Characterizing the lacunarity of random and deterministic fractal sets

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The notion of lacunarity makes it possible to distinguish sets that have the same fractal dimension but different textures. In this paper we define the lacunarity of a set from the fluctuations of the mass distribution function, which is found using an algorithm we call the gliding-box method. We apply this definition to characterize the geometry of random and deterministic fractal sets. In the case of self-similar sets, lacunarity follows particular scaling properties that are established and discussed in relation to other geometrical analyses.

I. INTRODUCTION

The geometry of structures grown in many physical processes is characterized by the existence of a fractal dimension that can be viewed as a measure of their irregularity [1]. In the case of self-similar sets, the fractal dimension describes the way in which the number of elements in a set, its mass, grows with linear size. Recently, much effort has been directed towards efficiently determining the fractal dimension in a great variety of systems. Generalization to geometrical multifractals has also been carried out [2]. In spite of the relevance of these metric properties, simple visual inspection shows that there can be several sets with the same fractal dimension but with different textures [3]. The notion of lacunarity, which is related to the degree of translational invariance, makes it possible to distinguish between these different sets.

Several methods have already been introduced in the literature for measuring the lacunarity of sets [3–5]. However, most of them appear to be impractical or encounter difficulties in characterizing a wide range of structures. In the present paper, we investigate a promising way of defining the lacunarity of a deterministic or a random set, following a general approach which consists of analyzing the mass distribution in the set. The paper is organized as follows. In Sec. II we present an algorithm we call the gliding-box algorithm, from which we characterize the lacunarity. In Sec. III we demonstrate the efficiency of the method in the case of generalized Cantor sets. In our definition, lacunarity is a function of scale r at which it is measured. For self-similar sets, it exhibits noteworthy scaling properties that are established and discussed in Sec. IV, in relation to other geometrical analyses.

II. A DEFINITION OF LACUNARITY

A. The gliding-box algorithm

A generally fruitful approach for reliably determining the lacunarity of a random or deterministic fractal set consists of analyzing the fluctuations of the mass distribution function. Different procedures have already been used in the literature to characterize the distribution of mass in a set; the box-counting method and the sandbox method are two well-known methods which will be revisited in Sec. IV. Here, we are concerned with an algorithm we call the gliding-box algorithm. In this method, the set under study is put on an underlying lattice with a mesh size equal to $2a$ (Fig. 1); a is lower or equal to particle radius ϵ. In the case of regular deterministic sets, such as Cantor sets or Sierpinski curves, this underlying lattice is simply the lattice on which the set is built. In the case of random sets, the existence of an underlying lattice arises naturally from discretization. For instance, in the particular case of two-dimensional experimental clusters, the underlying lattice is the array of pixels provided by the image processing system which is used to digitize the structure. Each particle is then identified by the site of the lattice which is the nearest to its center.

Now, we consider a box of radius r which "glides" on this lattice in all the possible manners, its center being placed successively on the different sites of the underlying lattice. This box can be any geometrical figure which has a radius r. In practice, when the set is of finite size, the movement of the gliding box is restricted by the existence of a boundary. There are several ways of bounding a set [6]. In Fig. 1, the boundary is the E parallelepiped which circumscribes the aggregate (E is the Euclidean dimension). The radius of the boundary defines the characteristic size of the set L. It must be noted that the different definitions of the characteristic size which result from the different choices of boundary are not necessarily equivalent. In particular, they can be more or less sensitive to the existence of a growing zone [7].

