one,

$$iu_t + u_{xx} + 2(n_1+n_2)u = 0 , \tag{1.3}$$

$$(n_1)_{tt} - C_1^2(n_1)_{xx} = -\mu_1(|u|^2)_{xx} , \tag{1.4a}$$

$$(n_2)_{tt} - C_2^2(n_2)_{xx} = -\mu_2(|u|^2)_{xx} , \tag{1.4b}$$

and consider the case $C_2^2 \gg C_1^2$. It is well known that the usual ZS can be approximately reduced to the NS equation if the group velocities V (e.g., the velocities of solitons) in the dispersive component are much smaller than the sound velocity. So, in the case when $V^2 \simeq C_1^2 \ll C_2^2$ the fast nondispersive component n_2 can be excluded by means of the relation $n_2 \approx (\mu_2/C_2^2)|u|^2$, and the system of Eqs. (1.3) and (1.4) turns into the system of Eqs. (1.1) and (1.2) with $\gamma=0$ and $\lambda\equiv\mu_2/C_2^2$.

Let us now comment on the dissipative term in Eq. (1.2). In a realistic physical system, dissipation must be included into each equation, the term added to Eq. (1.2) being the usual viscous dissipative term in the equation for sound propagation. In this work, we concentrate on the case when the direct dissipative losses in the dispersive component may be neglected. As will be demonstrated below, in this case the dynamics of a soliton is most interesting.

As is well known, although the ZS is not exactly integrable [7], it has the exact one-soliton solution, which takes the following form for the system of Eqs. (1.1) and (1.2) with $\gamma=0$:

$$u_{\mathrm{sol}}(x,t)=\lambda_{\mathrm{eff}}^{-1/2}U_{\mathrm{sol}}(x,t) \ . \tag{1.5a}$$

Here

$$U_{\mathrm{sol}}(x,t)=2i\eta\,\mathrm{sech}[2\eta(x-Vt)] \times\exp[(i/2)Vx+i(4\eta^2-V^2/4)t] \tag{1.5b}$$

is the standard form of the NS soliton with amplitude η and velocity V, and

$$\lambda_{\mathrm{eff}}\equiv\lambda+\mu(C^2-V^2)^{-1} \tag{1.5c}$$

is the effective coefficient of self-interaction of the dispersive wave component. The nondispersive component of the soliton is

$$n_{\mathrm{sol}}(x,t)=4\mu\eta^2\lambda_{\mathrm{eff}}^{-1}(C^2-V^2)^{-1}\,\mathrm{sech}^2(2\eta x) \ . \tag{1.5d}$$

The soliton exists at the values of V^2 for which λ_{eff} is positive. As for the sign of λ, it may be both positive and negative, which corresponds, respectively, to the attractive and repulsive self-interaction of the dispersive waves. At the same time, we will presume the coupling constant μ positive. Otherwise, the proper energy of the nondispersive waves in the full Hamiltonian of the undamped ZS,

$$H=\int_{-\infty}^{+\infty}dx[|u_x|^2-\lambda|u|^4-2\rho_{xx}|u|^2 +\mu^{-1}(\rho_{tx}^2+\rho_{xx}^2)], \quad \rho_{xx}\equiv n \tag{1.6}$$

is negative.

As follows from Eq. (1.5c), at $\lambda>0$ the solitons exist ($\lambda_{\mathrm{eff}}>0$) in the *subsonic* range

$$V^2<C^2 \tag{1.7}$$

and in the *transsonic* one (it may also be called *supersonic)*

$$V^2>C^2+\mu/\lambda \ , \tag{1.8}$$

while there are no soliton solutions ($\lambda_{\mathrm{eff}}<0$) in the *gap* $C^2<V^2<C^2+\mu/\lambda$. Note that, according to Eq. (1.5d), the nondispersive component of the solitonic wave field is positive in the subsonic case, and negative in the transsonic one.

If $\lambda<0$, the solitons exist only inside the gap

$$C^2=\mu/|\lambda|<V^2<C^2 \ , \tag{1.9}$$

provided $\mu/|\lambda|<C^2$, or in the whole subsonic range (1.7) if $\mu/|\lambda|>C^2$.

