Nonequilibrium electron transport near absorbing boundaries

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A modified moment method is used to predict the effect of absorbing boundaries on a continuous stream of electrons traveling in a gas in a steady, uniform electric field. Theoretical results are shown to be in good qualitative agreement with published Boltzmann and Monte Carlo calculations for the same physical problem.

I. INTRODUCTION

A. Overview

This paper deals with what is perhaps the simplest physical situation in gaseous electronics—the one-dimensional, steady-state Townsend (SST) discharge with steady, uniform electric field low enough that inelastic collisions can be ignored, but high enough that the average energy of the electrons is much greater than that of gas atoms. In this situation, it is expected that the central region of the SST discharge can be characterized by electron transport properties that are relatively independent of position, provided that electrode separation is sufficiently great, while regions near electrodes can be characterized by transport properties that vary with position. By custom, regions of space-independent transport properties are called equilibrium regions, and regions of space-dependent transport properties are called nonequilibrium regions. According to this terminology, *equilibrium* means that a balance exists between the energy gained by electrons from the electric field and that given up to gas atoms by collisions [1].

The purpose of this paper is to assess the validity of an approximate theoretical method by comparing electron motion near the absorbing boundary predicted by this approximate method with motion predicted by published Boltzmann and Monte Carlo calculations. The modification consists in making the two-term Legendre expansion in velocity space *before* taking velocity moments of the Boltzmann equation, then assuming an approximate electron-energy distribution function (EEDF) different from that normally assumed in moment theory. In particular, the approximate EEDF is chosen so that correct equilibrium values of transport properties are obtained in the steady, uniform case, as suggested previously [2,3]. Because this approximate method is a modification of the usual method of moments, the theory of velocity moments of the Boltzmann equation (BE) is summarized briefly to emphasize the difference.

B. Method of moments

Generally speaking, the words *moment method* are used to describe the derivation of conservation equations for measurable quantities—such as density, momentum, energy, etc.—based on velocity moments of the BE. The measurable quantities themselves are defined as certain velocity moments of the EEDF itself. Therefore, an approximation for the EEDF is needed to proceed with the derivation of conservation equations.

In the thirteen-moment approximation [4], for example, the EEDF is expanded in Hermite polynomials, the leading term of which is a so-called shifted Maxwellian function. In the standard five-moment approach, a simplification of the thirteen-moment approximation, only the leading term of this expansion is retained, giving five measurable quantities—density, three components of average velocity, and energy. The five equations that determine the space-time dependence of these measurable quantities are the particle balance, or continuity, equation, three components of the momentum balance equation, and the energy balance equation. Recently, this approach has been applied to the one-dimensional SST problem in He at high E/N (where E is the electric field and N the gas density) and compared with qualitative success to numerical and Monte Carlo solutions of the BE [5].

The shifted Maxwellian function can also qualitatively describe physical situations where the energy of drift motion becomes large compared with that of random motion, such as a vacuum diode [6], where the EEDF is close to a δ function in velocity. In many physical situations, however, the shifted Maxwellian function is not the best approximation to the actual EEDF. For example, the EEDF in a gas-filled diode can be closer to δ function in speed than to a δ function in velocity. Thus, the assumption of a spherical shell for the EEDF, where the radius of the shell is random velocity and the displacement of the shell is directed velocity, can be a better approximation than a shifted Maxwellian function in certain physical situations [7]. For another example, the shifted Maxwellian function gives incorrect equilibrium values for drift velocity and average energy as functions of E/N when the momentum transfer collision frequency varies with electron energy. Clearly, then, the particular assumption made about the EEDF for deriving moment equations must be tailored to the particular physical situation being addressed.

For the purpose of discussion, the moment method described above—namely, that based on a shifted Maxwellian EEDF, particularly the five-moment method—is

called the shifted Maxwellian distribution (SMD). Likewise, other moment methods are identified according to the form of the EEDF assumed. The present paper deals with a modification consisting of a slightly anisotropic EEDF having a Druyvesteyn-like distribution in energy for the isotropic part and an anisotropic part related to the isotropic part by the two-term Legendre expansion; this modification is called the modified Druyvesteyn distribution (MDD). Another modification is called the shifted shell distribution (SSD) [5]. For all intents and purposes, SSD is equivalent to the momentum transfer approximation (MTA) [8,9]. MTA consists in assuming that the instantaneous energy can be replaced by the average energy in analytic expressions for elastic and inelastic collision frequencies, a result that is derived, not assumed, in the SSD. The modification proposed in Ref. [2] cannot be classified according to the type of EEDF because the EEDF is left unspecified. The modification proposed in Ref. [3] is based on a modified Druyvesteyn distribution similar to the MDD of the present paper, but the anisotropic part of the EEDF is lacking spatial derivatives of the isotropic part.

C. Modified Druyvesteyn distribution (MDD)

The problem investigated in this paper is nonequilibrium transport near absorbing boundaries. The physical situation addressed is the one-dimensional SST experiment where the following conditions prevail.

(i) Electron density is low enough that both space-charge distortion of the applied electric field and electron-electron collisions are negligible.

(ii) E/N is low enough that inelastic collisions can be neglected compared with elastic collisions, and that the energy of drift motion can be neglected compared with the energy of random motion.

(iii) E/N is high enough that the average energy of the electrons is much higher than that of the scattering gas.