Let us define the distribution of mass in the collection of gliding boxes: $n(M,r)$ is the number of gliding boxes with radius r and mass M. Dividing $n(M,r)$ by the total number of boxes, which varies like $(L/a)^E$, provided that $\epsilon \le r \ll L$, we directly obtain the probability function $Q(M,r)$, such that a gliding box of radius r contains mass M. For a regular lattice that is translationally invariant, the mass embedded in a gliding box is independent of its position, and probability $Q(M,r)$ is a Dirac function. For a self-similar set, the value of $Q(M,r)$ depends on box radius r and on linear size L. Indeed, when L becomes

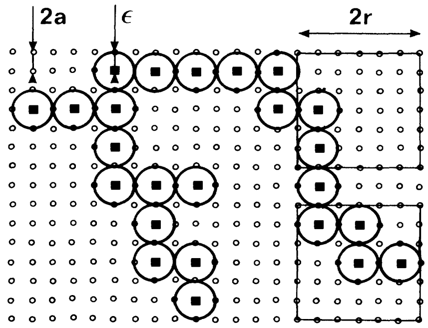

FIG. 1. Illustration of the gliding-box method; a and ϵ are the mesh size of the underlying lattice and the radius of a particle, respectively. The sites of the underlying lattice are represented by open dots; those which are occupied by the center of a particle are indicated by a solid square. The gliding box is a square of side $2r$.

infinite, an increasing number of gliding boxes are empty so that $Q(M,r)$ tends towards unity for $M=0$ and towards zero otherwise. Moreover, it must be noted that the true statistical behavior of $Q(M,r)$ is reached only for $r/L \ll 1$, i.e., when the set is large. On the contrary, the expression of $Q(M,r)$ must be corrected by a cutoff function that takes into account the finite size of the set. Such corrections will be omitted in the following sections.

B. Moments and lacunarity

It is convenient to analyze the properties of probability function $Q(M,r)$ starting from its statistical moments $Z_Q^{(q)}(r)$:

$$Z_Q^{(q)}(r)=\sum_M M^q Q(M,r) \ . \tag{1}$$

In general, a complete knowledge of all these different moments is necessary to achieve a proper physical characterization of the set (see Sec. IV). Now we focus on the second moment and we define the lacunarity at scale r by the mean-square deviation of the fluctuations of mass distribution probability $Q(M,r)$ divided by its square mean:

$$\Lambda(r)=\frac{Z_Q^{(2)}(r)}{[Z_Q^{(1)}(r)]^2} \ . \tag{2}$$

From this definition, lacunarity takes on a noteworthy physical meaning since it can be interpretated as the width of the mass distribution function $Q(M,r)$. In the case of a translationally invariant lattice, $Q(M,r)$ is a Dirac function, $Z_Q^{(2)}(r)=(Z_Q^{(1)})^2$, and lacunarity is independent of r and equal to 1. Nontranslationally invariant sets have a lacunarity larger than 1. Sets with voids of all sizes are expected to be very lacunar with lacunarities much larger than 1, while sets with single-sized voids have low lacunarities close to 1. In this respect, lacunarity measures to the extent at which a set is not translationally invariant. Definition (2) is general since it can be applied to any set which is not necessarily fractal at an arbitrary scale r. In the next section, we show that it suppresses all the shortcomings encountered in the previous definitions available in the literature.

III. THE LACUNARITY OF GENERALIZED CANTOR SETS

In this section we apply the general definition introduced above to characterize the lacunarity of uniform generalized Cantor sets. These sets have recursive properties that allow us to calculate analytically probability function $Q(M,r)$ and lacunarity $\Lambda(r)$. Cantor sets are one-dimensional sets, but the same procedure can be extended directly to any recursive sets in higher Euclidean spaces.

A. The generalized Cantor sets

The generalized Cantor sets represented in Fig. 2 are built according to a recursive inflation method which has been described elsewhere [8]. We start with a unit cell, here a particle of radius ϵ, and with N vectors $\underline{u}_n$ that form the generator and give the locations of the unit cells in the first iteration. In the following, we shall restrict ourselves to the case where the moduli of the vectors $\underline{u}_n$ are of the form $u_n=(2k+1)\epsilon$, k being an integer. A number ξ characterizes the growth at each step of the hierarchy. To get the ith iteration set, we scale up the vectors of the generator by the factor ξ^{i-1} and we arrange N sets identical to the $(i-1)$th iteration set at $\xi^{i-1}\underline{u}_n$. At the ith step of the construction, the resulting set includes N^i particles separated by voids of different sizes; its length $2L$ is equal to $2\xi^i\epsilon$. For a given choice of N and ξ, we can build $\binom{N}{\xi}$ generalized Cantor sets with the same fractal dimension $D=\log N/\log\xi$, some of them being identical. In Fig. 2, we have represented six particular sets obtained with $N=3$ and $\xi=5$. The different generators are shown in Table I. For instance, the initial set of type C_1 is made of three particles centered respectively at $u_1=\epsilon$, $u_2=5\epsilon$, and $u_3=9\epsilon$ (the origin is on the left end of each set). All these sets have the same fractal dimension but their lacunarity is different:

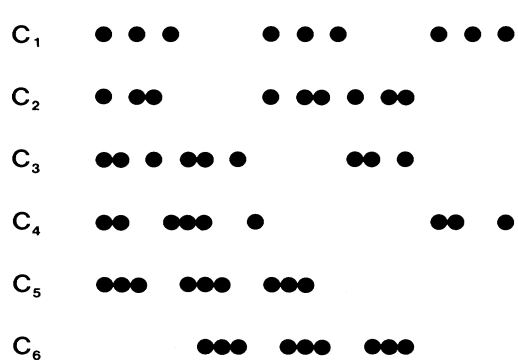

FIG. 2. Representation of six generalized Cantor sets obtained with $N=3$ and $\xi=5$ ($i=2$). The generators are displayed in Table I.

indeed, the voids between the particles can be scattered or condensed leading to the most homogeneous set (C_1), with low lacunarity, to the least homogeneous set (C_5 or C_6), with high lacunarity.

B. The lacunarity of generalized Cantor sets

To derive probability function $Q(M,r)$ and lacunarity $\Lambda(r)$ of a generalized Cantor set at step i, we measure the mass embedded in a gliding box of radius r which glides from the left end of the set to its right end in such a manner that its center is placed successively on the different sites of the underlying lattice which, here, has a mesh size equal to 2ϵ (Fig. 3). r varies between ϵ and $\xi^i\epsilon$; in this way, we get a collection including $\xi^i - r/\epsilon + 1$ boxes. The mass embedded in a box is equal to the number of particle centers situated inside the box or at its boundary. The mass distribution follows an exact recurrence relation which is simply obtained by noting that the ith iteration set results from the juxtaposition of N sets which are identical to the $(i-1)$th iteration set. Provided that r is lower that the radius of the $(i-1)$th iteration set ($r \le \xi^{i-1}\epsilon$), $n_i(M,r)$, is related to $n_{i-1}(M,r)$ through

$$n_i(M,r) = N n_{i-1}(M,r) + \sum_{j=1}^{\xi-1} \nu_j(M,r) , \tag{3}$$

where $\nu_j(M,r)$ is the number of boxes of radius r and mass M at the jth connection between two $(i-1)$th iteration sets, or between an $(i-1)$th iteration set and a void, or between a void and an $(i-1)$th iteration set. The second term in the right-hand side of Eq. (3) takes into account the contribution of the boxes situated at these connections (Fig. 3). It is worth noting that this relation is only valid when M is different from zero; however, since the zero-mass boxes make no contribution to the values of the different moments, it is not necessary to calculate $n_i(0,r)$.

Using relation (3), we can now derive an explicit expression for $n_i(M,r)$. Let w be the order of iteration of

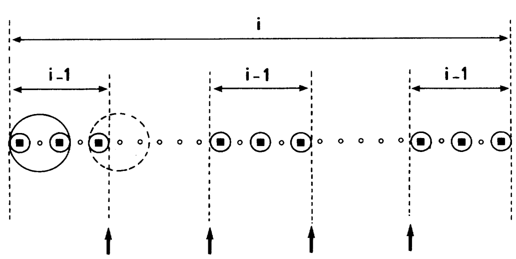

FIG. 3. Calculation of the lacunarity of a Cantor set of type C_1. The mesh size of the underlying lattice is 2ϵ. The sites occupied by the center of a particle are indicated by a solid square. The radius of the gliding box is $r = 3\epsilon$. We have represented one box, which is entirely included in the $(i-1)$th iteration set (solid-line), and one box at the connection between an $(i-1)$th iteration set and a void (dashed-line circle). There are four such connections, indicated by black arrows.