The objective of the present work is to study the evolution of the soliton under the action of the weak dissipation in the nondispersive subsystem [see Eq. (1.2)]. In Sec. II, this is done analytically by means of the simplest technique, based on the balance equations for the wave action and momentum, which are integrals of motion of the unperturbed one [Eqs. (1.5)], but the parameters η and V undergo a slow (adiabatic) evolution. The analysis demonstrates that in the case $\lambda>0$ an initial state in the form of the subsonic soliton always evolves into the quiescent soliton ($V=0$), which remains the obvious exact solution of the damped ZS system. As for the transsonic solitons, two different routes of evolution are possible. If the initial amplitude of the soliton is sufficiently small, the dependence of the soliton's momentum P on its velocity V is monotonous, and the dissipation-induced evolution results eventually in a complete decay of the soliton. In the opposite case, when the initial amplitude is sufficiently large, the dependence $P(V)$ proves to be nonmonotonous, see Fig. 2(b) below. In this case, during a finite time the transsonic soliton reaches a value $V=V_{\mathrm{cr}}$ corresponding to a local minimum of P. At this critical point, a "catastrophe" must happen with the soliton, as its full momentum must keep decreasing under the action of the dissipation, but this is impossible if the soliton retains a form close to the unperturbed one given by Eqs. (1.5), just because this is the state corresponding to the minimum of P. We demonstrate that the perturbative analysis becomes irrelevant for V very close to V_{cr}. To follow the further evolution, Eqs. (1.1) and (1.2) have been integrated numerically. The results of the simulations, presented in Sec. III, demonstrate that, at $V=V_{\mathrm{cr}}$, the transsonic soliton splits very quickly into a few new solitary pulses, one of them being a subsonic soliton which finally turns into the stable quiescent one in accordance with the analysis developed in Sec. II. In addition, two strong acoustic pulses, i.e., those which are salient in the n component, but have no counterparts in the u component, are generated. The acoustic pulses propagate with the velocities $\pm C$ and are slowly damped by the dissipation. At last, in some cases an additional transsonic soliton with a smaller amplitude is also formed after the splitting of the original one. This secondary transsonic soliton slowly decays as was predicted analytically in Sec. II.

In the case $\lambda<0$, the dependence $P(V)$ is always monotonous, and the initial soliton residing in the gap (1.9) always drifts to the left boundary of the gap. If $\mu/|\lambda|<C^2$, it decays like the transsonic soliton with the small initial amplitude in the case $\lambda>0$, and if $\mu/|\lambda|>C^2$, it turns into the quiescent soliton ($V=0$) like the subsonic soliton at $\lambda>0$.

At last, in Sec. IV we briefly discuss some problems for the generalized ZS related to soliton-soliton interactions in this system.

II. ANALYTICAL TREATMENT OF THE EVOLUTION OF THE SOLITON

A. The general analysis

To apply the balance-equation analysis to the slow dissipation-induced evolution of the soliton (1.5), let us note, first of all, that the damped ZS [Eqs. (1.1) and (1.2)] conserves the total wave action

$$N=\int_{-\infty}^{+\infty}|u(x,t)|^2dx \ . \tag{2.1}$$

In the aforementioned surface-wave problem [6], the conserved quantity (2.1) may be regarded as a full number of the surface phonons.

In various fields of physics, other examples of dissipative systems based on the NS equation are known that conserve the wave action despite the presence of dissipation. Important examples are the evolution equation for the envelope, of Langmuir waves in a plasma, which takes account of the nonlinear Landau damping (see e.g., Ref. [8]), and the equation for the envelope of electromagnetic waves in a nonlinear optical fiber with regard to the intrapulse Raman scattering [9]. The latter equation has the form

$$iu_z+u_{\tau\tau}+2|u|^2u=\gamma(|u|^2)_\tau u \ , \tag{2.2}$$

γ being a real perturbation parameters. It is straightforward to see that the perturbing term in Eq. (2.2) is dissipative, but, nonetheless, it conserves the wave action (2.1). It is known [9,10] that, in both physical systems mentioned, the nonlinear dissipative term, regarded as a small perturbation, acts upon a soliton like a constant accelerating force, i.e., the soliton's amplitude (which is proportional to the conserved wave action) remains constant, while its velocity grows linearly with time.