Under these conditions, the EEDF is expected to be almost isotropic in velocity space. Therefore, it seems reasonable to assume that the two-term Legendre expansion of the EEDF is valid. On the one hand, this assumption seems to be supported by good agreement between Monte Carlo and two-term Boltzmann calculations mentioned later. On the other hand, it should be noted that the MDD has not been fully tested even in the equilibrium case.

Use of the two-term Legendre expansion in predicting electron motion in gases was pioneered by Allis [10]. In equilibrium, the two-term Legendre expansion leads to analytic expressions for f_0 and f_1, the isotropic and anisotropic parts of the EEDF, respectively, when inelastic collisions are negligible and the momentum transfer collision frequency can be expressed as a power of electron energy. These analytic expressions do not contain the average energy explicitly, but implicitly through the electric field. The MDD approach is based on the premise that the explicit dependence on the electric field in an equilibrium situation can be interpreted as an explicit dependence on the average energy, which remains valid in a nonequilibrium situation. Consequently, MDD calculations consist of the following steps.

(i) Making the two-term Legendre expansion of the EEDF, then taking velocity moments of the BE to give conservation equations for electron density n, average velocity $\mathbf{c}_0$, and average energy ε, as in standard moment theory.

(ii) Assuming a functional form for the EEDF that depends on n, $\mathbf{c}_0$, and ε in a particular way that ensures correct values of transport quantities when electrons are in equilibrium with the electric field. The EEDF of the two-term Legendre expansion is known to satisfy this requirement.

(iii) Solving the moment (conservation) equations for $n(\mathbf{r})$, $\mathbf{c}_0(\mathbf{r})$, and $\varepsilon(\mathbf{r})$.

The MDD has been used recently to investigate nonequilibrium effects in the SST [11] and TOF [12] (time-of-flight) experiments, with reasonable success. It is pointed out in these papers that electron diffusion in the SST and TOF experiments is isotropic, but may appear to be anisotropic when analyzed by classical diffusion theory with spatially constant transport coefficients. In other words, the MDD predicts different pulse widths parallel and perpendicular to the electric field even though the diffusion coefficient is a *scalar* quantity.

The MDD falls into the $\mathbf{S}_2$ description [2], which includes all problems where the time rate of change of average velocity of electrons is small compared with their acceleration due to the electric field. Whereas the MDD depends explicitly on the functional form assumed for f_0, the general approach taken in Ref. [2] does not. While it appears that the MDD is similar to the lowest-order solution of the $\mathbf{S}_2$ description in that average collision frequencies are assumed to have the same dependence on the average energy in nonequilibrium as in equilibrium, the corresponding moment equations are different unless the two-term Legendre solution to Eq. (26) of Ref. [2] is used to evaluate Eqs. (2a) and (2b) of the same reference in terms of n, $\mathbf{c}_0$, and ε.

The MDD is also different from the method proposed in Ref. [3], for the same reason as that proposed in Ref. [2]: different moment equations. The final moment equations are different because of a difference in the anisotropic term of the two-term Legendre expansion: the anisotropic term has spatial gradients of the isotropic part of the EEDF in the MDD approach, whereas the second term has no spatial gradients in the approach of Ref. [3]. Consequently, momentum flow due to thermal diffusion and heat flow due to thermal conduction, both of which are proportional to the gradient of the average energy, are not taken into account in the approach of Ref. [3].

D. Plan of paper

The rest of this paper is organized into sections as follows: first, the MDD theory is outlined, including assumptions and method of solution; next, the MDD predictions are compared with published Boltzmann and Monte Carlo predictions [13–15] and shown to be in qualitative agreement; finally, the MDD results are summarized.

II. THEORY

A. Equations

The MDD theory is discussed in detail in Refs. [11] and [12]. The basic conservation equations are the steady-state continuity equation, sometimes called the diffusion equation, and the steady-state energy balance equation.

According to the MDD, the one-dimensional diffusion equation for electrons streaming with constant current density Γ in an electric field of magnitude E is

$$\Gamma=-\frac{d}{dz}(n\mathcal{D})+\mu En \ , \tag{1}$$

where $\mathcal{D}$ and μ are nonuniform diffusion and mobility coefficients given by the expressions

$$\mathcal{D}=\frac{2e}{3m}\int_0^\infty \frac{\epsilon^{3/2}f_0}{\nu(\epsilon)}d\epsilon \ , \tag{2}$$

$$\mu=-\frac{2e}{3m}\int_0^\infty \frac{\epsilon^{3/2}}{\nu(\epsilon)}d\epsilon \ , \tag{3}$$

in which m is the electronic mass, e is the electronic charge, $\nu(\epsilon)$ is the electron–neutral-particle momentum-transfer collision frequency, ϵ is the electron energy, and $f_0(\epsilon,\mathbf{r})$ is the isotropic part of the EEDF.

Expressed in terms of one-dimensional heat flow H, analogous to particle flow Γ, the corresponding energy balance equation is

$$\frac{dH}{dz}=\Gamma E-2\frac{m}{M}\nu_\epsilon n\varepsilon \ , \tag{4}$$

with

$$H=-\frac{d}{dz}(n\mathcal{G})+\mathcal{B}En \ , \tag{5}$$

where M is atomic mass and it is assumed that the average energy of electrons is much greater than that of the gas. The quantities $\mathcal{G}$ and $\mathcal{B}$ are additional nonuniform transport coefficients given by the expressions

$$\mathcal{G}=\frac{2e}{3m}\int_0^\infty \frac{\epsilon^{5/2}f_0}{\nu(\epsilon)}d\epsilon \ , \tag{6}$$

$$\mathcal{B}=-\frac{2e}{3m}\int_0^\infty \frac{\epsilon^{5/2}}{\nu(\epsilon)}\frac{\partial f_0}{\partial\epsilon}d\epsilon \ , \tag{7}$$

and ν_ϵ is the average energy-transfer collision frequency, also nonuniform, given by the expression

$$\nu_\epsilon n\varepsilon=-\int_0^\infty \epsilon\frac{\partial}{\partial\epsilon}[\nu(\epsilon)\epsilon^{3/2}f_0]d\epsilon \ . \tag{8}$$