the smallest set with a radius just greater than r: $\xi^{w-1}\epsilon \le r \le \xi^w \epsilon$. Beginning the recurrence at order w, we find

$$n_i(M,r) = N^{i-w} n_w(M,r) + \left[\sum_{k=0}^{i-w-1} N^k \right] \left[\sum_{j=1}^{\xi-1} \nu_j(M,r) \right] . \tag{4}$$

After summing the geometrical series which appears in the second term of the right-hand side of this expression, and assuming that the degree of iteration i is large enough to ensure that $N^{i-w} - 1 \simeq N^{i-w}$, $n_i(M,r)$ takes the form

$$n_i(M,r) = \Omega(M,r) N^i , \tag{5}$$

where the coefficient $\Omega(M,r)$ is independent of i and is simply a function of $n_w(M,r)$, N, ν_j, and w:

$$\Omega(M,r) = \frac{1}{N^w} \left[n_w(M,r) + \frac{1}{N-1} \sum_{j=1}^{\xi-1} \nu_j(M,r) \right] . \tag{6}$$

Figure 3 illustrates the determination of $\Omega(M,r)$ and $n_i(M,r)$ for $r = 3\epsilon$ in the case of the Cantor set C_1. Since r is lower than the first order set, we take $w = 1$. There are three gliding boxes in the first-order set, leading to $n_1(1,3\epsilon) = 1$, $n_1(2,3\epsilon) = 2$ and $n_1(M \ge 3, 3\epsilon) = 0$. In the second-order set, there are four connections between a first-order set and a void or between a void and a first-order set. The boxes situated at these connections are all identical and give $\nu_1(1,3\epsilon) = \nu_2(1,3\epsilon) = \cdots = 2$. Finally, $\Omega(M,r)$ and $n_i(M,r)$ follow directly from formulas (4) to (6).

The probability function that a box of radius r contains mass M is equal to the number of boxes of mass M, $n_i(M,r)$, divided by the total number of gliding boxes ($\simeq \xi^i$ for $r \ll \xi^i$):

$$Q(M,r) = \Omega(M,r) \left[\frac{N}{\xi} \right]^i . \tag{7}$$

$(N/\xi)^i$, which is equal to $(L/\epsilon)^{D-1}$, gives the dependence of $Q(M,r)$ on the linear size of the set. When it becomes infinitely large, $(L/\epsilon)^{D-1} \simeq 0$ and we recover the result that $Q(M,r) \simeq 0$ for $M > 0$, which means that most of the boxes are empty. The qth moment of $Q(M,r)$ results directly from relations (1) and (7):

$$Z_Q^{(q)}(r) = \left[\sum_{M>0} M^q \Omega(M,r) \right] \left[\frac{L}{\epsilon} \right]^{D-1} , \tag{8}$$

and lacunarity is

$$\Lambda(r) = \frac{\sum_{M>0} M^2 \Omega(M,r)}{\left[\sum_{M>0} M \Omega(M,r) \right]^2} \left[\frac{L}{\epsilon} \right]^{1-D} . \tag{9}$$

We can see immediately that the lacunarity of very large Cantor sets tends towards infinity, which means that they become less and less translationally invariant as they grow. In practice, when comparing sets which have the same fractal dimension and the same linear size, we can disregard the dependence on the linear scale L/ϵ and simply use the so-called reduced moments $Z_Q'^{(q)} = Z_Q^{(q)}(L/\epsilon)^{1-D}$ and reduced lacunarity $\Lambda'(r) = \Lambda(r)(L/\epsilon)^{D-1}$.