Inserting Eqs. (1.5) into Eq. (2.1) yields the value of N for the soliton of the generalized ZS: $N=4\eta/[\lambda+\mu(C^2-V^2)^{-1}]$, hence one of the evolution equations for the soliton can be written in the form

$$\eta=\tfrac{1}{4}N[\lambda+\mu(C^2-V^2)^{-1}] \ . \tag{2.3}$$

It is implied that N is the conserved quantity given by an initial condition, while the amplitude η and velocity V slowly evolve so that the relation (2.3) between them remains fulfilled in the adiabatic approximation.

To obtain the second evolution equation, we will consider the balance equation for the full momentum of the system. Note that the momentum-balance equation was effectively used to study the motion of a kink (topological soliton) in a perturbed sine-Gordon model [11]. To define the full momentum of the undamped ZS, we need to introduce the "potential" $v(x,t)$ of the nondispersive wave field: $n\equiv v_x$. In terms of $u(x,t)$ and $v(x,t)$, the full momentum is

$$P=\int_{-\infty}^{+\infty}[i(uu_x^*-u^*u_x)-4\mu^{-1}v_xv_t]dx \tag{2.4}$$

[cf. expression (1.6) for the Hamiltonian in terms of $\rho(x,t)$]. Differentiating Eq. (2.4) in time and inserting Eqs. (1.1) and (1.2), one arrives at the balance equation for the momentum:

$$\frac{dP}{dt}=(4\gamma/\mu)\int_{-\infty}^{+\infty}n_xn_tdx \ . \tag{2.5}$$

Next, assuming, as was said above, that the slowly evolving soliton retains a form close to the unperturbed one, we substitute Eqs. (1.5) into both the left-hand and right-hand sides of Eq. (2.5). Excluding the amplitude η in favor of N by means of Eq. (2.3), we find the soliton's momentum

$$P=NV+\tfrac{2}{3}\mu N^3V(C^2-V^2)^{-2}[\lambda+\mu(C^2-V^2)^{-1}] \ , \tag{2.6a}$$

where the first and second term on the right-hand side are contributions from the dispersive and nondispersive components, respectively [see Eq. (2.4)]. The dissipation-induced rate of change of the momentum is

$$\frac{dP}{dt}=-\tfrac{3}{20}\gamma\mu N^5V(C^2-V^2)^{-2}[\lambda+\mu(C^2-V^2)^{-1}]^5 \ . \tag{2.6b}$$

Thus, Eqs. (2.6) give the evolution equation for the velocity V in the closed form [recall that the amplitude η has been excluded by means of Eq. (2.3)].

For definiteness, in what follows we will consider positive values of V. As explained in Sec. I, we deal with positive μ only, and the existence of the soliton implies that the combination (1.5c) is positive too. Thus, it follows from Eq. (2.a) that $P>0$, and Eq. (2.6b) tells us that $dP/dt<0$. The positiveness of P and the negativeness of dP/dt will play an important role in the subsequent analysis.

B. $\lambda>0$, the subsonic range

For $\lambda>0$, the solitons exist in the subsonic range (1.7) and in the transsonic one (1.8). In the former range, the dependence $P(V)$ following from Eq. (2.6a) takes the form shown in Fig. 1. According to what was said above, the dissipation gives rise to the slow decrease of the momentum (breaking of the soliton). Thus the soliton's velocity must drift in the direction shown by the arrow in Fig. 1, and at $t\to\infty$ the velocity vanishes. This means that, asymptotically, the initial subsonic soliton turns into the one given by Eqs. (1.5) with $V=0$. Note that the maximum value of the soliton's wave field, equal to

$$\begin{aligned}|u|_{\max}&=2\eta/\sqrt{\lambda+\mu(C^2-V^2)^{-1}}\\&\equiv\tfrac{1}{2}N\sqrt{\lambda+\mu(C^2-V^2)^{-1}}\end{aligned} \tag{2.7}$$