B. Assumptions

1. Power law for $\nu(\epsilon)$

For the purpose of comparing results of this work with results of previous investigators, the energy dependence of ν is expressed by the power law

$$\nu(\epsilon)=A_l\epsilon^{(l+1)/2} \ , \tag{9}$$

where A_l is independent of ϵ. Values of l investigated are -1, 0, and 1, for comparison with Refs. [13], [14], and [15], respectively.

2. Energy distribution function

Short of actually solving the scalar equation for f_0, a formidable numerical task, the selection of the functional form for f_0 is arbitrary at this point. An intuitively obvious choice is the modified Druyvesteyn function

$$f_0=\frac{n(l+2)}{\theta^{3/2}\Gamma(3/2(l+2))}\exp\left[-\left[\frac{\epsilon}{\theta}\right]^{(l+2)}\right] , \tag{10}$$

where the gamma function $\Gamma(u)$ is defined by the equation

$$\Gamma(u)=\int_0^\infty x^{u-1}\exp(-x)dx \ ,$$

and the quantity $\theta(\mathbf{r})$, which is proportional to average energy $\varepsilon(\mathbf{r})$, determines the half-width of the energy distribution. This function is an obvious choice for f_0 because it reduces to the exact solution of the scalar equation for f_0 in the steady, uniform state, provided that the steady, uniform versions of Eqs. (1) and (4) are used to relate θ to E/N.

The complete nonequilibrium EEDF of the MDD is

$$f(\epsilon,\mu,z)=f_0(\epsilon,z)+\mu f_1(\epsilon,z) \ ,$$

where μ is the cosine of the polar angle in phase space and f_1 satisfies [10]

$$\nu(\epsilon)f_1=-\left[\frac{2\epsilon}{m}\right]^{1/2}\left[\frac{\partial f_0}{\partial z}+eE\frac{\partial f_0}{\partial\epsilon}\right] .$$

3. Transport coefficients

According to Eqs. (2), (3), and (6)–(9), the transport coefficients have the following dependence on θ for the modified Druyvesteyn f_0 given above:

$$\mu=k_l\theta^{-(l+1)/2} \ , \tag{11}$$

$$\mathcal{D}=d_l\theta^{(1-l)/2} \ , \tag{12}$$

$$\mathcal{B}=b_l\theta^{(1-l)/2} \ , \tag{13}$$

$$\mathcal{G}=g_l\theta^{(3-l)/2} \ , \tag{14}$$

$$\nu_\epsilon=f_l\theta^{(l+1)/2} \ . \tag{15}$$

The corresponding relation between ε and θ is

$$\varepsilon=a_l\theta \ . \tag{16}$$

Values of the constants k_l, d_l, b_l, g_l, f_l, and a_l are found by comparing Eqs. (11)–(16) above with Eqs. (19)–(23) and (25) of Ref. [12].

C. Method of solution

1. Dimensionless variables

By algebraic manipulation, Eqs. (1), (4), and (5) can be written in the following form, which is convenient for a

solution by a Runge-Kutta technique:

$$\frac{dN}{dS}=\frac{3-l}{2}\left[\frac{N}{U}-\frac{1}{U^{(1-l)/2}}\right]+\frac{1-l}{2}\rho_l\left[\frac{Q}{U^{(3-l)/2}}-\frac{N}{U}\right], \tag{17}$$

$$\frac{dQ}{dS}=\frac{2}{4-l}(1-NU^{(l+3)/2}), \tag{18}$$

$$\frac{dU}{dS}=\rho_l-1+\frac{1}{N}\left[U^{(l+1)/2}-\rho_l\frac{Q}{U^{(1-l)/2}}\right], \tag{19}$$

where $N\equiv n/n_{eq}$, $Q\equiv H/H_{eq}$, $U\equiv\varepsilon/\varepsilon_{eq}$, $S\equiv Wz/D$, and the constants ρ_l, H_{eq}, ε_{eq}, and W/D have the following values:

$$\rho_l=\frac{d_l b_l}{k_l g_l}=\frac{\Gamma[(4-l)/(2l+4)]\Gamma[(8+l)/(2l+4)]}{\Gamma[(6+l)/(2l+4)]\Gamma[(6-l)/(2l+4)]}, \tag{20}$$

$$H_{eq}=\frac{b_l}{k_l a_l}\Gamma\varepsilon_{eq}, \tag{21}$$

$$\frac{W}{D}=\frac{k_l a_l E}{d_l \varepsilon_{eq}}, \tag{22}$$

$$\varepsilon_{eq}=a_l\left[\frac{M}{2m}\frac{k_l}{f_l a_l}E^2\right]^{1/(l+2)}. \tag{23}$$

The parameter n_{eq} is arbitrary, depending on the strength of the electron source at the incoming boundary or the emitting boundary. The subscript eq means equilibrium value. The symbols W and D have their usual meaning of uniform, equilibrium values of mobility times electric field and diffusion coefficient, respectively.