In Table I, we have reported the values of the reduced lacunarity for $r=\epsilon$ of the six Cantor sets C_1-C_6. C_1, where the repartition of the voids is the most regular, has the lowest lacunarity, as expected. Sets C_2 and C_3, which are identical through mirror symmetry, have the same lacunarity, and so have C_5 and C_6, which only differ from one another in a translation. In addition, these two latter sets, where the voids are the most condensed, have the largest values, as expected. Thus, the classification established using the definition of lacunarity introduced in Sec. II is in perfect agreement with an intuitive inspection of the sets.

C. Comparison with other definitions of lacunarity

Let us compare these results with two other definitions of lacunarity which are available in the literature. The first one was introduced to measure the lacunarity of a collection of spins placed on the sites of Sierpinski carpets, in relation to their critical behavior [4]. In the case of generalized Cantor sets, this definition proceeds to the first iteration sets in the following manner: we calculate the mean-square deviation of the mass distribution function in a box which has a radius equal to the radius of the largest voids in the first iteration sets over all the configurations, $r=(\xi-N)\epsilon$, and which is translated on the first iteration sets in all the possible manners. The results for sets C_1-C_6 are compared with $\Lambda'(\epsilon)$ in Table I. C_1, which is the least lacunar set, still has the lowest lacunarity; similarly, one of the most lacunar sets, C_6, has the largest lacunarity. In addition, C_2 and C_3, which are identical, also have the same lacunarity. However, this characterization of lacunarity takes into account the voids situated at the ends of the sets, which gives different lacunarities to sets which are identical except for a translation, such as C_5 and C_6. This drawback arises from the fact that the lacunarities are determined arbitrarily on the first iteration set, contrary to the method defined in the preceding section.

TABLE I. This table compares the reduced lacunarity for $r=\epsilon$ of the six generalized Cantor sets represented in Fig. 2 with the lacunarities calculated from Refs. [4] and [5]. It must be noted that the reduced lacunarity $\Lambda'(\epsilon)$ is equal to the lacunarity parameter defined in Sec. IV A.

Generator	$\Lambda'(\epsilon)=\lambda$	Lacunarity[a]	Lacunarity[b]
$C_1(\epsilon,5\epsilon,9\epsilon)$	$\frac{1}{2}$	0	0.071
$C_2(\epsilon,5\epsilon,7\epsilon)$	$\frac{2}{3}$	$\frac{3}{16}$	0.137
$C_3(\epsilon,3\epsilon,7\epsilon)$	$\frac{2}{3}$	$\frac{3}{16}$	0.207
$C_4(\epsilon,3\epsilon,9\epsilon)$	$\frac{3}{4}$	$\frac{1}{2}$	0.268
$C_5(\epsilon,3\epsilon,5\epsilon)$	$\frac{5}{6}$	$\frac{11}{16}$	0.318
$C_6(3\epsilon,5\epsilon,7\epsilon)$	$\frac{5}{6}$	$\frac{1}{4}$	0.134

[a]Reference [4].
[b]Reference [5].

A second definition has been proposed as an attempt to generalize the notion of lacunarity at any scale [5,9]. It states that lacunarity is the average of the mean-square deviations of the mass distribution functions in boxes with radii varying between the radius of the largest voids, $r_{\min}=(\xi-N)\epsilon$, and the radius of the first iteration set, $r_{\max}=\xi\epsilon$. The results for sets C_1-C_6 are also shown in Table I. We find that the lacunarity of C_1 is now different from zero. The lacunarities of C_5 and C_6 are still different. In addition, a new inconsistency appears since C_2 and C_3, which are identical through mirror symmetry, now have unequal lacunarities.

In conclusion, the mean-square deviation of the mass fluctuations in the gliding boxes provides a robust characterization of the lacunarity of Cantor sets; it corrects all the drawbacks which arise from the preceding definitions.

IV. SCALING PROPERTIES OF MOMENTS AND LACUNARITY

In the preceding section, we demonstrated the efficiency of probability function Q derived from the gliding-box algorithm and of its moments in calculating the lacunarity of generalized Cantor sets. More generally, this definition can be applied to any set which is not necessarily fractal. Below, we focus specifically on uniform and multifractal self-similar sets. Relating the gliding-box method to other algorithms, such as the box-counting method and the sandbox method, we show that the dependence on scale r of $Q(M,r)$, $Z_Q^{(q)}$, and $\Lambda(r)$ follows particular scaling relations.