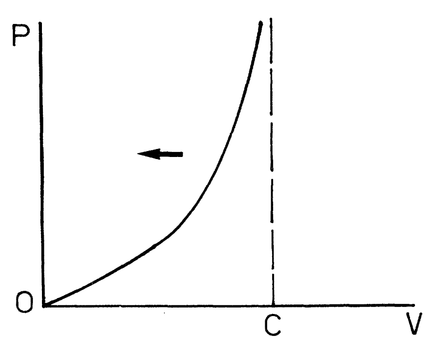

FIG. 1. Dependence of the soliton momentum P on its velocity V [Eq. (2.6(a)] in the subsonic range (at $\lambda>0$). The arrow indicates the direction of the adiabatic dissipation-induced drift of the velocity.

according to Eqs. (1.5) and (2.3), diminishes as V^2 decreases. So, the final state (the quiescent soliton) is less steep and more broad than the initial one.

C. $\lambda > 0$, the transsonic range

In the transsonic range (1.8), the dependence $P(V)$ may take two principally different forms shown in Figs. 2(a) and 2(b). The former case takes place if the wave action, or, in other words, the initial amplitude of the soliton [see Eq. (2.3)], is sufficiently small. In this case the momentum monotonously grows from the value $P_0 = N\sqrt{C^2+\mu/\lambda}$ [Fig. 2(a)] with the increase of V from $\sqrt{C^2+\mu/\lambda}$ to infinity. Accordingly, under the action of the dissipation the value of V monotonously decreases from some initial value V_0 to the boundary value $\sqrt{C^2+\mu/\lambda}$. At the asymptotic stage of the evolution ($t \to \infty$), the evolution equation (2.6) for the soliton's velocity takes the form

$$N\dot{V} = -\tfrac{3}{20}\gamma\mu V(C^2-V^2)^2\lambda_{\text{eff}}^5 , \tag{2.8}$$

where λ_{eff} is the quantity (1.5c). Finally, it follows from Eq. (2.8) that, at $t \to \infty$,

$$\lambda_{\text{eff}}^{-4} \sim \gamma N^4 t . \tag{2.9}$$

Substituting Eq. (2.9) into Eqs. (1.5a), (1.5b), and (2.3), we conclude that the soliton spreads unlimitedly, its width growing according to the law

$$\eta^{-1} \sim N^{-1}\lambda_{\text{eff}}^{-1} \sim (\gamma t)^{1/4} , \tag{2.10}$$

which does not depend on the value of N. The maximum value of the soliton's wave field decreases as follows [cf. Eq. (2.7)]:

$$|u|_{\max} \sim N\sqrt{\lambda_{\text{eff}}} \sim \sqrt{N}(\gamma t)^{-1/8} , \tag{2.11}$$

i.e., very slowly. As a matter of fact, the evolution of the transsonic soliton governed by Eqs. (2.10) and (2.11) may be called its decay.

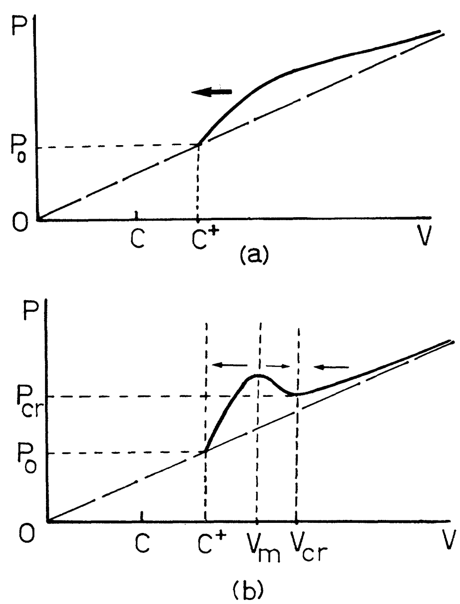

FIG. 2. Dependence $P(V)$ in the transsonic range (at $\lambda > 0$). The dashed asymptotic line, $P = NV$, gives the contribution of the dispersive component to the full momentum, see Eq. (2.6a). (a) The small amplitude of the soliton; (b) the large amplitude. The arrows have the same meaning as in Fig. 1. $C^+ = (C^2+\mu/\lambda)^{1/2}$.

