Undamped plasma waves

James Paul Holloway
University of Michigan, Ann Arbor, Michigan 48109-2104

J. J. Dorning
University of Virginia, Charlottesville, Virginia 22903-2442
(Received 20 August 1990)

In this paper we describe small-amplitude nonlinear plasma wave solutions to the one-dimensional Vlasov-Maxwell equations. The methods used to construct these waves rely on the decomposition of the distribution functions into odd and even parts and on using BGK forms to represent these pairs of functions; further manipulations using dimensional-reduction techniques from nonlinear functional analysis reduce the problem exactly to an algebraic equation that can be analyzed using bifurcation theory. Using these methods, we first develop a sufficient condition for waves of a given phase velocity to exist arbitrarily close to a given spatially uniform Vlasov equilibrium. Along with this condition we derive sufficient analytical information for the construction of approximate expressions for the electric potential and distribution functions, with exact knowlege of the asymptotic behavior of the error terms. These results have a very surprising physical implication: the Landau damping of small-amplitude waves is not inevitable. Instead, there exist plasma waves that trap particles even at arbitrarily small amplitude and do not damp.

I. INTRODUCTION

One of the central efforts in the study of collisionless plasmas has traditionally been the analysis of plasma waves, and especially plasma waves that are small-amplitude perturbations of one of the infinity of spatially uniform equilibria which collisionless plasmas possess. Much of our basic intuition about such plasmas is based on the properties of these small-amplitude waves, and they have provided the foundation on which much of the theory of collisionless plasmas is built. Even the nonlinear theories of plasma waves are based on the results of the linear analysis: we use the dispersion relations from linear theory to construct nonlinear dispersive wave equations; we characterize the nonlinear interactions of waves described by the linear theory; we try to model the effect of a broad spectrum of linear waves on the form of the spatially uniform plasma equilibria; we build Landau damping into theories of plasma turbulence; or we use the linear analysis to infer the stability of a plasma device.

Since first posed by Vlasov [1], the problem of small-amplitude waves has been extensively studied by linearizing the Vlasov-Maxwell equations. Despite extensive analysis even of just the one-dimensional problem [1–12], the basic conclusions of the linear analysis have not changed since the seminal paper of Landau [2] and the following developments of Van Kampen [3], Case [4], Jackson [5], and Backus [6]. When carefully reviewed, the analysis of the linearized equations that these authors have developed tells us that the time evolution of the electric field generated by a smooth initial perturbation (in fact analytic in a sufficiently wide strip in complex velocity) to a smooth spatially uniform equilibrium described by the functions F_α will be governed by the Landau dispersion relation, which is the analytic continuation of

$$D(\lambda,k,F_1,F_2,\ldots,F_N) = 1-\frac{4\pi}{k^2}\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}\frac{F'_\alpha(u)}{u+\lambda/ik}du \quad (1)$$

from $\mathrm{Re}(\lambda)>0$ to $\mathrm{Re}(\lambda)\leq 0$. From this dispersion relation the phenomena of Landau damping—the exponential decay of the electric field of a perturbation to certain plasma equilibria—can be derived. It is also clear from this classical analysis, although seldom emphasized, that initial distributions that are not so smooth (although not necessarily discontinuous) may lead to electric-field decay at rates much slower than the exponential rates of Landau damping. This was recognized by Van Kampen [3], and Weitzner [7] actually constructed an explicit example of a perturbation to the Maxwellian which gave rise to electric-field decay like t^{-3}. Van Kampen's analysis, and also Case's [4], was in fact based on the observation that very singular distribution functions (in fact, not functions at all but first-order distributions in the sense of Schwartz) would satisfy the linearized equations with periodic spatial and temporal dependence for the electric field, but without Landau damping.

It is interesting to note that in all of this analysis very little comment has been made on what exactly "small amplitude" means. Does it mean small electric potential? Does it mean distribution functions for wave and equilibrium that are close? Close how? The results that we describe here are based on exact nonlinear analysis of the equations using an old idea from plasma physics and some methods from modern nonlinear functional analysis; the solutions that have resulted from this marriage raise some interesting new questions concerning the meaning of small amplitude, the relevance of the linearized equations to nonlinear plasmas, and the validity of conclusions drawn from their analysis.

II. THE BASIC MODEL

In this paper we shall be concerned with a plasma that is sufficiently hot and rarefied to admit the Vlasov-Maxwell description, and we shall further only consider the one-dimensional form of the Vlasov-Maxwell equations, which are appropriate for longitudinal waves along a magnetic field. The density of a species α near a position x and velocity u at time t in an N-component plasma is denoted by $f_\alpha(x,u,t)$, while the self-consistent electric field in the x direction, which the plasma particles generate, is denoted by $E(x,t)$. These various quantities are then related by the one-dimensional Vlasov-Maxwell equations

$$\frac{\partial f_\alpha(x,u,t)}{\partial t}+u\frac{\partial f_\alpha(x,u,t)}{\partial x}+\frac{q_\alpha}{m_\alpha}E(x,t)\frac{\partial f_\alpha(x,u,t)}{\partial u}=0\ , \tag{2}$$

$$\frac{\partial E(x,t)}{\partial x}=4\pi\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}f_\alpha(x,u,t)du\ , \tag{3}$$

$$\frac{\partial E(x,t)}{\partial t}+4\pi\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}uf_\alpha(x,u,t)du=0\ . \tag{4}$$

Here m_α and q_α denote the mass and charge of particles of species α.

If we search for spatially uniform solutions of these equations we are led to

$$\frac{\partial f_\alpha(u,t)}{\partial t}+\frac{q_\alpha}{m_\alpha}E(t)\frac{\partial f_\alpha(u,t)}{\partial u}=0\ , \tag{5}$$

$$0=4\pi\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}f_\alpha(u,t)du\ , \tag{6}$$

$$\frac{dE(t)}{dt}+4\pi\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}uf_\alpha(u,t)du=0\ . \tag{7}$$

Differentiating Eq. (7) with respect to time and evaluating the resulting integral using Eq. (5) yields

$$\begin{aligned}\frac{d^2E(t)}{dt^2}&=\left[4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}u\frac{\partial f_\alpha(u,t)}{\partial u}du\right]E(t)\\&=-\left[4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}f_\alpha(u,t)du\right]E(t)\ ,\end{aligned} \tag{8}$$

where the integral has been evaluated by parts under the hypothesis that $f_\alpha(u,t)\to 0$ as $|u|\to\infty$. From Eq. (5) we can easily see that $n_\alpha=\int f_\alpha(u,t)du$ is a constant and so

$$E(x,t)=E_0\cos(\omega_p t)-\frac{4\pi}{\omega_p}J_0\sin(\omega_p t)\ , \tag{9}$$

where E_0 is the initial electric field and J_0 is the initial current. The frequency ω_p is defined by

$$\omega_p^2=4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}n_\alpha \tag{10}$$

and is well known from linearized fluid descriptions of the plasma, but in fact appears here in exact solutions to the nonlinear kinetic equations. In the case of a single-plasma species ($N=1$) and a fixed neutralizing background [modeled by the addition of a constant to the right-hand side of Eq. (3)] it can be shown more generally that the spatial average of the electric field oscillates with the plasma frequency ω_p [13,14], but this does not hold for the general multispecies case considered here.

Using Eq. (9) for the electric field in the Vlasov equation, Eq. (5), we find

$$f_\alpha(x,u,t)=F_\alpha\left[u+\frac{4\pi q_\alpha J_0}{m_\alpha\omega_p^2}[1-\cos(\omega_p t)]-\frac{q_\alpha}{m_\alpha\omega_p}E_0\sin(\omega_p t)\right]\ , \tag{11}$$

where $F_\alpha(u)$ is an arbitrary initial distribution function. Substituting this into Gauss's law, Eq. (6), implies that the functions F_α are not quite arbitrary, but are constrained by the zero-net-charge relation

$$0=\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}F_\alpha(u)du\ . \tag{12}$$

Finally, while the initial electric field is arbitrary, the initial current must be calculated from the initial distribution functions; substituting into Eq. (7) we see that

$$J_0=\sum_{\alpha=1}^{N}\int_{-\infty}^{\infty}uF_\alpha(u)du\ . \tag{13}$$

Thus we can completely solve the spatially uniform problem and discover that the spatially uniform solutions of the one-dimensional Vlasov-Maxwell equations simply describe plasma oscillations.

There are some special cases of these spatially uniform oscillations that deserve our special attention: the Vlasov equilibria. When $E_0=0$ and $J_0=0$ we see that the electric field becomes zero for all time and the distribution functions become independent of time. It is the dynamics near these spatially uniform equilibria that the linear theory attempts to describe, and which we shall study using nonlinear methods.

Before proceeding, one remark is called for concerning the Vlasov equilibria: usually the spatially uniform equilibria are described as being arbitrary save for satisfaction of the zero-net-charge constraint; here we require not only zero net charge but zero current as well. The difference comes about because we choose to solve the one-dimensional Vlasov-Maxwell equations rather than the one-dimensional Vlasov-Poisson equations, which are frequently considered in other works. These two systems are by no means directly equivalent [15,16], indeed the connection between them is rather subtle [16].

This development of the Vlasov equilibria brings us to the starting point for the linear theory. We have no need to review the details of this theory; they are well known.