A. Relation between the gliding-box method and the box-counting method

Let us first recall briefly the box-counting algorithm, which has been widely used in the literature to compute the dimension of fractal sets. In this method, the set is covered by a grid with a mesh size equal to $2r$. The orientation of the grid is fixed and chosen at random. Each element of the grid is a box with radius r. From the scaling of the number of nonempty boxes with radius $r(\simeq(r/L)^{-D})$ we determine box dimension D. We should note that this collection of boxes, which all have the same radius, does not form an optimal cover of the set. For other applications, for instance, to determine the Hausdorff dimension, a more general cover with unequal boxes is required. Here, we revisit the box-counting algorithm in the following manner. From the set of values of the mass in the kth box, M_k, we get the distribution function $n_B(M,r)$, which gives the number of boxes with radius r and mass M in the grid. Dividing $n_B(M,r)$ by the total number of boxes, $(L/r)^E$, we define the probability function $B(M,r)$ that a box with radius r belonging to the grid contains mass M: $B(M,r)=n_B(M,r)(r/L)^E$. The moments of $B(M,r)$ are given by

$$Z_B^{(q)}(r)=\sum_M M^q B(M,r)=\left[\frac{r}{\epsilon}\right]^E \sum_M M^q n_B(M,r) .$$

They are also equal to

$$Z_B^{(q)}(r)=M_0^q \left[\frac{r}{\epsilon}\right]^E \sum_M \left[\frac{M_k}{M_0}\right]^q ,$$

where M_0 is the total mass of the set and (M_k/M_0) is the relative portion of the set in the kth box. In general, the partition function $\sum_M (M_k/M_0)^q$ scales as $(r/L)^{(q-1)D(q)}$, where $D(q)$ are the generalized dimensions of the set [10]. Finally, we find that the moments of $B(M,r)$ follow the scaling relation (z_B is the prefactor of the power law):

$$Z_B^{(q)}(r)=z_B \left[\frac{r}{L}\right]^{(q-1)D(q)+E} . \tag{10}$$

Now, we notice that the collection of all the gliding boxes of radius r is equivalent to the collection of boxes obtained when the origin of the fixed grid used in the box-counting algorithm is translated on the underlying lattice in every possible manner. The different locations of the origin of the grid are labeled by index m which runs from 1 to $(r/a)^E$. Probability function $Q(M,r)$ and moments $Z_Q^{(q)}$ are, respectively, the average of probability functions $B_m(M,r)$ and of moments $[Z_B^{(q)}(r)]_m$ over the different locations of the grid. Consequently, we find that moments $Z_Q^{(q)}(r)$ also scale as power laws with exponents $(q-1)D(q)+E$. In the case of uniform sets where all the generalized dimensions coincide with the fractal dimension [$D(q)=D$ for any q]:

$$Z_Q^{(q)}(M,r)=z_Q \left[\frac{r}{L}\right]^{(q-1)D+E} , \tag{11}$$

where prefactor z_Q is related to prefactors $(z_B)_m$. Finally, lacunarity $\Lambda(r)$ varies according to the power law

$$\Lambda(r)=\lambda \left[\frac{r}{L}\right]^{D-E} . \tag{12}$$

In the case of multifractal sets, power law (12) still holds with the proviso that D is now the correlation dimension $D(2)$. In conclusion, the lacunarity of self-similar sets is entirely characterized by prefactor λ, which will be called the lacunarity parameter from now on. It must be noted that the lacunarity parameter is equal to the reduced lacunarity calculated for $r=\epsilon$.

