Amplification by globally coupled arrays: Coherence and symmetry

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We study the general problem of small-signal amplification by globally coupled oscillator arrays. The analysis is relevant, for example, to series-array Josephson-junction parametric amplifiers and to certain multimode laser systems. Maximum amplification is achieved near symmetry-preserving bifurcations; the response near symmetry-breaking bifurcations is substantially worse. In the presence of noise, we find a seemingly paradoxical phenomenon wherein individual array elements suffer large-scale random fluctuations, yet the total array response remains steady.

I. INTRODUCTION

There is a close connection between the bifurcations of a nonlinear system and its ability to act as a parametric amplifier [1–7]. This correspondence has been exploited to achieve a deeper, unified understanding of parametric amplifiers in a variety of physical domains, from superconducting Josephson-junction parametric amplifiers (SUPARAMPS) [7,8], to nuclear magnetic resonance masers [9], nuclear parametric spin waves [10], semiconductor lasers [11–13], and driven magnetostrictive ribbons [14]. The reason a unified description is possible is that, close to a bifurcation point, the relevant phase-space dynamics reduces to a low-dimensional "normal form" which describes the dominant response to small perturbations. Analysis of the normal form leads to scaling behavior which is independent of the detailed nature of the physical system.

The connection between bifurcations and the scaling behavior of amplified (random or periodic) perturbations has been developed for both one-dimensional iterative maps and ordinary differential equations [1–7]. Several scaling predictions may be derived from a linearized analysis, e.g., relating the output response as a function of perturbation frequency and bifurcation parameter. Beyond this, examination of the fully nonlinear normal form predicts additional phenomena [6], including the stabilizing shift of the (supercritical) bifurcation point, the onset of a multiplet of closely spaced lines in the power spectrum, and the so-called noise rise, a phenomenon which had plagued experiments on Josephson-junction parametric amplifiers [15]. Experimental observations of these phenomena have been reported for a variety of systems: In addition to all of the areas listed above [8–12,14,16,17], one can add experiments on a bouncing ball [18], charge-density waves [19,20], and *p*-*n* junction circuits [21,22]. In virtually all cases reported, the observed behavior follows closely the theoretical predictions. (A single experiment reporting deviation from these predictions is described in Ref. [17].)

The purpose of this paper is to consider a particular extension of the subject, namely the amplification properties of certain nonlinear *arrays* near their bifurcation points. Our immediate motivation stems from practical considerations that apply to Josephson-junction parametric amplifiers. In particular, a single Josephson junction is a low-power, low-impedance device; consequently, it is often desirable to use a large array of junctions in a single device. It is natural to expect there will still be a close link between bifurcations in the array dynamics and the regime where the array is most sensitive to perturbations; however, the theoretical considerations are bound to be more complicated than for the single-element problem. The main mathematical difficulty is that array systems typically involve many active degrees of freedom, so that the "essential" phase-space structure is not necessarily low dimensional. However, it may be possible to overcome this difficulty under certain circumstances. In this paper, we focus on a particular class of array system, namely globally coupled arrays [23–25]. We find that, for this class, the additional complexity of many degrees of freedom is rendered tractable by an appropriate coordinate transformation. As a result, we are able to draw a fairly complete picture of the small-signal amplification properties of such arrays.

In the final analysis, we find that the amplification properties of arrays is fairly straightforward. The most important distinction to draw is between symmetry-breaking and symmetry-preserving bifurcations. (All possible instabilities can be classified as one of these two types.) Only in the latter case does the output power scale as N^2, which is the best-case scenario for oscillator arrays, corresponding to perfect constructive coherence between the individual oscillating array elements. This has important implications for the design of parametric amplifier arrays, as discussed in Sec. VI.

The contents of this paper are as follows. In Sec. II, we introduce the dynamical equations for the class of systems to be studied, namely arrays of identical elements subject to global coupling. Though our starting point is very general, we subsequently restrict our attention to some subset of situations: as a guide, we choose those situations which seem most important to the Josephson-junction application. (Even with this reduced focus, the analysis remains fairly general.) In Sec. III, we carry out as far as possible the general aspects of the calculation, including the linearization and diagonalization of the dynamical equations. Sections IV and V pursue various special cases: the response to homogeneous, periodic perturbations is developed in Sec. IV, while Sec. V deals with inhomogeneous, random perturbations. In both sections, both symmetry-preserving and symmetry-breaking bifurcations are considered; we find this particular distinction is crucial, and leads to very different results. Throughout this paper, the emphasis is on the scaling behavior of the output power as a function of the array size N and the bifurcation parameter. The latter measures how close the system is tuned to the bifurcation point. Many of the results are "unspectacular" in the sense that they are easily understood on physical grounds. However, the analysis also uncovers a couple of surprises: these are also discussed in Sec. VI, as are the prospects for observing them in future experiments.