2. *Boundary conditions*

The system of Eqs. (17), (18), and (19) requires three boundary conditions for a unique solution. The nature of these equations is such that if each dependent variable N, U, and Q takes on the equilibrium value of unity at precisely the same location, then equilibrium prevails everywhere. Therefore, equilibrium conditions cannot be specified as starting values in a nonequilibrium situation.

Before discussing boundary conditions for the MDD further, it is helpful to discuss boundary conditions for the SMD, because one less boundary condition is required. The SMD leads to a system of two first-order equations satisfied by N and U [16] rather than a system of one first-order and one second-order equation as in the present work [17]. This situation comes about because the assumption of a shifted Maxwellian EEDF results in no momentum flow due to thermal diffusion or heat flow due to thermal conduction, both of which are proportional to the gradient of the average energy. With two first-order differential equations, two boundary conditions are required to give a unique solution. In Ref. [16], which deals with a SST problem similar to the one discussed in this paper, the required conditions are assumed to be the following: (i) $N=0$ at absorbing boundary; (ii) $U=0$ at emitting boundary. The second condition is arbitrary—any other physically reasonable value of U at the emitting boundary is mathematically acceptable. The choice $U=0$ at the emitting boundary corresponds to a physical situation where electrons emitted at the incoming place have very low energy, such as electrons produced by thermionic emission. For electrons ejected by uv photons impinging on the emitting boundary, U at the emitting boundary might be significantly larger than 1, depending on the value of ε_{eq}.

For the MDD, an additional boundary condition is required. It seems reasonable to require that heat flow H—given by Eq. (5)—be finite at the absorbing boundary. By Eq. (19), this requirement implies that $U=\rho_l Q$ at the absorbing boundary where $N\to 0$. Therefore, boundary conditions for the MDD are the following: (i) $N\to 0$ at the absorbing boundary, located at $S=0$; (ii) $U=U_d$ at the emitting boundary, located at $S=S_d$; (iii) dU/dS finite at the absorbing boundary, implying $U=\rho_l Q$ there.

The Runge-Kutta technique of solving a system of first-order differential equations requires starting values for all dependent variables at the same value of the independent variable S; consequently, integration is started at the absorbing boundary in the present paper. By necessity, therefore, the starting value of U at the absorbing boundary is adjusted repeatedly until boundary condition (ii)—$U(S_d)=U_d$— is satisfied. Three examples are chosen for illustration. Case 1 is characterized by $l=-1$, the case of constant collision frequency studied in the infinite half-space [13] and in a finite space [16]. Case 2 is characterized by $l=0$, the case of constant cross section studied in the infinite half-space [9,14]. Case 3 is characterized by $l=1$, which was also studied in the

TABLE I. Parameters for cases 1–3.

Case	l	S_d	U_d	U_0	ε_∞	ε_d	Method	Reference
1	−1	38	0	1.32		0	MDD	This work
		∞		1.67			BE	[13]
		38	0	1.35			SMD	[16]
2	0	16.5	4.33	1.56	0.462	2	MDD	This work
		∞		1.8[a]	0.462		BE	[14]
		∞		1.3			SSD	[9]
3	1	16.5	6.33	1.67	0.316	2	MDD	This work
		∞		1.85[a]	0.316		BE	[15]

[a] By extrapolation.

infinite half-space [15]. In case 1, the average energy at the emitting boundary is assumed to be near zero and S_d is assumed to be 38, in correspondence with previous work [16]. In cases 2 and 3, the average energy at the emitting boundary is arbitrarily assumed to be 2 eV and the distance d between absorbing and emitting boundaries is arbitrarily assumed to be 5.2 cm. Because $\mu_{eq}E/D_{eq}=3.18\ \text{cm}^{-1}$ in Refs. [14] and [15], the corresponding value of S_d is $3.18\times5.2=16.5$. Likewise, because $\varepsilon_{eq}=0.462$ eV in Ref. [14] and 0.316 eV in Ref. [15], the corresponding values of U at the emitting boundary are $2/0.462=4.33$ in case 2 and $2/0.316=6.33$ in case 3. These parameters are summarized for all three cases in Table I. All entries in Table I are input values—boundary conditions, etc.—except those in the column headed U_0, the values of which are solution results. Solution results of previous calculations of U_0 are also given in Table I.

III. RESULTS AND DISCUSSION

In this section, MDD results for density, average energy, and drift velocity μE, calculated according to the method outlined above, are compared with the corresponding Boltzmann and Monte Carlo calculations of Refs. [13–15]. It is shown that MDD results are in semiquantitative agreement with BE results for the spatial variation of density in the nonequilibrium region near the absorbing boundary. Furthermore, MDD results predict increasing average energy in the nonequilibrium region near the absorbing boundary where the electron density goes to zero, in qualitative agreement with BE results. In case 1, the MDD predicts a nonequilibrium region near the absorbing boundary that is the same size as that predicted by the analytic solution of the BE. In cases 2 and 3, however, the MDD predicts a somewhat larger region of nonequilibrium than that predicted by the numerical solution of the BE. This result suggests that the influence of the absorbing boundary extends further than the distance of 1 cm assumed in the Boltzmann calculations of Ref. [14] and [15]. It is tempting to speculate that closer agreement between the numerical Boltzmann solution and the MDD solution might be obtained with a Boltzmann calculation extending all the way to the emitting boundary, as was recently done for the SST in He at high E/N [18].