But we should like to make one observation concerning the linearization; to produce a linearized description of the plasma we would write $f_\alpha = F_\alpha + g_\alpha$ where F_α describes the spatially uniform Vlasov equilibrium in whose neighborhood we are interested. Introducing this into the Vlasov equation for f_α yields

$$\frac{\partial g_\alpha(x,u,t)}{\partial t} + u\frac{\partial g_\alpha(x,u,t)}{\partial x} + \frac{q_\alpha}{m_\alpha}E(x,t)\frac{dF_\alpha(u)}{du} + \frac{q_\alpha}{m_\alpha}E(x,t)\frac{\partial g_\alpha(x,u,t)}{\partial u} = 0 \ . \tag{14}$$

The linear analysis would now proceed with the assumption that the nonlinear plasma-field interaction term is small compared to the linear plasma-field interaction term, namely

$$\left|\frac{\partial g_\alpha(x,u,t)}{\partial u}\right| \ll \left|\frac{dF_\alpha}{du}(u)\right| \ . \tag{15}$$

Using this assertion to justify the omission of the nonlinear term then produces the linear equations. Thus, in order to arrive at the linear description it is necessary to assume that the velocity gradient of the deviation g_α is small compared to the velocity gradient of the equilibrium F_α. This assumption—which must in some way constitute a part of the meaning of "small amplitude" as far as the linear theory is concerned—is certainly not necessary in order to ensure a small electric potential or in order to ensure that the deviation g_α itself is small. So, to derive the linear equations a rather strong assumption must in fact be made concerning the deviations from the Vlasov equilibrium; we shall show presently that it is this assumption which leads to the relative scarcity of undamped waves in the linear theory.

It is straightforward to find the undamped periodic traveling-wave solutions with phase velocity V of the linearized equations; these solutions are sums, possibly infinite, over wave numbers k of Van Kampen–Case modes [3,4]

$$g_\alpha(x,u,t,k) = \left[-\frac{q_\alpha}{m_\alpha}\frac{E_k}{ik}\mathrm{P}\frac{F'_\alpha(u)}{u-V} + C_\alpha^k\delta(u-V)\right]e^{ik(x-Vt)} \ , \tag{16}$$

$$E(x,t,k) = E_k e^{ik(x-Vt)} \ , \tag{17}$$

where P denotes the principle-value distribution and the nonunique constants E_k and C_α^k are related by

$$ikE_k\left[1 - \frac{4\pi}{k^2}\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\mathrm{P}\int_{-\infty}^{\infty}\frac{F'_\alpha(u)}{u-V}du\right] = 4\pi\sum_{\alpha=1}^{N}q_\alpha C_\alpha^k \ . \tag{18}$$

While these expressions do provide a traveling-wave solution of the linear equations in a mathematical sense, physically they must be suspect. The $2\pi/k$-periodic traveling-wave distribution function described by these solutions would be a linear superposition

$$f_\alpha(x,u,t) = F_\alpha(u) + \sum_{m=1}^{\infty} A_m g_\alpha(x,u,t,mk) \ , \tag{19}$$

where A_m are some mode amplitudes. But this function is neither everywhere positive nor absolutely integrable over velocity. Indeed, it does not even assign a finite number of particles to finite regions of phase space—in any range of particle velocities including and to one side of the phase velocity V there are an infinite number of particles. Furthermore, the deviation $\sum_{m=1}^{\infty} A_m g_\alpha(x,u,t,mk)$ is not small in a mean (L^1 or absolutely integrable) sense for any nonzero value of amplitudes A_m. It is small only in less physically motivated senses, such as the weak topology of the space of first-order tempered distributions, meaning that for every continuously differentiable function $\Upsilon(u)$ with at worst polynomial growth in u

$$\int_{-\infty}^{\infty}\Upsilon(u)[f_\alpha(x,u,t) - F_\alpha(u)]du \to 0 \tag{20}$$

as $A_m \to 0$. But physically meaningful quantities such as

$$\int_V^{\infty}[f_\alpha(x,u,t) - F_\alpha(u)]du \tag{21}$$

cannot be assigned any consistent and finite value, and do not tend to zero with the amplitudes A_m. While the presence of the principle value in $g_\alpha(x,u,t,k)$ provides a prescription for integrating across the singularity at the phase velocity $u = V$, it does so by canceling a positive infinity of particles on one side of the phase velocity with a negative infinity of particles on the other side. Given these rather nonphysical properties, it is rather difficult to establish that these singular traveling-wave solutions in fact represent any kind of approximation to an actual solution of the correct original nonlinear equations.

The trend of the linear analysis is clear: for equilibrium distribution functions F_α which yield no roots of the Landau dispersion relation on the imaginary axis or in the right half plane, the electric field predicted by the linear equations damps exponentially if the initial distribution function is analytic in a wide strip in complex velocity [2,3], damps at a slower polynomial rate if the distribution function is smooth but not analytic in a strip [7], and does not damp only if the distribution function is unphysically singular [3,4]. The primary goal of this paper is the construction of smooth but undamped small-amplitude plasma wave solutions of the original nonlinear equations. The existence of such solutions, which will be smooth but not analytic in velocity, does not imply that Landau's solution of the linear equations is incorrect; instead, it supports our contention that the linear equations do not contain a physically complete picture of small-amplitude waves, and that there are indeed physically realizable undamped plasma waves of arbitrarily small amplitude.

III. AN INADEQUACY OF THE LINEAR THEORY

In this section we shall show that the linear theory is, in general, incapable of describing undamped waves of any amplitude, no matter how small or large, with smooth distribution functions. The implication is that the only undamped waves that the linear theory can describe are those with pathological distribution functions, such as the Van Kampen and Case solutions mentioned above, or those few corresponding to eigenvalues embedded in the Van Kampen–Case continuum.

To demonstrate this inadequacy of the linear theory we must briefly examine the equations describing a traveling-wave solution: suppose that the particle distribution functions $f_\alpha(x,u,t)$, $\alpha=1,2,\ldots,N$, and electric field $E(x,t)$ are C^1 (that is, have continuous first derivatives) and represent traveling-wave solutions that satisfy the one-dimensional Vlasov-Maxwell equations, Eqs. (2)–(4). Then there are functions $\tilde{f}_\alpha(\chi,v)$, $\alpha=1,2,\ldots,N$, and $\tilde{E}(\chi)$ that represent the distributions and field in the wave frame, and a phase velocity V, such that

$$f_\alpha(x,u,t)=\tilde{f}_\alpha(x-Vt,u-V)\ , \tag{22}$$

$$E(x,t)=\tilde{E}(x-Vt)\ . \tag{23}$$

Since, by hypothesis, the functions $f_\alpha(x,u,t)$, $\alpha=1,2,\ldots,N$, and $E(x,t)$ satisfy Eqs. (2)–(4), we can substitute these expressions into those equations to find

$$v\frac{\partial\tilde{f}_\alpha(\chi,v)}{\partial\chi}+\frac{q_\alpha}{m_\alpha}\tilde{E}(\chi)\frac{\partial\tilde{f}_\alpha(\chi,v)}{\partial v}=0\ , \tag{24}$$

$$\frac{d\tilde{E}(\chi)}{d\chi}=4\pi\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}\tilde{f}_\alpha(\chi,v)dv\ , \tag{25}$$

and

$$4\pi\sum_{\alpha=1}^{N}q_\alpha\int_{-\infty}^{\infty}v\tilde{f}_\alpha(\chi,v)dv=0\ . \tag{26}$$

Thus, in the wave frame the traveling wave simply represents a stationary solution of the one-dimensional Vlasov-Maxwell equations.

We now want to show that any undamped periodic traveling-wave solution of the one-dimensional Vlasov-Maxwell equation has a qualitative property that is at odds with the assumptions of the linearized theory. Specifically, we wish to show that for any such solution the distribution functions must satisfy

$$\frac{\partial f_\alpha(x,V,t)}{\partial u}=0\ , \tag{27}$$

where V is the phase velocity of the wave. If we can show that this is the case then the assumption of the linearization, namely Eq. (15), will be violated because Eq. (27) implies that

$$\left|\frac{\partial g_\alpha(x,V,t)}{\partial u}\right|=\left|\frac{dF_\alpha(V)}{du}\right|\ . \tag{28}$$

To show that undamped traveling waves have a zero velocity gradient at the wave phase velocity, that is, that Eq. (27) holds, we need only show that the function $\tilde{f}_\alpha$ corresponding to the distribution in the wave frame satisfies

$$\frac{\partial\tilde{f}_\alpha(\chi,0)}{\partial v}=0\ . \tag{29}$$

But Eq. (24) makes it clear that this is true for any point χ at which $\tilde{E}(\chi)\neq0$, and so we must only verify Eq. (29) at those points χ where $\tilde{E}(\chi)=0$. The wave is interesting only if there is some point χ_0 where $\tilde{E}(\chi_0)\neq0$ and we therefore assume this to be the case. Those points where $\tilde{E}(\chi)=0$ are then of two classes: (a) either every neighborhood of χ contains a point χ_0 where $\tilde{E}(\chi_0)\neq0$ or else (b) there is a closed interval about χ on which $\tilde{E}$ is identically zero (the interval must be closed because $\tilde{E}$ is continuous). At points of the first sort we know that $\partial\tilde{f}_\alpha/\partial v|_{(\chi,0)}=0$ because, by hypothesis, $\partial\tilde{f}_\alpha/\partial v$ is continuous and this velocity gradient is zero at some point in every neighborhood of χ. To treat a point of the second class we note that on a closed interval where $\tilde{E}$ is zero the distribution function must satisfy

$$v\frac{\partial\tilde{f}_\alpha(\chi,v)}{\partial\chi}=0\ . \tag{30}$$

The continuity of $\partial\tilde{f}_\alpha/\partial\chi$ then tells us that

$$\frac{\partial\tilde{f}_\alpha(\chi,v)}{\partial\chi}=0 \tag{31}$$

on any such interval, and this implies that $\tilde{f}_\alpha$ must in fact be independent of χ over any interval of χ where $\tilde{E}(\chi)=0$; it follows that $\tilde{f}_\alpha(\chi,v)=\tilde{f}_\alpha(\chi_b,v)$, where χ_b is any boundary point of the interval (which is guaranteed to exist because we have assumed that $\tilde{E}$ is nonzero somewhere). Since, by definition, a boundary point χ_b is of class (a) [in every neighborhood of it there is a point χ_0 at which $\tilde{E}(\chi_0)\neq0$] we know that

$$0=\frac{\partial\tilde{f}_\alpha(\chi_b,0)}{\partial v}=\frac{\partial\tilde{f}_\alpha(\chi,0)}{\partial v} \tag{32}$$

for every point χ in the interval over which $\tilde{E}$ is zero. This completes all possible cases and Eq. (29) is verified.