Finally, in view of the relation between the gliding-box method and the box-counting method, it seems interesting to characterize the lacunarity of a fractal set from the mean-square deviation of probability function $B_m(M,r)$ derived for a particular location of the grid. In practice, we applied the box-counting algorithm to study the generalized Cantor sets represented in Fig. 2. The calculations were done numerically on high iteration sets ($L=5^{20}$) and for boxes with radii varying between 2ϵ and very large values (up to $5^4\epsilon$). First, for a given location of the origin of the grid, we checked that the variations of moments $(Z_B^{(q)})_m$ agree with the scaling behavior expected from (10). Secondly, for any value of the grid mesh size $2r$, we found that the moment values fluctuate when the origin of the grid takes the different possible locations, even for very large sets and very large boxes. Thus, lacunarity cannot be reliably determined from the box-counting algorithm.

B. Relation between the gliding-box method and the generalized sandbox method

The sandbox algorithm is another basic way of determining the fractal dimension of a set. In the original sandbox method, D is found from the scaling of the mass $M(r)$ embedded within a region of radius r centered on a fixed point belonging to the fractal [$M(r)\simeq r^D$]. We have generalized the sandbox method in the following manner [11]. We consider the collection of the boxes of radius r which are centered on the different elements of the set. $P(M,r)$ is the probability that a given box belonging to this collection contains mass M. When the set is uniform and self-similar, we expect that $P(M,r)$ varies as $P(M,r)=f_P(M/r^D)$, where f_P is a scaling function. At first sight, the mean-square deviation of the probability function $P(M,r)$ divided by the square mean may be an alternative characterization of lacunarity. In the case of the generalized Cantor sets in Sec. III, the values calculated in this way lead to the same lacunarity for all sets, which obviously is not correct. Thus, from this counterexample we conclude that lacunarity cannot be determined reliably from the fluctuations of sandbox probability function $P(M,r)$.

Now, let us establish the relation between $P(M,r)$ and $Q(M,r)$. We note that $P(M,r)$ is the conditional probability that a gliding box of radius r contains mass M provided that it is centered on an element of the set. We write

$$P(M,r)=Q(M,r)C(M,r) , \tag{13}$$

where $C(M,r)$ is the probability that the center of a gliding box of mass M and radius r is situated on an element of the set. We assume that $C(M,r)$ is equal to the total number of elements in the box divided by its volume multiplied by a scaling function of variable M/r^D: $C(M,r)=(r/\epsilon)^{D-E}f_C(M/r^D)$. By reporting the expressions of $P(M,r)$ and $C(M,r)$ inside relation (13) and calculating the moments $Z_Q^{(q)}$, we recover scaling relations (11) and (12) derived in Sec. IV A.

C. Applications to deterministic and random sets

In Fig. 4, we have represented the first- and second-order reduced moments of sets C_1, C_2, C_5, and C_6 calculated from the analytical expressions established in Sec. III B, for $r\geq\epsilon$. We recall that the reduced moments do not take into account the variation in the linear size of the set, L, and depend only on scale r. We observe that $Z_Q^{\prime(1)}$ and $Z_Q^{\prime(2)}$ follow the scaling behavior which is expected from (11). The first-order moments are all equal; the different values are aligned on a straight line with a slope $E=1$. In the same manner, the points representing the values of $Z_Q^{\prime(2)}$ fall on parallel lines with a slope close

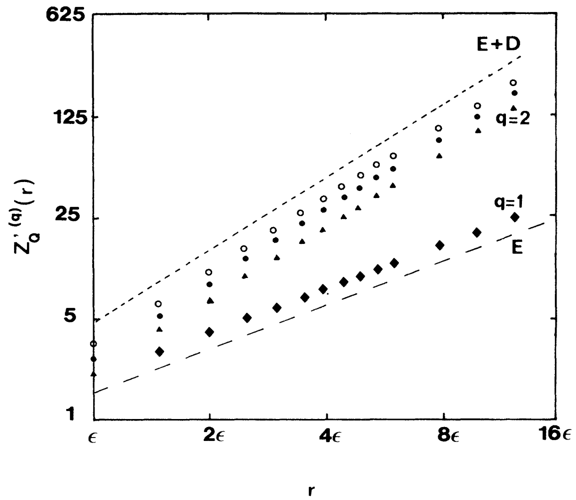

FIG. 4. Representation in log-log coordinates of the variations of the first- and second-order reduced moments $[Z'^{(q)}_Q(r)=Z^{(q)}_Q(r)(L/\epsilon)^{1-D}]$ of Cantor sets C_1, C_2, C_3, C_5, and C_6. The points corresponding to the first order reduced moments (◆) are all aligned on a straight line. The points representing the second-order moments (▲, C_1; ●, C_2 and C_3; ○, C_5 and C_6) are aligned on parallel lines.