It is also shown that MDD predictions of $N(S)$ for energy-dependent collision frequency agree much more closely with those of the numerical solution of the BE than do density-gradient-expansion [19] (DGE) predictions. It has been shown previously that DGE predictions of $U(S)$ do not agree even qualitatively with any of the other methods discussed in this paper [16]. Therefore, it is suggested that DGE analysis is simply not applicable to nonequilibrium problems of the type discussed in this paper.

A. Case 1: $l=-1$

MDD results for the relative density and the average energy in the case of constant collision frequency are shown graphically in Fig. 1. Note that the absorbing boundary is located at $S=0$ in this figure, the emitting boundary is located at $S=38$, and U at the emitting boundary is nearly zero. These conditions are imposed for comparison with SMD moment calculations made previously [16]. The results shown in Fig. 1 are hardly distinguishable from those presented in Fig. 1 of Ref. [16]. The results of Ref. [16] are obtained by solving the system of SMD equations consisting of Eqs. (1), (4) and the following relation for H:

$$H=\tfrac{5}{3}\Gamma\varepsilon\ ,$$

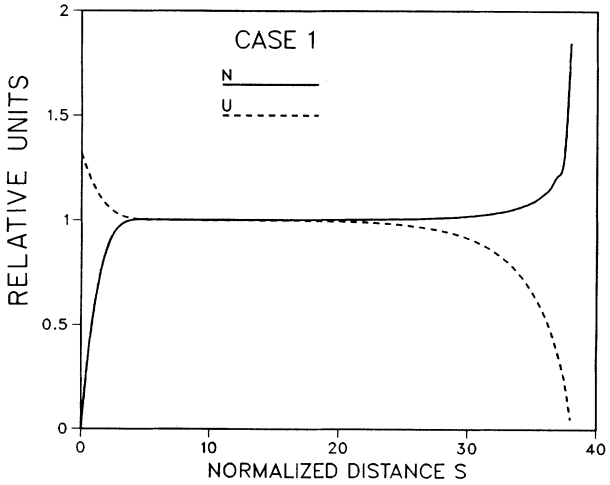

FIG. 1. Relative density N and average energy U plotted against dimensionless distance S, measured from the absorbing boundary. Emitting boundary located at $S_d=38$. Case 1, $l=-1$. Boundary conditions are $N(0)=0$, $U(S_d)\approx0$, and $Q(0)=U(0)$.

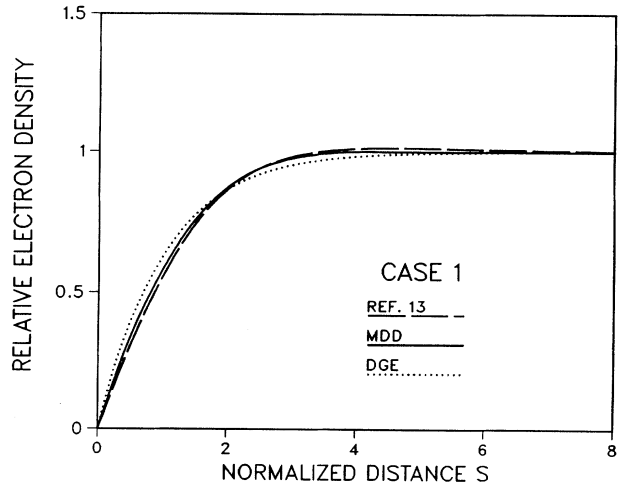

FIG. 2. Comparison of MDD results for the relative density $N(S)$ with the analytic solution of the Boltzmann equation of Ref. [13] and the DGE result of Ref. [19] near the absorbing boundary. Case 1, $l=-1$.

which follows from the assumption of a shifted Maxwellian EEDF when the energy of drift motion is small compared with the energy of random motion.

Figure 2 shows a comparison of the present results for $N(S)$ with the analytic solution of the BE [13] near the absorbing boundary. Shown also in this figure is the DGE result for N, which is $N=1-\exp(-S)$. According to this figure, the agreement between the MDD and the analytic Boltzmann solution of Ref. [13] is quite good, as is the agreement between the MDD and the DGE.

Figure 3 shows a comparison of the present results for $U(S)$ with the analytic solution of the BE [13] near the absorbing boundary. As listed in Table I, the value of U at the absorbing boundary is 1.32 for the MDD, whereas the analytic solution of the BE gives 1.67 [13]. However, the extent of the nonequilibrium region next to the absorbing boundary is the same for both calculations. In cases 2 and 3 discussed below, the extent of the nonequilibrium region based on the MDD is somewhat larger than that based on the numerical solution of the BE.

B. Case 2: $l=0$

MDD results for the relative density and the average energy in the case of a constant cross section are shown graphically in Fig. 4. As before, the absorbing boundary is located at $S=0$, but the emitting boundary is located at $S=16.5$. In addition, the value of U at the emitting boundary is 4.33, while that at the absorbing boundary is 1.564.

Figure 5 shows a comparison of the present results for $N(S)$ with the numerical solution of the BE [14] near the absorbing boundary. Shown also in this figure is the DGE result for N, which is $N=1-\exp(-2S)$. According to this figure, the agreement between the MDD and the numerical Boltzmann solution of Ref. [14] is much better than the agreement between the MDD and DGE calculations. The influence of the absorbing boundary extends over a somewhat larger distance for the MDD. Clearly, the DGE solution is not a very good approximation.