Hence the linear theory, which requires that Eq. (15) be satisfied, is inconsistent with undamped traveling plasma waves with distribution functions that are smooth in velocity, since they must satisfy Eq. (27). The paucity of physically realistic traveling-wave solutions with constant amplitude—neither damped nor growing—in the linear theory is therefore no surprise. However, this scarcity need not reflect a physical truth; instead it may only represent a mathematical effect of a crude approximation. This article suggests that this is indeed the case.

IV. A REQUIREMENT FOR THE EXISTENCE OF UNDAMPED PERIODIC WAVES

While in this paper we shall show that there are many periodic small-amplitude traveling-wave solutions of the one-dimensional Vlasov-Maxwell equations, it is not true that such waves can have an arbitrary phase velocity. Suppose that we consider a fixed Vlasov equilibrium described by the spatially uniform distribution functions F_α. We shall examine traveling waves with a phase velocity V that are small perturbations of this equilibrium, and see what conclusion we can draw about the minimum "amplitude" of these waves.

A traveling-wave solution with phase velocity V is described by distributions $f_\alpha(x,u,t)=\tilde{f}_\alpha(x-Vt,u-V)$ and a field $E(x,t)=\tilde{E}(x-Vt)$, and we are interested in solutions of this form when the electric field is small and the distribution functions f_α are close to F_α in some sense. By transforming into the wave frame we see that we are interested in functions $\tilde{f}_\alpha(\chi,v)$ and $\tilde{E}(\chi)$, which satisfy Eqs. (24)–(26), and for which $\tilde{E}(\chi)$ is small and $\tilde{f}_\alpha(\chi,v)$ is close to

$$\tilde{F}_\alpha(v)=F_\alpha(v+V)\ , \tag{33}$$

the Vlasov equilibrium distribution function for species α shifted into the wave frame.

We now introduce the even part of the wave distribution function in the wave frame $\tilde{f}^e_\alpha$, defined by

$$\tilde{f}^e_\alpha(\chi,v)=\tfrac{1}{2}[\tilde{f}_\alpha(\chi,v)+\tilde{f}_\alpha(\chi,-v)]\ . \tag{34}$$

Using Eqs. (24)–(26) we see that

$$v\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial\chi}+\frac{q_\alpha}{m_\alpha}\tilde{E}(\chi)\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial v}=0\ , \tag{35}$$

$$\frac{d\tilde{E}(\chi)}{d\chi}=4\pi\sum_{\alpha=1}^{N} q_\alpha\int_{-\infty}^{\infty}\tilde{f}^e_\alpha(\chi,v)dv\ . \tag{36}$$

It is from these equations relating the even part of the distribution functions to the electric field that we shall derive a necessary condition for the existence of plasma waves of arbitrarily small amplitudes.

To proceed, however, we shall need some hypotheses on the distribution functions $\tilde{f}_\alpha$. In a previous paper [17] we have presented our sharpest result in some detail, so in this paper we shall describe a slightly less refined, and therefore less complicated, set of hypotheses which will allow us to derive the desired result more easily. Specifically, suppose that the functions $\tilde{f}^e_\alpha$ are C^2 (twice continuously differentiable) and that $\partial\tilde{f}^e_\alpha/\partial\chi$ is bounded by an integrable function of v uniformly in χ. It then follows (from the Lebesgue dominated convergence theorem) that

$$\frac{d}{d\chi}\int_{-\infty}^{\infty}\tilde{f}^e_\alpha(\chi,v)dv=\int_{-\infty}^{\infty}\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial\chi}dv \tag{37}$$

and therefore, from Eq. (36), we have that

$$\frac{d^2\tilde{E}(\chi)}{d\chi^2}=4\pi\sum_{\alpha=1}^{N} q_\alpha\int_{-\infty}^{\infty}\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial\chi}dv\ . \tag{38}$$

Now from Eq. (35) we see that

$$\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial\chi}=-\frac{q_\alpha}{m_\alpha}\tilde{E}(\chi)\frac{1}{v}\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial v} \tag{39}$$

for $v\neq 0$. But since $\partial\tilde{f}^e_\alpha/\partial v|_{(\chi,0)}=0$ [since $\tilde{f}^e_\alpha(\chi,v)$ is even in v] and we have assumed that $\tilde{f}^e_\alpha$ is C^2, we in fact can conclude that

$$\lim_{v\to 0}\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial\chi}=-\frac{q_\alpha}{m_\alpha}\tilde{E}(\chi)\frac{\partial^2 \tilde{f}^e_\alpha(\chi,0)}{\partial v^2} \tag{40}$$

is well defined. Therefore the electric field and distribution functions are related by

$$\frac{d^2\tilde{E}(\chi)}{d\chi^2}(\chi)=-4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}\frac{1}{v}\frac{\partial \tilde{f}^e_\alpha(\chi,v)}{\partial v}dv\,\tilde{E}(\chi)\ . \tag{41}$$

Let us now introduce a decomposition similar to that used in the linear theory; namely, let us write $\tilde{f}^e_\alpha=\tilde{F}^e_\alpha+\tilde{g}^e_\alpha$, where $\tilde{F}^e_\alpha(v)=\frac{1}{2}[\tilde{F}_\alpha(v)+\tilde{F}_\alpha(-v)]$ is the symmetric part of the equilibrium distribution in the wave frame. Introducing this into Eq. (41) yields

$$\frac{d^2\tilde{E}}{d\chi^2}+\kappa^2\tilde{E}(\chi)+\gamma(\chi)\tilde{E}(\chi)=0\ , \tag{42}$$

where we have defined

$$\kappa^2=4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}\frac{1}{v}\frac{d\tilde{F}^e_\alpha(v)}{dv}dv \tag{43}$$

and

$$\gamma(\chi)=4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}\frac{1}{v}\frac{\partial \tilde{g}^e_\alpha(\chi,v)}{\partial v}dv\ . \tag{44}$$

Note that the integrands appearing in these integrals are not singular because only the even parts of the distribution functions are being used.

Suppose that $\tilde{E}(\chi)$ is periodic with wavelength λ and satisfies Eq. (42); multiplying this equation by $\tilde{E}(\chi)$ and integrating from $\chi=0$ to λ yields, after an integration by parts and using the periodicity,

$$\int_0^\lambda\left[\frac{d\tilde{E}}{d\chi}\right]^2 d\chi=\int_0^\lambda[\kappa^2+\gamma(\chi)][\tilde{E}(\chi)]^2d\chi\ , \tag{45}$$

which implies that $\kappa^2+\gamma(\chi)>0$ for some value of χ. Therefore we know that in order for there to be a periodic traveling-wave solution of the Vlasov-Maxwell equations it must be that $\gamma(\chi)>-\kappa^2$ for some value of χ.

If it should happen that κ^2 is negative, then this means that γ cannot be too small. Thus, for those phase velocities such that κ^2 is negative it is not possible to have periodic traveling-wave solutions of the Vlasov-Maxwell equations which are arbitrarily close to the Vlasov equilibrium F_α, at least in the sense that $\gamma(\chi)$ cannot be too small. By contrast, when $\kappa^2\geq 0$ there is no lower bound forced on γ for a traveling-wave solution and it might be possible to find undamped traveling-wave solutions, with phase velocity V, arbitrarily close to the equilibrium F_α. Thus $\kappa^2\geq 0$ is a requirement for the existence of undamped spatially periodic traveling waves of velocity V to have arbitrarily small amplitude.

V. THE CONSTRUCTION OF TRAVELING WAVES OF BGK FORM

In this section we should like to show that when the quantity κ^2 is positive it is possible to explicitly construct families of periodic traveling-wave solutions of the Vlasov-Maxwell equations with phase velocity V, and that these families of traveling-wave solutions include waves of arbitrarily small amplitude. In order to accomplish this construction we shall exploit the decomposition of the distribution function into its odd and even part (in the wave frame) and use the standard Bernstein-Greene-Kruskal (BGK) representation [18] to describe the even part. The distribution of particles in energy which arises in the BGK representation can then be treated as a "parameter" and a bifurcation analysis undertaken to construct periodic traveling waves of all wave numbers near κ.

To begin this analysis recall that if the distribution function and electric field are of the form

$$\tilde{f}_\alpha^e(\chi,v)=g_\alpha^e\left[\frac{v^2}{2}+\frac{q_\alpha}{m_\alpha}\phi(\chi)\right], \tag{46}$$

$$\tilde{E}(\chi)=\frac{\partial\phi(\chi)}{\partial\chi} \tag{47}$$

for any smooth functions g_α^e and ϕ, then the Vlasov equation, Eq. (24), will be automatically satisfied. Therefore, in order to determine the even part of the distribution function and the electric potential in the wave frame we need only satisfy Eq. (25), which can be written as

$$\frac{d^2\phi}{d\chi^2}+4\pi\sum_{\alpha=1}^{N}\int_{-\infty}^{\infty}g_\alpha^e\left[\frac{v^2}{2}+\frac{q_\alpha}{m_\alpha}\phi(\chi)\right]dv=0\ . \tag{48}$$