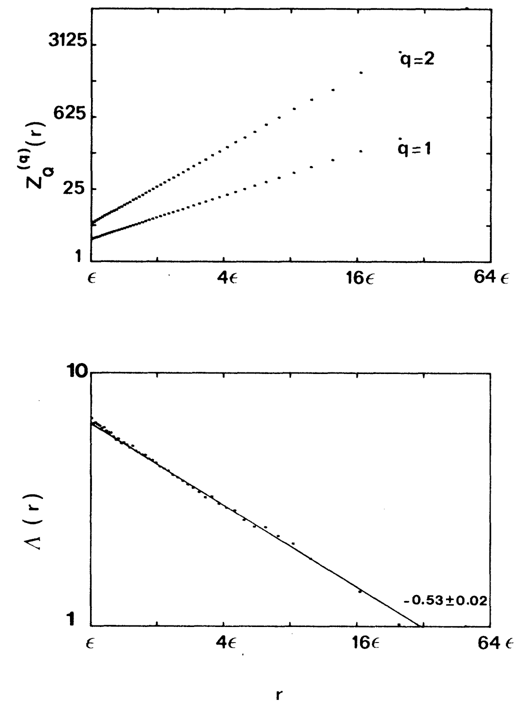

FIG. 5. Representation in log-log coordinates of the first- and second-order moments $Z^{(q)}_Q(r)$ and of the lacunarity of an aggregate grown in a simulation of diffusion-limited cluster-cluster aggregation (1024 particles, $E=2$).

to $E+D$. The line corresponding to C_5 and C_6 is above the line representing C_2, which is itself above the line obtained for C_1, thus reflecting the fact that C_5 and C_6 are the most lacunar, that C_1 is the least lacunar, and that C_2 is intermediate. The lacunarity of each set is entirely determined by the lacunarity parameter which is identical to the reduced lacunarity for $r=\epsilon$. The values are reported in Table I.

We have also applied our definition of lacunarity for characterizing the lacunarity of random sets. Figure 5 gives the variation of the first- and second-order moments and of the lacunarity of an aggregate which has been generated in a simulation of cluster-cluster aggregation with Brownian trajectories, in two dimensions. The variations of $Z^{(1)}_Q(r)$, $Z^{(2)}_Q(r)$, and $\Lambda(r)$ are in good agreement with the scaling behavior expected from (11) and (12). On the one hand, these results establish that the aggregate is self-similar with $D=1.47\pm0.02$. This value of the fractal dimension is in perfect agreement with other geometrical analyses of the same aggregate. On the other hand, the lacunarity parameter is determined from the intercept with the vertical axis of the straight line which best fits the experimental points $\lambda=6.5\pm0.1$.

V. CONCLUSION

The scope of the results reported in this paper is twofold. First, lacunarity appears to be a new tool for characterizing the geometry of deterministic and random sets. It quantifies the elusive notion of texture. The definitions of probability function $Q(M,r)$ and of lacunarity $\Lambda(r)$ which have been developed are general in the sense that they apply at any scale r to any set which is not necessarily fractal. Second, checking that the qth moments of $Q(M,r)$ scale as power laws with exponents $(q-1)D+E$ provides an explicit demonstration of self-similarity and a new way of determining fractal dimension D. When self-similarity holds, lacunarity also exhibits its noteworthy scaling properties which have been established in relation to other counting procedures. We have tested our results on deterministic sets and simulated clusters. In practice, we have applied the gliding-box algorithm extensively in order to study the morphology of experimentally grown aggregates and to measure their lacunarity; the results are planned to be published elsewhere.

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