Figure 6 shows a comparison of the present results for $U(S)$ with the numerical solution of the BE [14] near the absorbing boundary. As listed in Table I, the value of U at the absorbing boundary is 1.564 for the MDD, whereas the numerical solution of the BE gives 1.8 [14]. Consistent with Fig. 5, the nonequilibrium region extends

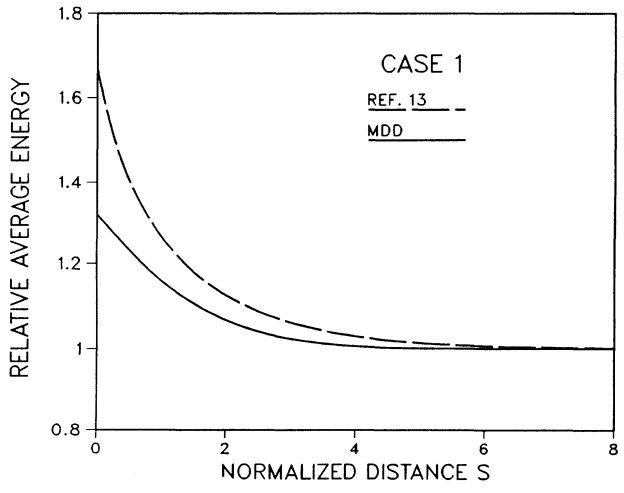

FIG. 3. Comparison of MDD results for the relative average energy $U(S)$ with the analytic solution of the Boltzmann equation of Ref. [13] near the absorbing boundary. Case 1, $l=-1$. Note that the extent of the nonequilibrium region is about the same, even though agreement for $U(0)$ is poor.

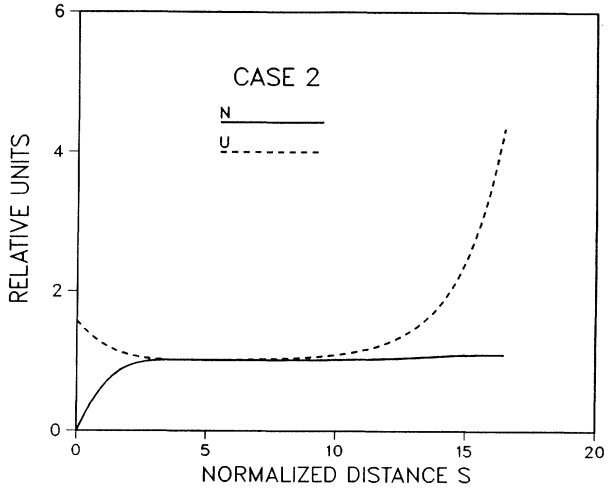

FIG. 4. Relative density N and average energy U plotted against dimensionless distance s, measured from the absorbing boundary. Emitting boundary located at $S_d=16.5$. Case 2, $l=0$. Boundary conditions are $N(0)=0$, $U(S_d)=4.33$, and $Q(0)=1.273U(0)$.

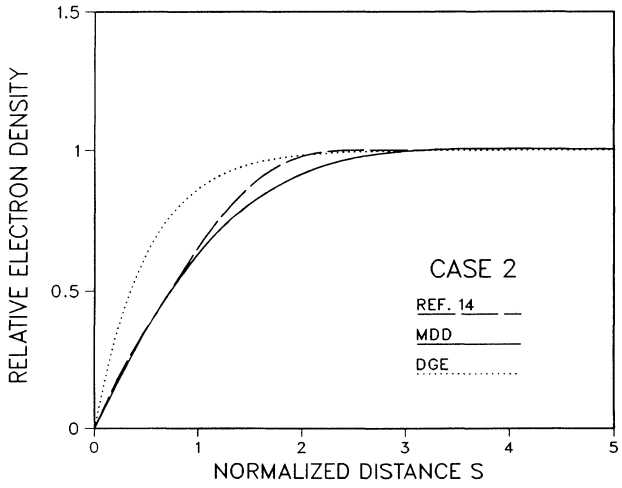

FIG. 5. Comparison of MDD results for the relative density $N(S)$ with the numerical solution of the Boltzmann equation of Ref. [14] and the DGE result of Ref. [19] near the absorbing boundary. Case 2, $l=0$.

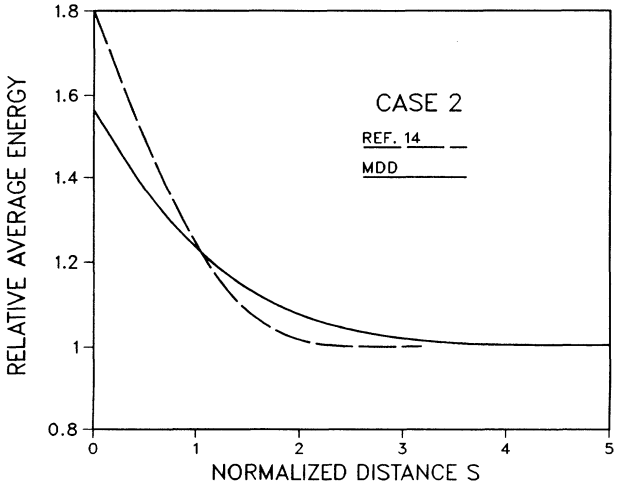

FIG. 6. Comparison of MDD results for the relative average energy $U(S)$ with the numerical solution of the Boltzmann equaton of Ref. [14] near the absorbing boundary. Case 2, $l=0$. Note that the extent of the nonequilibrium region is larger for the MDD.

over a somewhat larger distance for the MDD.