To use this equation as the basis for a bifurcation analysis of the potential we can treat the arbitrary functions g_α^e as parameters, and examine how $2\pi/\kappa$-periodic solutions of this equation change as the functions g_α^e are varied. However, our goal is actually more specific; we wish to study only those solutions corresponding to waves near the equilibrium F_α. We shall therefore need the functions g_α^e to capture the essential features of $\tilde{F}_\alpha^e$, so that when the potential is small our local nonlinear analysis will be describing the waves of interest. To achieve this, let us define the function

$$G_\alpha^e(\eta)=\tilde{F}_\alpha^e(\sqrt{2\eta})\ , \tag{49}$$

which has the property that

$$\tilde{F}_\alpha^e(u)=G_\alpha^e\left[\frac{u^2}{2}\right]. \tag{50}$$

Our desire is then that the functions g_α^e should look much like the function G_α^e. There is one difficulty however; the functions $g_\alpha^e(\eta)$ must be defined for negative values of η because the electric potential of the wave might be negative, while the functions G_α^e are defined only for $\eta\geq 0$. However, G_α^e is a smooth function having at least half as many continuous derivatives as F_α itself; there is therefore a smooth (having any finite number of continuous derivatives) extension of G_α^e to negative values of η. Hence we can use

$$g_\alpha^e=(1+\mu)\mathcal{G}_\alpha^e\ , \tag{51}$$

where $\mathcal{G}_\alpha^e$ is any smooth and non-negative function that satisfies $\mathcal{G}_\alpha^e(\eta)=G_\alpha^e(\eta)$ for $\eta\geq 0$. Then when $\mu=0$ and $\phi=0$ we have that

$$\tilde{f}_\alpha^e(\chi,v)=g_\alpha\left[\frac{v^2}{2}\right]=\mathcal{G}_\alpha\left[\frac{v^2}{2}\right]=G_\alpha\left[\frac{v^2}{2}\right]=\tilde{F}_\alpha^e(v)\ , \tag{52}$$

and so for small values of the parameter μ and potential ϕ we expect that the even part of the wave distribution function will be close to $\tilde{F}_\alpha^e(v)$. Using this form in Eq. (48) we find that the potential should satisfy

$$\frac{d^2\phi}{d\chi^2}+\mathcal{H}(\phi,\mu)=0\ , \tag{53}$$

where

$$\mathcal{H}(\phi,\mu)=4\pi\sum_{\alpha=1}^{N}\int_{-\infty}^{\infty}\left[\mathcal{G}_\alpha^e\left[\frac{v^2}{2}+\frac{q_\alpha}{m_\alpha}\phi(\chi)\right]+\mu\mathcal{G}_\alpha^e\left[\frac{v^2}{2}+\frac{q_\alpha}{m_\alpha}\phi(\chi)\right]dv\right]. \tag{54}$$

This is then a nonlinear equation for ϕ depending on the parameter μ; it has one known solution when $\mu=0$, namely $\phi=0$. Our task now is to explore the $2\pi/\kappa$-periodic solutions of this equation for small values of the parameter μ.

By varying the parameter μ in the appropriate way we shall be able to exactly adjust the distribution function so that its spatial wavelength is $2\pi/\kappa$ independent of the wave amplitude. In a previous paper [17] we treated this problem differently; in that paper we did not use a parameter, and instead we set $g_\alpha^e=\mathcal{G}_\alpha^e$ to describe the even part of the wave distribution function. This led to a case in which the wave number of the waves was dependent on amplitude and only tended toward κ as the amplitude went to zero. The analysis here, in which the wave number is independent of amplitude, complements that earlier analysis and emphasizes an important point: there is no exact dispersion relation for small-amplitude nonlinear plasma waves. Indeed, another analysis, sketched in Buchanan, Holloway, and Dorning [19] shows that for a fixed phase velocity there is actually a range of allowable wave numbers and that the size of this wave-number band grows with amplitude.

An exact nonlinear analysis of $2\pi/\kappa$-periodic solutions of Eq. (53) begins with an examination of the derivative

$$\mathcal{H}_\phi(0,0)=4\pi\sum_{\alpha=1}^{N}\frac{q_\alpha^2}{m_\alpha}\int_{-\infty}^{\infty}\frac{d\mathcal{G}_\alpha^e}{d\eta}\left[\frac{v^2}{2}\right]dv\ . \tag{55}$$

Because $\mathcal{G}_\alpha^e$ is a smooth extension of G_α^e to negative arguments, we have that

$$\frac{d\mathcal{G}_\alpha^e}{d\eta}\left[\frac{v^2}{2}\right]=\frac{dG_\alpha^e}{d\eta}\left[\frac{v^2}{2}\right]=\frac{1}{v}\frac{d\tilde{F}_\alpha^e}{dv}(v) \tag{56}$$

and so $\mathcal{H}_\phi(0,0)=\kappa^2$. Equation (53) can therefore be written as

$$\frac{d^2\phi}{d\chi^2}+\kappa^2\phi+\mathcal{N}(\phi,\mu)=0\ , \tag{57}$$

where

$$\mathcal{N}(\phi,\mu)=\mathcal{H}(\phi,\mu)-\kappa^2\phi \tag{58}$$

and

$$\mathcal{N}(\phi,0)=o(\|\phi\|) \text{ as } \|\phi\|\to 0 \ . \tag{59}$$

With the equation written in this form we can apply the Liapunov-Schmidt method [20–22] to reduce the nonlinear differential equation to an algebraic problem. In the present case the application of this method is based on an orthogonal decomposition of ϕ into two parts: those functions that are a linear combination of sines and cosines, and their orthogonal complement. The reason for this particular choice is that the nonlinearity of the problem is only nontrivially involved in that part of ϕ that looks like sine and cosine; the remainder of ϕ is, we shall presently discover, uniquely determined by its projection onto the sine and cosine.

With these ideas in mind, we write every $2\pi/\kappa$-periodic function ϕ in the form

$$\phi(\chi)=A\cos(\kappa\chi)+B\sin(\kappa\chi)+\psi(\chi) \ , \tag{60}$$

where ψ is $2\pi/\kappa$ periodic and orthogonal to $\cos(\kappa\chi)$ and $\sin(\kappa\chi)$. Multiplying Eq. (57) by $\cos(\kappa\chi)$ and $\sin(\kappa\chi)$ respectively, and integrating over the χ interval $[0,2\pi/\kappa]$ reveals that, for a function ϕ that satisfies Eq. (57), A, B, μ and ψ must be related by

$$H_{0c}(A,B,\mu,\psi) \equiv \int_0^{2\pi/\kappa}\cos(\kappa\chi)\mathcal{N}(A\cos(\kappa\chi)+B\sin(\kappa\chi)+\psi(\chi),\mu)d\chi=0 \ , \tag{61}$$

$$H_{0s}(A,B,\mu,\psi) \equiv \int_0^{2\pi/\kappa}\sin(\kappa\chi)\mathcal{N}(A\cos(\kappa\chi)+B\sin(\kappa\chi)+\psi(\chi),\mu)d\chi=0 \ . \tag{62}$$

If we now also consider the problem

$$\frac{d^2\psi}{d\chi^2}+\kappa^2\psi=-\mathcal{N}(A\cos(\kappa\chi)+B\sin(\kappa\chi)+\psi(\chi),\mu) +\cos(\kappa\chi)\frac{\kappa}{\pi}H_{0c}(A,B,\mu,\psi) +\sin(\kappa\chi)\frac{\kappa}{\pi}H_{0s}(A,B,\mu,\psi)=0 \ , \tag{63}$$

we see that Eqs. (61)–(63) are equivalent to Eq. (57), and hence to Eq. (53). At first glance it is not clear that this is progress: we began with one nonlinear differential equation and we now have one nonlinear differential equation coupled to two algebraic equations. However, for the purpose of showing that solutions do exist and understanding their nonuniqueness, this new formulation is perfect. The right-hand side of Eq. (63) is orthogonal to both $\cos(\kappa\chi)$ and $\sin(\kappa\chi)$ and for $(A,B,\mu)=(0,0,0)$ is $o(\|\psi\|)$ as $\psi\to 0$. The linear operator on the left-hand side can therefore be uniquely inverted and the resulting equation used as the basis of a convergent iterative procedure to prove the existence of a smooth function $\Psi(A,B,\mu)$ such that $\psi(\chi)=\Psi(A,B,\mu)(\chi)$ is the unique solution of Eq. (63) for each (A,B,μ) near zero; the details of this procedure are formalized in the well known implicit function theorem [21] (see Holloway and Dorning [17] for a detailed application to the present problem). Note that this result deals with Eq. (63) alone and is independent of the question of the existence of a nontrivial solution. It is Eqs. (61) and (62) that contain the essential nonlinearities of the problem for the potential, and which must be examined to determine the existence of the desired nontrivial solutions.

Before taking up Eqs. (61) and (62) we should explore a few useful properties of Ψ. Notice first that for each μ near zero and $(A,B)=(0,0)$ there must be (again by the implicit function theorem) a constant value of ψ which satisfies Eq. (63). But since $\Psi(A,B,\mu)(\chi)$ is the unique solution of this equation, it must therefore be that $\Psi(0,0,\mu)(\chi)$ is independent of χ. Thus, if the small potential does not have a sine or cosine part [i.e., $(A,B)=(0,0)$], then the potential is constant and has no electric field; in this case the corresponding equilibrium of the Vlasov-Maxwell equations is just another one of the infinity of spatially uniform equilibria. As a special case also note that $\Psi(0,0,0)(\chi)=0$. Indeed, by using Eq. (63) we can compute the derivatives with respect to A and B as $\Psi_A(0,0,0)(\chi)=\Psi_B(0,0,0)(\chi)=0$ and hence conclude that $\Psi(A,B,\mu)(\chi)=O(|(A,B)|^2)$ as $(A,B)\to(0,0)$; thus Ψ is a unique second-order correction to the sine and cosine part of the potential.

We can now substitute the function $\Psi(A,B,\mu)(\chi)$ into Eqs. (61) and (62), leaving us with two algebraic relations among the three quantities A, B, and μ, namely

$$H_c(A,B,\mu)\equiv H_{0c}[A,B,\mu,\Psi(A,B,\mu)]=0 \ , \tag{64}$$

$$H_s(A,B,\mu)\equiv H_{0s}[A,B,\mu,\Psi(A,B,\mu)]=0 \ . \tag{65}$$

This algebraic problem can be further simplified, however, by using the symmetries of the problem; solutions with $B\neq 0$ can in fact be recovered from solutions with $B=0$ by phase shifts $\chi\to\chi+\theta$ and/or spatial inversions $\chi\to-\chi$. These symmetry properties also imply that $H_s(A,0,\mu)=0$ and that $H_c(A,0,\mu)=-H_c(-A,0,\mu)$, as may be verified by an examination of Eqs. (61)–(63). Thus, since $H_s(A,0,\mu)=0$ is automatically satisfied, we really only need to examine one algebraic relation $H(A,\mu)=0$ between A and μ, with $H(A,\mu)\equiv H_c(A,0,\mu)$.