Let us proceed to the case corresponding to Fig. 2(a), when the dependence $P(V)$ is nonmonotonous. This case can be surely realized if N is sufficiently large. Analyzing the dependence $P(V)$ given by Eq. (2.6a), one can see that, provided $N^2 \gg C^4/\mu\lambda$, the momentum attains the local minimum at the point

$$V_{\min} \simeq (2\mu\lambda N^2)^{1/4} . \tag{2.12}$$

It is worth mentioning that a nonmonotonous dependence of the full soliton's momentum on its velocity has been recently revealed [12] in the system consisting of the sine-Gordon equation coupled to one or two D'Alembert equations (according to Ref. [13], this system describes dynamics of elastic ferromagnets and ferroelastics).

Again, the momentum must monotonously decrease under the action of the dissipation. If the initial soliton's velocity V_0 lies between the local maximum V_m and the boundary value $\sqrt{C^2+\mu/\lambda}$ [Fig. 2 (a)], the evolution of the soliton will be qualitatively the same as in the preceding case, i.e., it will eventually decay according to Eqs. (2.10) and (2.11). However, if V_0 lies to the right of V_m, the soliton must adiabatically drift to the state with $V = V_{\text{cr}}$ corresponding to the minimum value $P = P_{\text{cr}}$, see Fig. 2(a). Note that the drift from $V = V_0$ to $V = V_{\text{cr}}$ takes finite time, as the rate of change of the momentum dP/dt does not vanish at $V = V_{\text{cr}}$ [see Eq. (2.6b)]. When the velocity attains the value V_{cr}, the soliton must keep decreasing its momentum according to Eq. (2.6b), but, being in the state with the minimum momentum, it has no way to do this adiabatically. Thus we can expect that the soliton cannot retain its nearly unperturbed form, and some "catastrophe" must abruptly happen with it at V close to V_{cr}. The "catastrophe" will be investigated, by means of the direct numerical integration of the underlying equations (1.1) and (1.2), in the next section. Here it is pertinent to note that the perturbative analysis gets invalid at small values of $\delta V \equiv V - V_{\text{cr}}$. Indeed, as the dependence $P(V)$ has a minimum at $V = V_{\text{cr}}$, we have $P - P_{\text{cr}} \simeq (\delta V)^2$ at $\delta V \to 0$, hence we must insert into the evolution equation (2.6b)

$$\frac{dP}{dt} \equiv \frac{dP}{dV}\dot{V} \sim \delta V \delta\dot{V} \tag{2.13}$$

($\delta\dot{V} \equiv \dot{V}$). Since the right-hand of Eq. (2.6b) has no peculiarity at $V = V_{\text{cr}}$, Eqs. (2.6b) and (2.13) yield

$$(\delta V)^2 \sim \gamma(t_{\text{cr}} - t) , \tag{2.14}$$

where t_{cr} is the critical moment when $V = V_{\text{cr}}$. According to Eq. (2.14), the derivative

$$\delta\dot{V} \sim \gamma/\sqrt{t_{\text{cr}} - t} \tag{2.15}$$

diverges at $t_{\text{cr}} - t \to 0$. In the same time, the perturbative analysis is applicable provided the velocity changes sufficiently slowly, i.e., $\delta\dot{V}$ is small. Thus Eq. (2.15) implies that the perturbation theory does not apply at small values of $t_{cr}-t$, i.e., at small $(\delta V)^2$, see Eq. (2.14). Equation (2.15) suggests also that the rearrangement of the soliton near $V=V_{cr}$ must be rather quick, which is corroborated by the numerical results presented in Sec. III.

D. The case $\lambda<0$

If λ is negative, the solitons exist inside the gap (1.9). If $\mu/|\lambda|>C^2$, the left boundary of the gap is, in fact, the point $V=0$. In this case, the situation is qualitatively similar to that shown in Fig. 1, i.e., the soliton will stop eventually. If $\mu/|\lambda|<C^2$, the dependence $P(V)$ takes the form shown in Fig. 3, where $P_0 \equiv N\sqrt{C^2-\mu/|\lambda|}$. In this case, the soliton drifts to the boundary point $V=\sqrt{C^2-\mu/|\lambda|}$ where λ_{eff} vanishes, see Eq. (1.5c). So, the soliton decays asymptotically according to Eqs. (2.10) and (2.11).