Figure 7 shows a comparison of the present results for the relative drift velocity $\mathcal{W}(S)\equiv\mu/\mu_{\rm eq}$ with the numerical solution of the BE [14] near the absorbing boundary. Clearly, the numerical Boltzmann solution for the relative drift velocity does not follow the relation

$$\mathcal{W}\equiv\frac{\mu}{\mu_{\rm eq}}=U^{-(l+1)/2}\ , \tag{24}$$

which is inherent in the MDD.

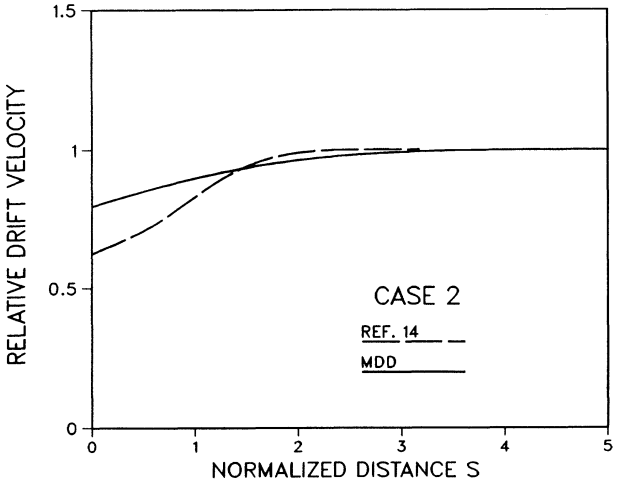

FIG. 7. Comparison of MDD results for the relative drift velocity $\mathcal{W}(S)$ with the numerical solution of the Boltzmann equation of Ref. [14] near the absorbing boundary. Case 2, $l=0$.

C. Case 3: $l=1$

MDD results for the relative density and the average energy in the case of a cross section proportional to the square root of the electron energy are shown graphically in Fig. 8. As in case 2, the absorbing boundary is located at $S=0$ and the emitting boundary is located at $S=16.5$. Now, however, the value of U at the emitting boundary is 6.33 and that at the absorbing boundary is 1.67. The general features of nonequilibrium regions near the boundaries are the same as in case 2.

Figure 9 shows a comparison of the present results for $N(S)$ with the numerical solution of the BE [15] near the absorbing boundary. Shown also in this figure is the DGE result for N, which is $N=1-\exp(-3.45S)$. As in case 2, the agreement between the MDD and the numerical Boltzmann solution of Ref. [15] is better than that between the MDD and the DGE. Likewise, the nonequilibrium region extends over a somewhat larger distance for the MDD.

Figure 10 shows a comparison of the present results for $U(S)$ with the numerical solution of the BE [15] near the absorbing boundary. As listed in Table I, the value of U at the absorbing boundary is 1.67 for the MDD, whereas the numerical solution of the BE gives 1.85 [15]. Consistent with Fig. 9, the nonequilibrium region extends over a somewhat larger distance for the MDD.

Figure 11 shows a comparison of the present results for the relative drift velocity $\mathcal{W}(S)\equiv\mu/\mu_{\rm eq}$ with the numerical solution of the BE [15] near the absorbing boundary. As in case 2, the numerical Boltzmann solution for the relative drift velocity does not follow Eq. (24).

IV. SUMMARY AND CONCLUSIONS

A modified-moment method, called the MDD, is used to predict the effect of absorbing boundaries on a con-

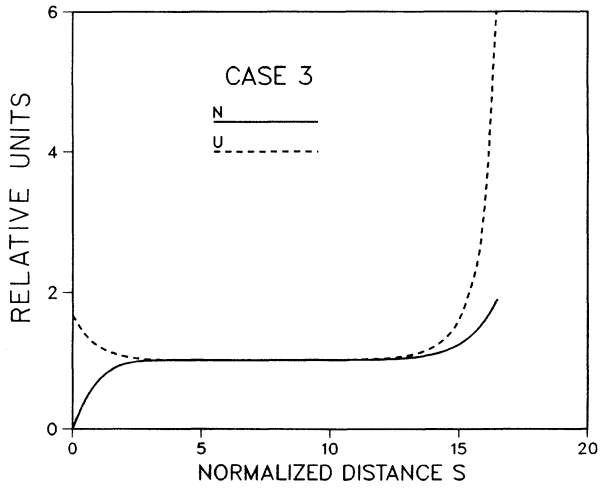

FIG. 8. Relative density N and average energy U plotted against dimensionless distance S, measured from the absorbing boundary. Emitting boundary located at $S_d=16.5$. Case 3, $l=1$. Boundary conditions are $N(0)=0$, $U(S_d)=6.33$, and $Q(0)=1.5U(0)$.

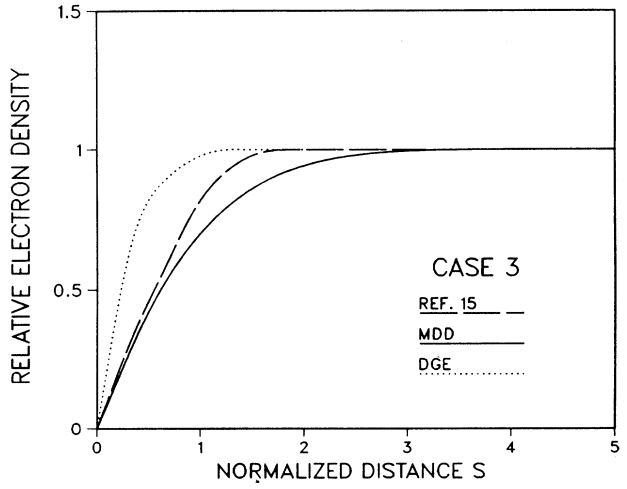

FIG. 9. Comparison of MDD results for the relative density $N(S)$ with the numerical solution of the Boltzmann equation of Ref. [15] and the DGE result of Ref. [19] near the absorbing boundary. Case 3, $l=1$.