We are already familiar with one solution of $H(A,\mu)=0$, namely $A=0$ for any value of μ. Again, this solution corresponds to a spatially uniform potential. But if we use $\Psi(0,0,0)=0$ and $\mathcal{N}_\phi(0,0)=0$ to compute

$$\frac{\partial H(0,0)}{\partial A}=\int_0^{2\pi/\kappa}\cos(\kappa\chi)\mathcal{N}_\phi(0,0)\cos(\kappa\chi)d\chi=0 \ , \tag{66}$$

we immediately realize that the trivial solution $A=0$ may not be the unique solution near $(A,\mu)=(0,0)$. Nontrivial solutions are most conveniently separated from the zero solution by considering the equation $A^{-1}H(A,\mu)=0$; for a given value of μ, $A\neq 0$ is a solution of $H(A,\mu)=0$ if and only if $A^{-1}H(A,\mu)=0$. Consider therefore the equation $\hat{H}(A,\mu)=0$, where $\hat{H}$ is the smooth function defined by

$$\hat{H}(A,\mu)=\begin{cases}\dfrac{H(A,\mu)}{A}, & A\neq 0\\[2ex] \dfrac{\partial H(0,\mu)}{\partial A}, & A=0 \ .\end{cases} \tag{67}$$

We see that $\hat{H}(0,0)=0$ and that

$$\frac{\partial\hat{H}(0,0)}{\partial\mu}=\frac{\partial^2 H(0,0)}{\partial A\,\partial\mu}=\int_0^{2\pi/\kappa}\cos^2(\kappa\chi)[\mathcal{N}_{\phi\phi}(0,0)\Psi_\mu(0,0,0)(\chi)+\mathcal{N}_{\phi\mu}(0,0)]d\chi \tag{68}$$

or, using $\Psi_\mu(0,0,0)(\chi)=-\kappa^{-2}\mathcal{N}_\mu(0,0)$ for all χ and evaluating the derivatives of $\mathcal{N}$,

$$\frac{\partial\hat{H}(0,0)}{\partial\mu}=\pi\kappa\neq 0 \ . \tag{69}$$

Thus $\hat{H}_\mu(0,0)\neq 0$ and the implicit function theorem guarantees that there is a smooth function $M(A)$ with $M(0)=0$ such that $\hat{H}(A,\mu)=0$ if and only if $\mu=M(A)$. Therefore, for any (sufficiently small) value of A we can find the corresponding value of the parameter μ which makes $\hat{H}(A,\mu)=0$. Since this implies the existence of small nonzero solutions A to $H(A,\mu)=0$, we can immediately conclude that there are small solutions of Eq. (53) which are $2\pi/\kappa$ periodic and spatially nonuniform.

We now know that such nontrivial solutions exist and that for the parameter value $\mu=M(A)$ they can be expressed (modulo the symmetries that allow the recovery of the sine terms from the cosine terms) in the form

$$\phi(\chi,A)=A\cos(\kappa\chi)+\Psi(A,0,M(A))(\chi) \ , \tag{70}$$

where all of the functions are smooth—and hence Taylor expandable—in A, and where $\Psi(A,0,M(A))(\chi)$ is orthogonal to $\cos(\kappa\chi)$. We can therefore justify and explicitly solve for the coefficients in the expansions

$$\phi=A\cos(\kappa\chi)+A^2\psi_2(\chi)+A^3\psi_3(x)+o(A^3) \ , \tag{71}$$

$$\mu=A\mu_1+A^2\mu_2+o(A^2) \ . \tag{72}$$

The implicit function theorem has assured us that $\psi_2(\chi)$ and $\psi_3(\chi)$, orthogonal to $\cos(\kappa\chi)$, exist, and we can use this fact to determine the coefficients μ_1 and μ_2 by applying the appropriate solvability condition to the right-hand sides of the linear equations which $\psi_2(\chi)$ and $\psi_3(\chi)$ will satisfy. Thus the elimination of secular terms, which is commonly applied when making such expansions, is rigorously justified in this case by the nonlinear analysis embodied in the implicit function theorem. Performing the analysis to find the coefficients in these expansions yields

$$\psi_2(\chi)=\frac{1}{4\kappa^2}\mathcal{H}_{\phi\phi}(0,0)[-1+\tfrac{1}{3}\cos(2\kappa\chi)] \ , \tag{73}$$

$$\psi_3(\chi)=\frac{1}{8\kappa^2}\left[\frac{1}{24\kappa^2}[\mathcal{H}_{\phi\phi}(0,0)]^2+\tfrac{1}{24}\mathcal{H}_{\phi\phi\phi}(0,0)\right]\cos(3\kappa\chi) \ , \tag{74}$$

and

$$\mu_1=0 \ , \tag{75}$$

$$\mu_2=\frac{1}{\mathcal{H}_{\phi\mu}(0,0)}\left[\frac{5}{24\kappa^2}[\mathcal{H}_{\phi\phi}(0,0)]^2-\tfrac{1}{8}\mathcal{H}_{\phi\phi\phi}(0,0)\right] . \tag{76}$$

The derivatives of $\mathcal{H}$ can be explicitly evaluated as

$$\mathcal{H}_{\phi\phi}(0,0)=4\pi\sum_{\alpha=1}^{N}q_\alpha\left[\frac{q_\alpha}{m_\alpha}\right]^2\int_{-\infty}^{\infty}\frac{d^2\mathcal{G}_\alpha^e}{d\eta^2}\left[\frac{v^2}{2}\right]dv \ , \tag{77}$$

$$\mathcal{H}_{\phi\phi\phi}(0,0)=4\pi\sum_{\alpha=1}^{N}q_\alpha\left[\frac{q_\alpha}{m_\alpha}\right]^3\int_{-\infty}^{\infty}\frac{d^3\mathcal{G}_\alpha^e}{d\eta^3}\left[\frac{v^2}{2}\right]dv \ , \tag{78}$$

$$\mathcal{H}_{\phi\mu}(0,0)=\kappa^2 \ . \tag{79}$$

At this point we know that there exist pairs (ϕ,μ) near (0,0) which satisfy Eq. (53), and that these can be parametrized by an amplitude A; furthermore, we have approximate expressions for these quantities, given by Eqs. (71)–(76). From these solutions we know that a corresponding distribution function has its even part given by Eq. (46) with $g_\alpha^e(\eta)=\mathcal{G}_\alpha^e(\eta)+M(A)\mathcal{G}_\alpha^e(\eta)$, and that this distribution function tends toward $\tilde{F}_\alpha^e$ as $A\to 0$.

To complete our construction we must now develop the odd part of the wave distribution function; we can represent this function in the BGK form

$$\mathcal{f}_\alpha^o(\chi,v)=\begin{cases} g_\alpha^o\left[\dfrac{v^2}{2}+\dfrac{q_\alpha}{m_\alpha}\phi(\chi)\right], & v\geq 0\\ -g_\alpha^o\left[\dfrac{v^2}{2}+\dfrac{q_\alpha}{m_\alpha}\phi(\chi)\right], & v\leq 0 \ .\end{cases} \tag{80}$$

To make certain that the distribution function will be close to the spatially uniform background we first introduce

$$\tilde{F}_\alpha^o=\tfrac{1}{2}[\tilde{F}_\alpha(v)-\tilde{F}_\alpha(-v)] \tag{81}$$

and

$$\mathcal{G}_\alpha^o(\eta)=\begin{cases}\tilde{F}_\alpha^o(\sqrt{2\eta}), & \eta\geq 0\\ 0, & \eta\leq 0\end{cases} \tag{82}$$

which have the property that

$$\tilde{F}_\alpha^o(v)=\mathcal{G}_\alpha^o\left[\frac{v^2}{2}\right] . \tag{83}$$

We must now describe families of smooth functions $g_\alpha^o(\eta,A)$, parametrized by the wave amplitude A, that approach $\mathcal{G}_\alpha^o(\eta)$ uniformly in $\eta\geq 0$ as $A\to 0$. There is no unique way to describe such a family, although they are constrained by various requirements.

(i) $g_\alpha^o(\eta,A)\to G_\alpha^o(\eta)$ as $A\to 0$, uniformly in η.

(ii) $|g_\alpha^o(\eta,A)|\leq g_\alpha^e(\eta)$. This will ensure that the resulting distribution function is everywhere non-negative.

(iii) $g_\alpha^o(\eta,A)=0$ for $\eta\leq\Phi_\alpha^{\max}\equiv\sup_\chi[(q_\alpha/m_\alpha)\phi(\chi)]$. This ensures that the distribution function is well defined and that the distribution of trapped particles is even in velocity, as it must be for a solution of the stationary Vlasov equation.