II. FORMULATION OF THE PROBLEM

The starting point for our problem is the set ordinary differential equations, given by

$$\dot{x}_k = F_k(x_k, Z; \mu) \ , \quad k = 1, \ldots, N \tag{2.1}$$

where x_k is a real vector quantity representing the state of the kth oscillator, the overdot denotes differentiation with respect to time, F_k is a vector function of x_k, and Z is given by

$$Z = \frac{1}{N} \sum_{j=1}^{N} x_k \ . \tag{2.2}$$

The notation here is intended to emphasize that the function F_k has explicit dependence only on the local variable x_k and on the specific combination of variables given by Z. Of course, the F_k also depends on one or more control parameter(s) μ. Moreover, we assume that the functional form of F_k is independent of k. Thus Eq. (2.1) describes a set of identical degrees of freedom, whose evolution depends on both "local dynamics" (i.e., terms involving x_k, alone), and "coupling dynamics" (i.e., terms that describe the interaction between degrees of freedom, involving Z). The interaction term chosen is of the type known as "global coupling" [23–25]; physicists will also recognize the form of Eq. (2.1) as describing "mean-field" coupling. We emphasize that, while the mean-field form is often used as a computationally expedient approximation, equations of the form (2.1) may also arise as the physically correct description. This fact often comes as a surprise to people when they first here of it; nevertheless, this type of global interaction correctly describes certain linear series-array Josephson-junction circuits [26,27], as well as a type of multimode laser system first described by Baer [28–30].

As an example, consider the series array of Josephson junctions depicted in Fig. 1. Such configurations have been studied in some detail, for a variety of different loads: the physical significance of the load is that it acts to dynamically couple the junctions. For the case shown, the load is a single resistor, and the governing equations of motion are [26]

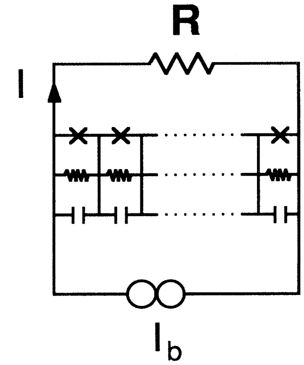

FIG. 1. Circuit schematic of a current biased, resistively shunted Josephson-junction series-array parametric amplifier.

$$\frac{\hbar C}{2e}\ddot{\phi}_k + \frac{\hbar}{2er}\dot{\phi}_k + I_0 \sin\phi_k + \frac{\hbar}{2eR}\sum \dot{\phi} = I_b \ , \qquad k = 1, \ldots, N \tag{2.3}$$

where ϕ_k is the phase difference of the macroscopic wave function across the kth junction, I_0 is the critical current, I_b is the bias current, R is the load resistance, r is the junction resistance, C is the junction capacitance, $\hbar$ is Planck's constant divided by 2π, and e is the electron charge. Equation (2.3) follows directly from Kirchkoff's lump circuit laws, with no further approximations [26]. Equation (2.3) is clearly of the form of Eq. (2.1), with $x_k = (\phi_k, \dot{\phi}_k)$.

As a second example, we mention the problem of a multimode laser with intracavity doubling crystal; the physical details of the process are described elsewhere [28,29]. The basic idea is that in a long cavity, many different longitudinal modes satisfy the condition for positive gain, and so may be active simultaneously. The presence of an intracavity nonlinear crystal couples the modes via a second-harmonic-generation process. The dynamics can be described by a set of ordinary differential equations for the intensity I_k and population inversion G_k corresponding to the kth active mode. In the case where all of the modes have the same linear polarization, one has [29]

$$\begin{aligned} \tau_c \frac{dI_k}{dt} &= \left[G_k - \alpha - g\epsilon I_k - 2\epsilon g \sum_{(j \neq k)} I_j \right] I_k \ , \\ \tau_f \frac{dG_k}{dt} &= \gamma - \left[1 + \beta \sum_{j(\neq k)} I_j \right] G_k \ , \end{aligned} \tag{2.4}$$