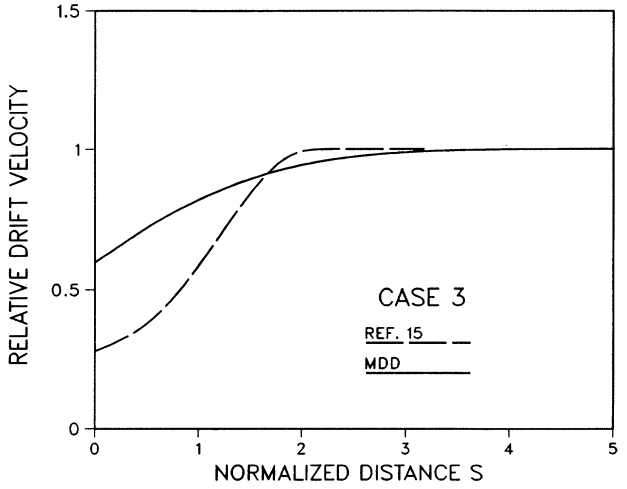

FIG. 11. Comparison of MDD results for the relative drift velocity $\mathcal{W}(S)$ with the numerical solution of the Boltzmann equation of Ref. [15] near the absorbing boundary. Case 3, $l=1$.

tinuous stream of electrons traveling in a gas in a steady, uniform electric field. Theoretical results for the density variation in the nonequilibrium region near the absorbing boundary are shown to be in semiquantitative agreement with published Boltzmann and Monte Carlo calculations for the same physical problem. Furthermore, the average energy increases in the nonequilibrium region near the absorbing boundary where the electron density goes to zero, in qualitative agreement with published Boltzmann and Monte Carlo calculations.

The MDD prediction of the spatial extent of the nonequilibrium region near an absorbing boundary is in good agreement with that predicted by the analytic solution of the Boltzmann equation [13] for constant collision frequency. The MDD results for the variation of the electron density in the nonequilibrium region are in close agreement with the analytic solution and with the DGE solution of the Boltzmann equation, while results for the variation of the average energy do not agree as well with the analytic Boltzmann solution.

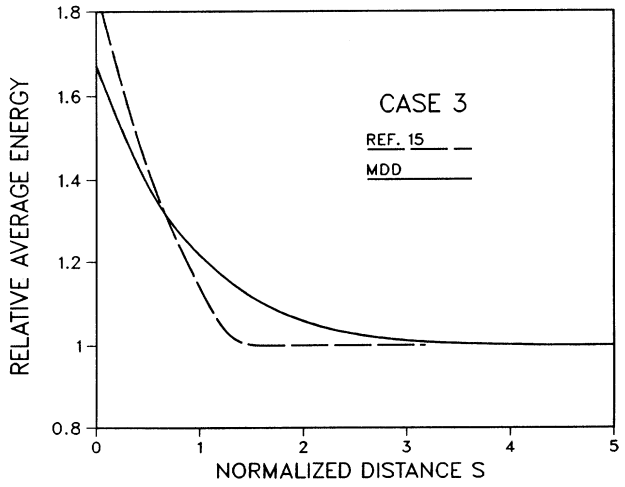

FIG. 10. Comparison of MDD results for the relative average energy $U(S)$ with the numerical solution of the Boltzmann equation of Ref. [15] near the absorbing boundary. Case 3, $l=1$. Note that the extent of the nonequilibrium region is larger for the MDD.

The MDD prediction of the spatial extent of the nonequilibrium region near an absorbing boundary agrees somewhat less with that predicted by the numerical solution of the Boltzmann equation for the energy-dependent collision frequency [14,15]. These results suggest that the influence of the absorbing boundary extends further than the distance of 1 cm assumed in the Boltzmann calculations of Refs. [14] and [15].

The lack of more quantitative agreement can be ascribed to the approximate nature of the moment methods on the one hand, and to the possible incompleteness of the numerical Boltzmann solutions on the other. It should be noted that closer agreement is obtained between the MDD and the analytic solution of the Boltzmann equation than between the MDD and the numerical Boltzmann solutions. It is speculated that the numerical Boltzmann solutions may be misleading because of the assumption that the equilibrium conditions prevail 1 cm away from the absorbing boundary.

It is concluded that the modified-moment method described in this paper is adequate for qualitative predictive modeling of nonequilibrium effects near absorbing boundaries. Furthermore, it is simpler and far less time consuming to explore nonequilibrium behavior with moment equations than with either Boltzmann or Monte Carlo calculations. It should be kept in mind, however, that the main assumption regarding the form of the EEDF in

the nonequilibrium region has not been formally justified. Therefore, it is not clear just how much significance can be placed on the predictions of the MDD method. A self-consistent Monte Carlo simulation amplifying the work of Ref. [15], a complete Boltzmann analysis similar to that of Ref. [18], or a complete analysis based on the theory outlined in Ref. [2], might provide a better test of the MDD method.

[1] Perhaps a less ambiguous term is *drift equilibrium,* to distinguish the present situation from the situation where *local thermodynamic equilibrium* exists, meaning that all particles have nearly the same average energy, as in arcs at high pressure.

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