(iv) In order to ensure that the net current of the wave is zero as required by Eq. (26) we need

$$\sum_{\alpha=1}^{N}q_\alpha\int_0^\infty g_\alpha^o(\eta,A)d\eta=0 \ .$$

We shall now describe a particular choice of g_α^0 that satisfies all of these requirements. This will not be the only possible family—indeed a slightly different description was used in Buchanan, Holloway, and Dorning [19]—but it will suffice. Let $R(\eta)$ be any infinitely smooth function that satisfies $0 \le R(\eta) \le 1$ for all η, $R(\eta)=1$ for $\eta \le 1$, and $R(\eta)=0$ for $\eta \ge 2$. As an example of such a function, we mention

$$R(\eta)=\begin{cases} 1, & \eta \le 1 \\ \frac{1}{2}\left[1-\tanh\left[\dfrac{3-2\eta}{2(\eta-1)(\eta-2)}\right]\right], & 1 \le \eta \le 2 \\ 0, & \eta \ge 2 \end{cases} \tag{84}$$

which we have actually used in the numerical calculations described in Sec. VI. This function has derivatives of all orders—even at the points $\eta=1$ and 2—but it is not analytic in any strip about the real η axis. Indeed, it is not possible to construct an analytic function with the properties required of $R(\eta)$; we shall say more on this point later. With these prescriptions in mind, consider the function $g_\alpha^0(\eta, A)$ defined by

$$g_\alpha^o(\eta, A)=[1+M(A)][1-R(\eta/\Phi_\alpha^{\max})] \times\{[1-\beta_\alpha(A)]\mathcal{G}_\alpha^o(\eta)+\beta_\alpha(A)\mathcal{G}_\alpha^e(\eta)\} . \tag{85}$$

For A so small that $|M(A)|<1$, and provided that $0<\beta_\alpha(A)<1$, we see that

$$\begin{aligned}|g_\alpha^o(\eta, A)| &\le [1+M(A)]\{[1-\beta_\alpha(A)]|\mathcal{G}_\alpha^o(\eta)|+\beta_\alpha(A)\mathcal{G}_\alpha^e(\eta)\} \\ &\le [1+M(A)]\{[1-\beta_\alpha(A)]\mathcal{G}_\alpha^e(\eta)+\beta_\alpha(A)\mathcal{G}_\alpha^e(\eta)\}=g_\alpha^e(\eta) ,\end{aligned} \tag{86}$$

where we have used the positivity of the spatially uniform equilibrium to note that $|\mathcal{G}_\alpha^0(\eta)| \le \mathcal{G}_\alpha^e$. Thus condition (ii) is satisfied. The second factor in the expression for g_α^o ensures that condition (iii) is met, and if

$$\beta_\alpha(A)=\left[\int_0^\infty R(\eta/\Phi_\alpha^{\max})\mathcal{G}_\alpha^o(\eta)d\eta\right] \times\left[\int_0^\infty [1-R(\eta/\Phi_\alpha^{\max})] \times[\mathcal{G}_\alpha^e(\eta)-\mathcal{G}_\alpha^o(\eta)]d\eta\right]^{-1} . \tag{87}$$

then we can readily verify that

$$\sum_{\alpha=1}^N q_\alpha \int_0^\infty g_\alpha^o(\eta, A)d\eta =[1+M(A)]\sum_{\alpha=1}^N q_\alpha \int_0^\infty \mathcal{G}_\alpha^o(\eta, A)d\eta=0 \tag{88}$$

because the spatially uniform equilibrium has zero current; thus condition (iv) is accommodated. Unfortunately, this value of β is only positive if the numerator in its definition is also positive. If this is not the case then instead we can define

$$g_\alpha^o(\eta, A)=[1+M(A)][1-R(\eta/\Phi_\alpha^{\max})] \times\{[1-\beta_\alpha(A)]\mathcal{G}_\alpha^o(\eta)-\beta_\alpha(A)\mathcal{G}_\alpha^e(\eta)\} . \tag{89}$$

and

$$\beta_\alpha(A)=\left[-\int_0^\infty R(\eta/\Phi_\alpha^{\max})\mathcal{G}_\alpha^o(\eta)d\eta\right] \times\left[\int_0^\infty [1-R(\eta/\Phi_\alpha^{\max})] \times[\mathcal{G}_\alpha^e(\eta)+\mathcal{G}_\alpha^o(\eta)]d\eta\right]^{-1} , \tag{90}$$

and thereby again satisfy conditions (ii)–(iv). In order to verify the limit in condition (i) we need only note that $M(A)\to 0$ and $\beta_\alpha(A)\to 0$ as $A\to 0$, and that this limit is uniform in η because $\mathcal{G}_\alpha^o(0)=0$.

We have now proven that if $\kappa^2>0$, then there exist exact $2\pi/\kappa$-periodic solutions to the nonlinear Vlasov-Maxwell equations in the wave frame. These solutions are smooth functions and remain so as the amplitude of the wave tends toward zero. Thus, in the original laboratory frame there exist exact nonlinear undamped spatially periodic traveling-wave solutions arbitrarily close to the spatially uniform Vlasov equilibrium F_α used to compute κ^2.

The manner in which these undamped wave solutions approach the spatially uniform equilibrium is of interest. Assuming polynomial decay of the equilibrium functions $F_\alpha(u)$ at large velocity, the waves are small amplitude in the sense that for any p, $1 \le p \le \infty$,

$$\lim_{A\to 0}\int_0^{2\pi/\kappa}|\phi(\chi)|^p d\chi=0 , \tag{91}$$

$$\lim_{A\to 0}\int_0^{2\pi/\kappa}\int_{-\infty}^{\infty}|\tilde{f}_\alpha(\chi,v)-\tilde{F}_\alpha(v)|^p d\chi\, dv=0 . \tag{92}$$

Physically the case $p=1$ in Eq. (92) means that the wave distribution functions represent only a small rearrangement from the equilibrium distribution of particles. This, it seems to us, is all that it is physically appropriate to require.

Conversely, the undamped Van Kampen–Case waves that follow from the linearized equations describe positive and negative infinite numbers of particles above and below the phase velocity. Hence they cannot provide a qualitatively accurate picture of the distribution function; as described in Sec. II, they can approximate the distribution function only in unphysical topologies such as the weak topology of the space of distributions. Physically this means that individual Van Kampen–Case modes do not approximate the distribution function of undamped waves, although they may generate smooth velocity moments that approximate those of the distribution function.

In order to establish how the Van Kampen–Case modes do not approximate the distribution functions we examine the rate of change of $\tilde{f}_\alpha(\chi,v)$ with wave amplitude A as A goes to zero. Because $\mathcal{G}^e_\alpha(\eta)$ is a smooth function we easily compute

$$\left.\frac{d}{dA}\tilde{f}^e_\alpha(\chi,v)\right|_{A=0}=\frac{d\mathcal{G}^e_\alpha}{d\eta}\left[\frac{v^2}{2}\right]\frac{q_\alpha}{m_\alpha}\cos(\kappa x)$$
$$=\frac{1}{v}\frac{d\tilde{F}^e_\alpha(v)}{dv}\frac{q_\alpha}{m_\alpha}\cos(\kappa x) \tag{93}$$

for all values of χ and v. A similar derivative of the odd part of the distribution will not exist for all of phase space, but, for the form of $g^o_\alpha(\eta,A)$ defined in Eqs. (85) or (89), will be well defined for those χ and v satisfying $v^2/2+(q_\alpha/m_\alpha)\phi(\chi)\geq 2\Phi^{\max}_\alpha$. For these values we have

$$\left.\frac{d}{dA}\tilde{f}^o_\alpha(\chi,v)\right|_{A=0}=\frac{1}{v}\frac{d\tilde{F}^o_\alpha(v)}{dv}\frac{q_\alpha}{m_\alpha}\cos(\kappa x)\ . \tag{94}$$

These derivatives lead to a first-order expansion of the distribution function

$$\tilde{f}_\alpha(\chi,v)=\tilde{F}_\alpha(v)+A\frac{1}{v}\frac{d\tilde{F}_\alpha(v)}{dv}\frac{q_\alpha}{m_\alpha}\cos(\kappa x)+o(A) \tag{95}$$

valid *well outside the trapped particle region* [strictly for $v^2/2+(q_\alpha/m_\alpha)\phi(\chi)\geq 2\Phi^{\max}_\alpha$]. Thus, referring to Eqs. (16) and (19) shifted into the wave frame, we see that the first-order correction derived from the nonlinear analysis agrees with a Van Kampen–Case mode outside of a phase-space region around the phase velocity whose width is proportional to $\sqrt{A}$ in velocity. However, inside the trapped particle region and a layer surrounding it the exact solutions—which are locally integrable, positive, and locally even in velocity—differ significantly from the undamped linear theory modes—which are singular, negative and not even.

Although the distribution function that results from the linear theory is qualitatively incorrect, it nevertheless contains quantitatively correct integral information. In particular, it leads to the correct small-amplitude behavior for smooth moments. To show this we follow a procedure similar to that used by Bernstein, Greene, and Kruskal [18], but we exploit our more precise results on the electric potential to show that smooth moments are independent of the details of the trapped particle distribution at first order. Let $\Upsilon(v)$ be any C^1 function and define, for fixed χ, $v_m\geq 0$ by $v_m^2/2+(q_\alpha/m_\alpha)\phi(\chi)=2\Phi^{\max}_\alpha$ and $v_t\geq 0$ by $v_t^2/2+(q_\alpha/m_\alpha)\phi(\chi)=\Phi^{\max}_\alpha$, respectively; v_t is the speed of a particle on the separatrix between trapped and untrapped particles at χ, while v_m is a somewhat larger speed outside of which Eq. (95) holds. Then

$$\int_{-\infty}^{\infty}\Upsilon(v)\tilde{f}_\alpha(\chi,v)dv=\int_{-\infty}^{\infty}\Upsilon^e(v)[\tilde{F}^e_\alpha(v)+A\frac{q_\alpha}{m_\alpha}\frac{1}{v}\frac{d\tilde{F}^e_\alpha(v)}{dv}\cos(\kappa x)+o(A)]dv+2\int_{v_t}^{v_m}\Upsilon^o(v)\tilde{f}^o_\alpha(\chi,v)dv$$
$$+2\int_{v_m}^{\infty}\Upsilon^o(v)[\tilde{F}^o_\alpha(v)+A\frac{q_\alpha}{m_\alpha}\frac{1}{v}\frac{d\tilde{F}^o_\alpha(v)}{dv}\cos(\kappa x)+o(A)]dv\ , \tag{96}$$

where $\Upsilon^e(v)$ and $\Upsilon^o(v)$ are the odd and even parts of $\Upsilon(v)$, respectively. We now show that the second term $2\int_{v_t}^{v_m}\Upsilon^o(v)\tilde{f}^o_\alpha(\chi,v)dv$ is $o(A)$ as $A\to 0$. To do this we use the differentiability of $\Upsilon(v)$ to note that there is a constant C such that $|\Upsilon^o(v)|\leq Cv$ for $0\leq v\leq v_m$, and so

$$\left|2\int_{v_t}^{v_m}\Upsilon^o(v)\tilde{f}^o_\alpha(\chi,v)dv\right|\leq C\int_{\Phi^{\max}_\alpha}^{2\Phi^{\max}_\alpha}|g^o_\alpha(\eta,A)|d\eta\ . \tag{97}$$