III. NUMERICAL RESULTS: REARRANGEMENT OF THE SOLITON AT THE CRITICAL POINT

To analyze the further evolution of the soliton after the critical point $V=V_{cr}$ has been reached, we integrated numerically Eqs. (1.1) and (1.2) (for $\lambda>0$) with initial conditions corresponding to a transsonic soliton. The simulations were performed with the values of the parameters $C=1$, $\lambda=1$, $\mu=1$, $\gamma=0.1$ [in fact, one can always set $C=\mu=1$, making an obvious scale transformation in Eqs. (1.1) and (1.2)]. An Euler implicit scheme was used for the NS equation [14] while a simple explicit-difference scheme is quite sufficient for the wave equation. The numerical plots give successive snapshots every 70–100 time computational steps (the precise number depends on the figure).

First of all, we have taken the initial values

$$N=1, \quad V_0^2=3\;, \tag{3.1}$$

which, as one can check, correspond to the situation of Fig. 2(a). The evolution of the dispersive and nondispersive components of the wave field is shown in Fig. 4. As is seen (with regard to the linear perspective artificially introduced by the computer plotter) from Fig. 4, the soliton slowly spreads in accordance with the prediction drawn in the preceding section.

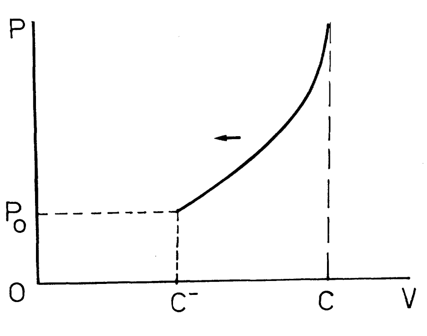

FIG. 3. Dependence $P(V)$ in the case $\lambda<0$, $\mu/|\lambda|<C^2$. The arrow has the same meaning as in Figs. 1 and 2. $C^-=(C^2-\mu/|\lambda|)^{1/2}$.

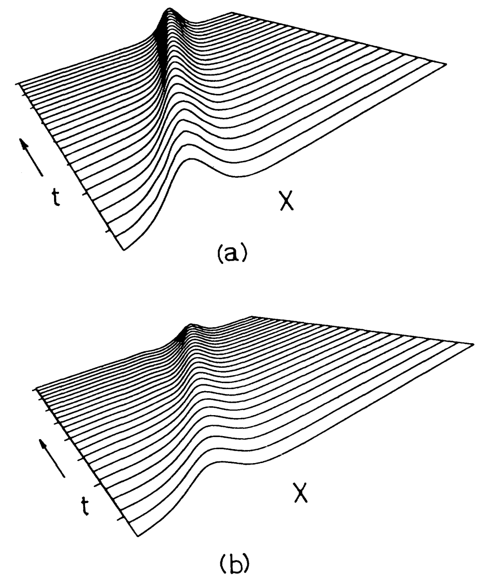

FIG. 4. Results of the numerical integration of Eqs. (1.1) and (1.2) with the initial data (3.1): (a) the dispersive component $|u|^2$; (b) the nondispersive one, $-n$. Note that the graphs are plotted with an artificially introduced linear perspective; $N=1$, $V^2=3$.

The opposite case, when the local minimum is well pronounced on the dependence $P(V)$ [Fig. 2(b)], corresponds to the initial data

$$N=5, \quad V_0^2=10\;, \tag{3.2}$$

when the initial point lies well on the right of V_{cr}. The evolution of the soliton is shown in Fig. 5. At the initial stage, which proves to be very short in this case, the soliton evolves adiabatically, and then, at the critical point, it abruptly splits into three pulses. Comparing the signals in the u and n components [Figs. 5(a) and 5(b)], we conclude that the central signal ($n>0$) corresponds to the stable quiescent ($V=0$) soliton, while the left-going and right-going ones ($n<0$), which have no counterparts in the u component, are acoustic pulses that propagate at the velocities $-C$ and $+C$ and slowly fade under the action of the dissipation.