Now observe that for $\Phi^{\max}_\alpha\leq\eta\leq 2\Phi^{\max}_\alpha$ we have $|1-R(\eta/\Phi^{\max}_\alpha)|\leq 1$, $|\mathcal{G}^o_\alpha(\eta)|\leq K\sqrt{2\eta}\leq K(4\Phi^{\max}_\alpha)^{1/2}$ and $\mathcal{G}^e_\alpha(\eta)\leq K'$ for some constants K and K'. Therefore, using either Eq. (85) or (89) to describe g^o_α we find that

$$\left|2\int_{v_t}^{v_m}\Upsilon^o(v)\tilde{f}^o_\alpha(\chi,v)dv\right|$$
$$\leq C\Phi^{\max}_\alpha[1+M(A)]$$
$$\times[|1-\beta_\alpha(A)|2K(\Phi^{\max}_\alpha)^{1/2}+K'|\beta_\alpha(A)|]\ . \tag{98}$$

Since $\Phi^{\max}_\alpha=O(A)$, $M(A)=O(A)$ and $\beta_\alpha(A)\to 0$ as $A\to 0$ it follows that

$$2\int_{v_t}^{v_m}\Upsilon^o(v)\tilde{f}^o_\alpha(\chi,v)dv=o(A) \tag{99}$$

as $A\to 0$. Therefore, as $A\to 0$,

$$\int_{-\infty}^{\infty}\Upsilon(v)\tilde{f}_\alpha(\chi,v)dv$$
$$=\int_{-\infty}^{\infty}\Upsilon(v)\tilde{F}_\alpha(v)dv$$
$$+A\frac{q_\alpha}{m_\alpha}\cos(\kappa x)\mathrm{P}\int_{-\infty}^{\infty}\Upsilon(v)\frac{1}{v}\frac{d\tilde{F}_\alpha(v)}{dv}dv+o(A)\ , \tag{100}$$

where the principle-value integral follows from the limit $v_m\to 0$ as $A\to 0$ and from the fact that the even part of $\tilde{F}_\alpha$ does not contribute to the principle-value integral. But, to first order in A, Eq. (100) is equal to the corresponding integral of the Van Kampen–Case solution of the linearized equations. Hence the linear theory Van Kampen-Case modes with $k=\kappa$ and $C^\kappa_\alpha=0$ in Eq. (16) do correctly describe the small-amplitude behavior of smooth velocity moments of the distribution functions, although they do not correctly describe the distribution function itself. These developments show that the linear theory does capture some quantitative features of exact undamped traveling waves, but nevertheless its predictions are incomplete because it does not describe the distribution function well and cannot quantify the number of particles in all regions of phase space.

VI. AN EXPLICIT CASE

Using the approximate expressions for the potential developed above it is possible to explicitly, but still only approximately, write down the wave distribution function in the wave frame by using the functions g_α^e and g_α^o just described. In this section we shall present a concrete example of the waves described in the preceding section, with the aim of showing graphically how the wave distribution functions tend towards the spatially uniform equilibrium distribution functions. In particular, we shall consider a two species plasma in which $-q_1=m_1=1$ and, formally, $q_2/m_2=0$ so that the second species is merely a neutralizing background. The spatially uniform distribution of species 1 will be

$$F_1(u)=\frac{1}{\sqrt{2\pi}}e^{u^2/2}\ , \tag{101}$$

so that the plasma is essentially an electron-heavy-ion plasma with the electrons in thermal equilibrium. If we numerically compute κ^2 as a function of phase velocity we find that $\kappa^2<0$ for $|V|<V_c$, where $V_c\approx 1.3$; therefore there are no small-amplitude undamped plasma waves with phase speed below this value. But for $|V|>V_c$ we have $\kappa^2>0$ and according to our nonlinear results there will be undamped plasma waves with wave number κ. Let us fix our attention on a phase velocity of $V=2$, at which $\kappa^2=0.2799/\lambda_D^2$ ($\lambda_D=1/\sqrt{4\pi}$ is the Debye length). For this phase velocity the nonlinear analysis has provided periodic traveling-wave solutions of wavelength $2\pi/\kappa=11.87\lambda_D$ whose amplitude can be made arbitrarily small.

Figure 1 shows the distribution function of one species for a wave for $A=0.1$. The trapped-particle region at the phase velocity $V=2$ can be seen clearly. Indeed, while the potential at this amplitude is dominated by the cosine term, it is difficult to justify labeling this a small-amplitude wave because the disturbance to the background distribution function seems so significant; this rather large amplitude was chosen for this figure so that it would be possible to see the influence of the trapped particles on the distribution function; at this scale a smaller amplitude would have made the figure indistinguishable from the spatially uniform equilibrium. But even at a smaller amplitude the trapped particles have a major influence on the velocity gradient of the distribution function, producing a rather flat shoulder—actually a crater—on the side of the distribution function. This region of trapped particles and the zero velocity gradient at the phase velocity cannot be avoided at any amplitude, no matter how small. The trapped-particle region becomes narrower and the deviation from the spatially uniform distribution $F_1(u)$ becomes smaller as $A\to 0$, but $\partial f_1/\partial u|_{(x,V,t)}=0$ at all amplitudes, no matter how small, while $\partial F_1/\partial u|_V\neq 0$. Figure 2 shows a sequence of plots which illustrate this point, and shows how the wave distribution function can nevertheless approach $F_1(u)$ as the amplitude goes to zero; Fig. 2(a) shows $\tilde{f}_1(\pi/\kappa,u)$ for a series of decreasing values of A and Fig. 2(b) shows $\tilde{g}_1(\pi/\kappa,u)=\tilde{f}_1(\pi/\kappa,u)-\tilde{F}_1(u)$ for these same values. The slope $\partial\tilde{g}_1(\pi/\kappa,0)/\partial u$ represents the deviation of the velocity gradients at the wave phase velocity and is seen to be essentially independent of amplitude A. Nevertheless, as $A\to 0$ the deviation $\tilde{g}_1$ does not diverge or develop larger derivatives. The deviation is clearly most significant within the trapped-particle region and undergoes a fairly rapid transition to almost zero just outside. This transition is effected by the function $R(\eta)$ described in Sec. V, and while the velocity distribution function was thereby constructed to have continuous velocity deriva-

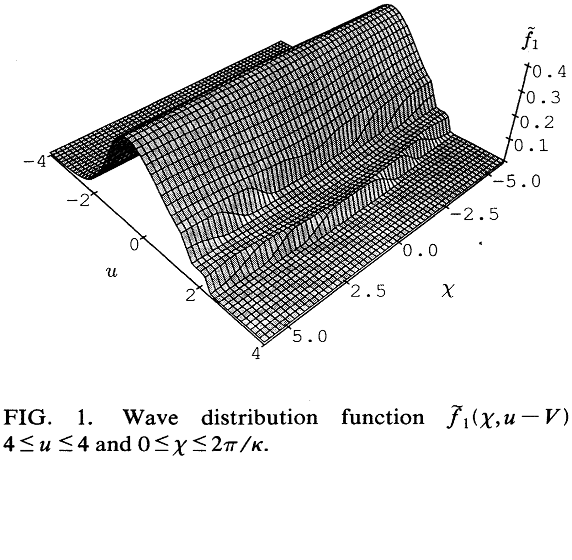

FIG. 1. Wave distribution function $\tilde{f}_1(\chi,u-V)$ for $-4\le u\le 4$ and $0\le\chi\le 2\pi/\kappa$.

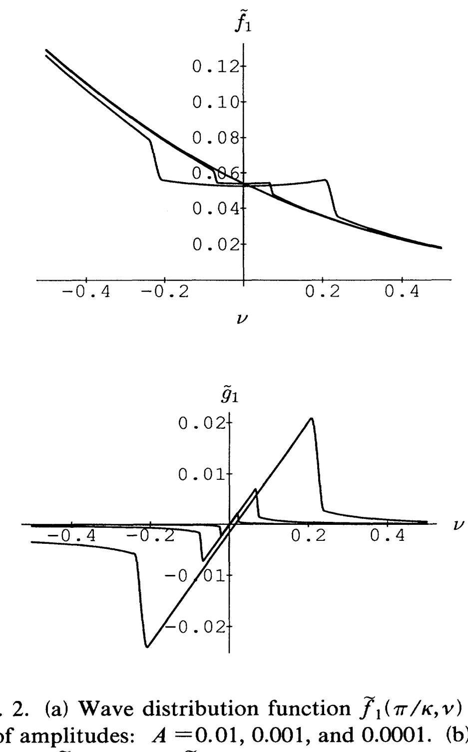

FIG. 2. (a) Wave distribution function $\tilde{f}_1(\pi/\kappa,\nu)$ vs ν for a series of amplitudes: $A=0.01$, 0.001, and 0.0001. (b) deviation $\tilde{g}_1(\pi/\kappa,\nu)=\tilde{f}_1(\pi/\kappa,\nu)-\tilde{F}_1(\nu)$ vs ν for a series of amplitudes: $A=0.01$, 0.001, and 0.0001.

tives of all orders through this transition, it is not analytic in velocity. But because of the influence of the trapped particles the linear equations [at least the equations linearized about $F_1(u)$] are not valid, and the wave, no matter how small its amplitude and no matter how close to $F_1(u)$, will not damp. This prediction has been confirmed recently by numerical simulations [23] of the one species thermal equilibrium plasma by using initial data derived from Eqs. (43), (46), (71), (72), and (85) or (89).