To see the rearrangement of the soliton in the intermediate case, we have also taken the initial values

$$N=2, \quad V_0^2=3 \tag{3.3}$$

[cf. the initial data (3.1)]. In this case, the numerical simulation demonstrates a rather long adiabatic evolution of the initial soliton (Fig. 6), which is changed by the abrupt (but less abrupt than in Fig. 5) splitting into four pulses. These pulses may be identified as the stable quiescent soliton (the central signal, $n>0$, in the n component and its counterpart in the u component), the slowly spreading transsonic soliton, similar to that shown in Fig. 4 (the ultimate right signal in the n component and its

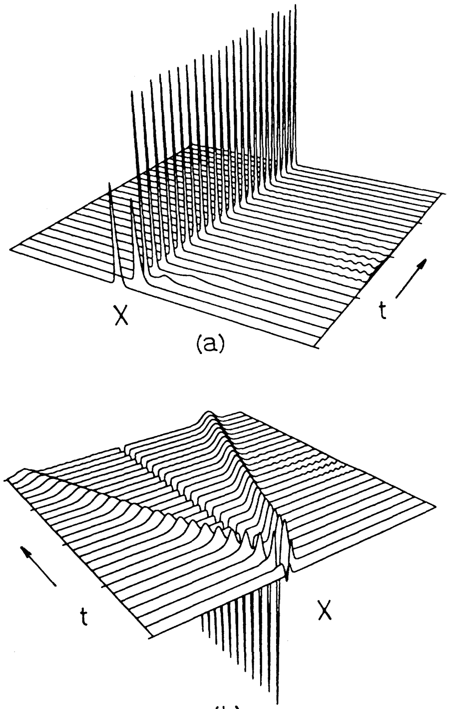

FIG. 5. The same as in Fig. 4 for the initial data (3.2); $N=5$, $V^2=10$.

counterpart in the u subsystem), and two slowly fading acoustic pulses with no u counterparts.

The effect revealed in the numerical simulation, viz., the abrupt splitting of the soliton under the action of the *weak* dissipation, seems to be a new type of inelastic process for a soliton induced by small perturbations (see the review of the perturbation theory for solitons given in Ref. [15]).

So far, we ignored the possible presence of dissipation in the dispersive subsystem. If that dissipation is present, it will give rise to a decrease of the wave action N. According to the results obtained above, we expect that, if the additional dissipation is sufficiently strong, it will prevent the splitting of the soliton into the secondary pulses. If this dissipation is weak, it is natural to expect that it will render the splitting less abrupt.

IV. CONCLUSION

In this work, we have investigated, analytically and numerically, the evolution of one soliton in the damped generalized ZS. Another interesting class of problems takes its origin in soliton-soliton interactions. First of all, a collision of two solitons in the presence of the dissipation in the nondispersive subsystem may result in a fusion of the solitons into the so-called breather (a bound state),

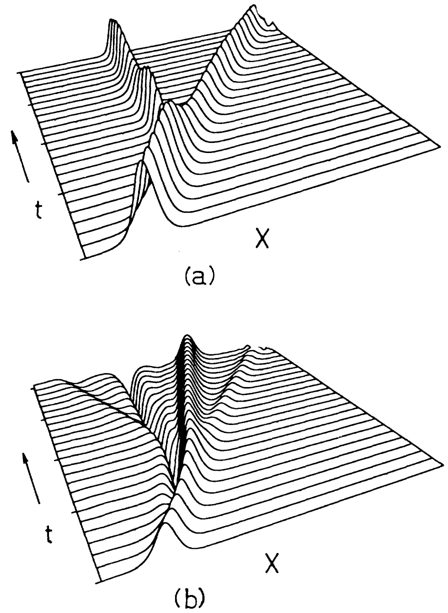

FIG. 6. The same as in Fig. 4 for the initial data (3.3); $N=2$, $V^2=3$.

provided their relative velocity is sufficiently small [16]. Due to the specificity of this dissipative perturbation, the total wave action must be conserved in the course of the fusion. If the relative velocity is not small, it is natural to expect that the collision will give rise to an *exchange* of the wave action between the two solitons, the total wave action (i.e., the sum of the solitons' amplitudes) being conserved [17]. It is necessary to note that, since the undamped ZS is not exactly integrable, the soliton-soliton collision may be accompanied by conspicuous *radiative* losses even in the absence of dissipation, provided the system is not close to its exactly integrable NS limit, i.e., if the relative velocity of the colliding solitons is not small in comparison with the sound velocity C of the nondispersive subsystem. This problem deserves a detailed analysis which is under way now.

At last, let us note that the undamped ZS admits, alongside the exact one-soliton solutions, the obvious exact solutions in the form of linear acoustic (nondispersive) waves with no dispersive component. Inelastic interactions between the soliton and the linear acoustic wave have been investigated in detail (by means of the perturbative analysis in the near-NS regime) in Ref. [18] for the ZS proper, i.e., the one with $\lambda=0$ [see Eq. (1.1)]. It is straightforward to extend the results obtained in Ref. [18] to the generalized system with $\lambda\neq0$.

ACKNOWLEDGMENT

The Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, is "associé au CNRS."

*Permanent address: P. P. Shirshov Institute for Oceanology, U.S.S.R. Academy of Sciences, 23 Krasilov Street, Moscow, 117259, U.S.S.R.

[1] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys.—JETP **35**, 908 (1972)].

[2] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevsky, *Theory of Solitons* (Nauka, Moscow, 1980) (English translation published by Consultants Bureau, New

York, 1984).
[3] A. S. Davydov, Phys. Scr. **20**, 387 (1979).
[4] L. Stenflo, Phys. Scr. **33**, 156 (1986).
[5] D. R. Shukla, in *Nonlinear Waves,* edited by L. Debnath (Cambridge University Press, Cambridge, England, 1983); M. V. Goldman, Rev. Mod. Phys. **56**, 709 (1984); H. L. Pécseli, IEEE Trans. Magn. **13**, 53 (1985).
[6] H. Hadouaj and G. A. Maugin, C. R. Acad. Sci. (Paris) **II-309**, 1877 (1989); G. A. Maugin and K. Hadouaj, Phys. Rev. B **44**, 1266 (1991).
[7] E. I. Schulman, Dokl. Akad. Nauk SSSR **259**, 579 (1981) [Sov. Phys. Dokl. **26**, 691 (1981)].
[8] Y. H. Ichikawa and T. Taniuti, J. Phys. Soc. Jpn. **34**, 513 (1973); H. L. Pécseli and K. B. Dysthe, J. Plasma Phys. **19**, 931 (1977).
[9] J. P. Gordon, Opt. Lett. **11**, 662 (1986); Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. **QE-23**, 510 (1987). Note that in the NS equation governing the evolution of envelopes of the electromagnetic waves in a fiber [e.g., Eq. (2.2)], the traveling coordinate Z plays the role of the evolutional variable, while the reduced time τ is a spatial-like one.
[10] Y. H. Ichikawa, Phys. Scr. **20**, 296 (1979).
[11] P. L. Christiansen and O. H. Olsen, Wave Motion **4**, 163 (1982); D. J. Bergman, E. Ben-Jacob, Y. Imry, and K. Maki, Phys. Rev. A **27**, 3345 (1983); O. H. Olsen and M. R. Samuelsen, Phys. Rev. B **28**, 210 (1984); O. A. Levring, M. R. Samuelsen, and O. H. Olsen, Physica D **11**, 349 (1984).
[12] Yu. S. Kivshar and B. A. Malomed, Phys. Rev. B **42**, 8561 (1990).
[13] J. Pouget and G. A. Maugin, Phys. Rev. B **30**, 5304 (1984); **31**, 4633 (1985); Phys. Lett. A **109**, 389 (1985); G. A. Maugin and A. Miled, Phys. Rev. B **33**, 4830 (1986).
[14] M. Delfour, M. Fortin, and G. Payre, J. Comput. Phys. **44**, 277 (1981).
[15] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[16] B. A. Malomed, Physica D **15**, 374 (985).
[17] The soliton-soliton collision in the presence of the wave-action-conserving dissipation perturbation of Eq. (2.2) has been investigated in detail by numerical methods in S. Chi and S. Wen, Opt. Lett. **14**, 1216 (1989).
[18] B. A. Malomed, Phys. Scr. **38**, 66 (1988).