VII. APPROXIMATE DISPERSION RELATIONS

In this section we shall present a few calculations of κ^2 as a function of phase velocity for some electron-proton plasma equilibria. Such calculations reveal over what range of phase velocities these equilibria support undamped waves, and recognizing κ as an approximation to the wave number of the corresponding waves also provides an approximate dispersion relation for the waves.

The first example, whose dispersion relation has appeared before [24], is a thermal equilibrium electron-proton plasma in which the electrons and ions have equal temperatures. A plot of κ^2 versus phase velocities is shown in Fig. 3. We see from this that, again, $\kappa^2<0$ for $V<V_c\approx 1.3v_e^{\text{th}}$, indicating that undamped waves of such low phase velocities are not possible (v_e^{th} is the electron thermal velocity). In contrast for $V>V_c$ we have $\kappa^2>0$ and such waves do exist. Using $\omega=\kappa V$ we can convert this figure to a dispersion diagram such as that shown in Fig. 4. This dispersion diagram shows a branch of waves which are essentially Langmuir waves, but unlike the traditional Langmuir waves for such a plasma they do not damp, not even slowly. Another branch of waves at low frequency and long wavelength call to mind the ion-acoustic waves, which are predicted to be strongly damped in this plasma, but here this branch describes undamped, small-amplitude nonlinear waves. It will be observed that these two branches are in fact connected, so that there is both a frequency cutoff and a wave-number cutoff; apparently it is not possible to produce the necessary population of trapped particles in a small potential unless the potential well is sufficiently long and oscillating sufficiently slowly.

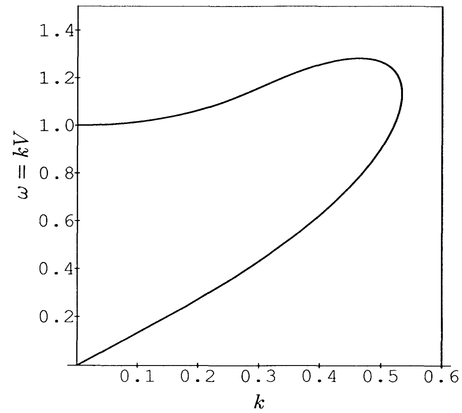

FIG. 4. Dispersion diagram for undamped nonlinear longitudinal waves for an electron-proton plasma in thermal equilibrium with equal electron and ion temperatures. The scale for the wave number is the inverse-electron Debye length and the scale for the frequency is the electron plasma frequency.

As a second example Fig. 5 shows κ^2 for a system which consists of a Maxwellian beam of electrons injected into an electron-proton plasma in thermal equilibrium, the entire system being charge neutral and having zero current. For this plasma κ^2 is positive for $1.3v_e^{\text{th}}\lesssim V\lesssim 19v_e^{\text{th}}$ and for $21v_e^{\text{th}}\lesssim V$, but it becomes negative for phase velocities in the neighborhood of the beam velocity. The corresponding dispersion diagram, shown in Fig. 6, therefore has additional branches due to the presence of the beam.

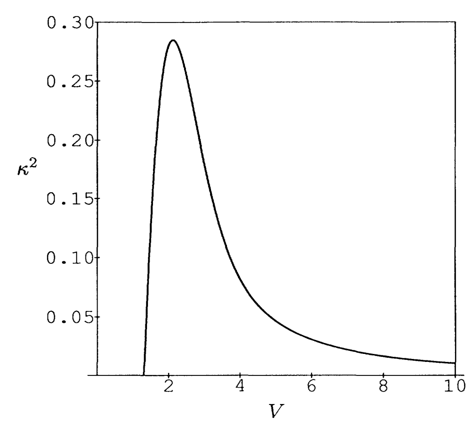

FIG. 3. Function κ^2 vs positive phase velocity for an electron-proton plasma in thermal equilibrium with equal electron and ion temperatures. The scale for κ is the inverse-electron Debye length and the scale for the phase velocity is the electron thermal velocity.

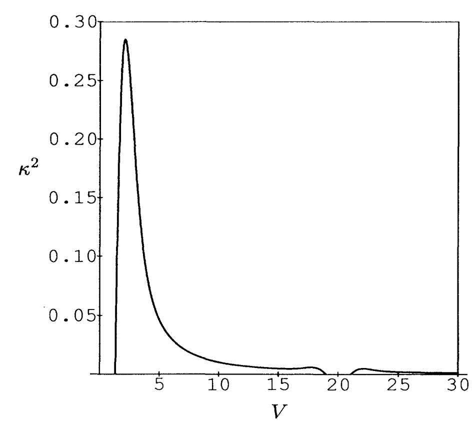

FIG. 5. Function κ^2 vs positive phase velocity for an electron-proton plasma with a high-energy electron beam at $20v_e^{\text{th}}$. The beam and both of the main distributions are Maxwellian with equal temperatures, and $n_b=0.01n_e$. The scale for κ is the inverse-electron Debye length and the scale for the phase velocity is the electron thermal velocity.

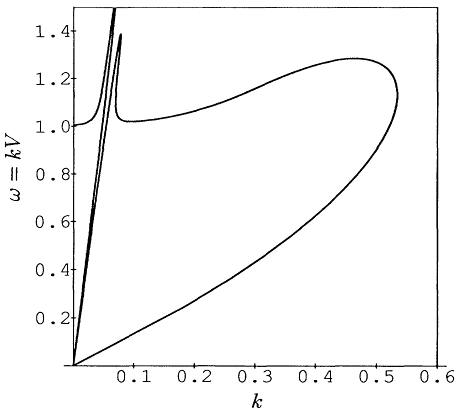

FIG. 6. Dispersion diagram for an electron-proton plasma with a high-energy electron beam at $20v_e^{th}$. The beam and both of the main distributions are Maxwellian with equal temperatures, and $n_b = 0.01n_e$. The scale for the wave number is the inverse-electron Debye length and the scale for the frequency is the electron plasma frequency.

VIII. CONCLUSIONS

In this paper we have presented some exact nonlinear analysis of small-amplitude nonlinear plasma waves. The results of this analysis show that the standard linear theory does not capture the physics of all small-amplitude plasma waves, and demonstrate that the nonlinear effects of particle trapping can arise at arbitrarily small amplitudes in the electric potential. While these results do not invalidate the linear theory—nothing done here disallows the Landau damping of the electric field generated by *some* perturbation of the Maxwellian for example—they do explicitly exhibit solutions which behave differently. These solutions include appropriately smooth physical distribution functions and show that undamped small-amplitude waves are not the result of unphysical singularities, as the linear theory might suggest. Finally, since the results of the linear analysis have provided many of the foundations upon which collisionless plasma physics is constructed, the implications for basic plasma theory of the failure of the linear theory to account for these small-amplitude effects should be examined.

ACKNOWLEDGMENTS

The authors would like to acknowledge a fruitful correspondence with Dr. John Greene. This work was supported by the National Aeronautics and Space Administration under Grants Nos. NGT-50183 and NAGW-1669.

[1] A. Vlasov, J. Phys. **9**, 25 (1945).
[2] L. Landau, J. Phys. **10**, 25 (1946).
[3] N. G. Van Kampen, Physica **21**, 949 (1955).
[4] K. M. Case, Ann. Phys. (N.Y.) **7**, 349 (1959).
[5] J. D. Jackson, J. Nucl. Energy, Part C **1**, 171 (1960).
[6] George Backus, J. Math. Phys. **1**, 178 (1960).
[7] Harold Weitzner, Phys. Fluids **6**, 1123 (1963).
[8] M. D. Arthur, William Greenberg, and P. F. Zweifel, Phys. Fluids **20**, 1296 (1977).
[9] K. M. Case, Phys. Fluids **21**, 249 (1978).
[10] P. Degond, Centre de Mathematiques Appliquees, Ecole Polytechnique, Palaiseau CEDEX, France Internal Report No. 100, 1983 (unpublished).
[11] Vladimir Protopopescu, C. R. Acad. Sci. Paris **302**, 271 (1986).
[12] John David Crawford and Peter D. Hislop, Ann. Phys. (N.Y.) **189**, 265 (1989).
[13] David Fyfe and David Montgomery, Phys. Fluids **21**, 316 (1978).
[14] Alexander J. Klimas, J. Math. Phys. **20**, 2131 (1979).
[15] B. Abraham-Shrauner, Phys. Fluids **27**, 197 (1984).
[16] Alexander J. Klimas and Jeffery Cooper, Phys. Fluids **26**, 478 (1983).
[17] James Paul Holloway and J. J. Dorning, in *Modern Mathematical Methods in Transport Theory,* edited by W. Greenberg (Burkhauser, Basel, 1991), Vol. 51.
[18] Ira B. Bernstein, John M. Greene, and Martin D. Kruskal, Phys. Rev. **108**, 546 (1957).
[19] Mark Buchanan, James Paul Holloway, and J. J. Dorning, *Research Trends in Nonlinear and Relativistic Effects in Plasmas,* Proceedings of the Research Trends in Nonlinear and Relativistic Effects in Plasma Workshop, La Jolla, CA, edited by V. Stefan (American Institute of Physics, New York, in press).
[20] Lamberto Cesari, in *Nonlinear Functional Analysis and Differential Equations,* edited by L. Cesari, R. Kannan, and J. D. Schuur, Lecture Notes in Pure and Applied Mathematics Vol. 19 (Dekker, New York, 1976).
[21] Shui-Nee Chow and Jack K. Hale, *Methods of Bifurcation Theory* (Springer-Verlag, New York, 1982).
[22] Jack K. Hale, in *Bifurcation Theory and Applications,* edited by L. Salvadori, Lecture Notes in Mathematics Vol. 1057 (Springer-Verlag, Berlin, 1984).
[23] Lucio Demeio and James Paul Holloway, J. Plasma Phys. (to be published).
[24] James Paul Holloway and J. J. Dorning, Phys. Lett. A **138**, 279 (1989).