where τ_c and τ_f are the cavity round-trip time and fluorescence time, respectively; α is the cavity loss parameter; γ is the gain parameter, β is the saturation parameter, and ϵ is a parameter that depends on the nature of the second-harmonic-generating crystal. As written, these assume that the gain γ, loss α, and saturation parameter β are the same for all modes, which is analogous to the assumption of identical Josephson junctions in Eq. (2.3). One sees again the form of Eq. (2.1), now with $x_k=(I_k,G_k)$.

Mathematically, the significance of the two properties inherent in Eq. (2.1)—identical degrees of freedom and global coupling—is the existence of an invariance with respect to any permutation of the N indices. This is crucial for determining the types of bifurcation that occur generically [31]. It also allows us to transform the problem tackled in this paper into a tractable form, as we describe in Sec. III.

The basic idea we pursue is this. We assume that Eq. (2.1) has a stable periodic solution and that as the control parameter μ is increased past its critical value μ^*, the system undergoes a bifurcation, so that this solution loses stability. We want to see how small perturbations affect the system for μ just less than μ^*; in general, we expect that the response will be substantial and that the response will grow larger the closer μ gets to μ^*. In what follows, we will call the perturbation the "signal," the presence of which is included by modifying Eq. (2.1).

At this point, there are many ways to proceed. To begin with, the unperturbed dynamical system could be autonomous or explicitly time dependent. Then the signal might enter additively to the right-hand side of Eq. (2.1), or multiplicatively. The signal might be homogeneous—that is to say, independent of k—or it might be inhomogeneous. The signal could be periodic or random. The unperturbed stable periodic orbit may share the full symmetry of the underlying dynamics, it may have a somewhat lower symmetry, or no symmetry at all. The bifurcation in question might be symmetry breaking or symmetry preserving, and in addition, any one of several types: a period-doubling bifurcation, a Hopf bifurcation, etc. Finally, we have to decide what quantity or quantities we wish to monitor as output.

In view of the great number of possibilities—and this even after having specified the structure inherent in Eq. (2.1)—we now narrow our focus. In doing so, we are guided by those aspects which we feel are most significant for practical applications of parametric amplifier arrays. (The questions we focus on may also be relevant to other situations as well, but may or may not be the most important ones in these other contexts.) Thus we assume that (i) the underlying stable state is fully symmetric, corresponding to the in-phase dynamical state of oscillator arrays, so that $x_1(t)=x_2(t)=\cdots=x_N(t)$; and (ii) this orbit is the result of a time-periodic forcing, so that the unperturbed system is nonautonomous.

In what follows, we consider only two specific types of signal: homogeneous signals which are periodic (which corresponds to a time-dependent current source $I_B(t)$ in Fig. 1), and random signals which are statistically uncorrelated for different elements (corresponding physically to Johnson noise generated by each junction resistor in Fig. 1). Finally, there are two quantities we will monitor as the system response: the output of a single degree of freedom x_k and that of the "bulk" or "average" quantity Z (corresponding in Fig. 1 to the voltage across a single Josephson junction and the average voltage across the entire array, respectively). The analysis we present can be extended to a variety of situations other than the ones we focus on here, and in some cases the technical details may change; however, we will not consider them further.

III. GENERAL ANALYSIS AND DIAGONALIZATION

Our plan is to augment Eq. (2.1) by a small perturbation, and study the system response when its control parameter is tuned near a bifurcation point. We thus change Eq. (2.1) to

$$\dot{x}_k=F_k(x_k,Z)+\xi_k(t)\ , \quad k=1,\ldots,N \tag{3.1}$$

where the variables x_k are n dimensional, and we have dropped explicit reference to the control parameter μ. The small-signal terms ξ_k have been written as additive perturbations. The corresponding problem with multiplicative signal may be treated in an analogous manner: as has been shown in the single-oscillator problem, one expects that the most important results are essentially unchanged in the perturbative limit considered here [4].

The goal of this section is to carry out a general analysis of Eq. (3.1) as far as is practical, before moving on to special cases in Secs. IV and V. The main result of this section is that the linearized dynamics can be effectively "diagonalized" by an appropriate linear transformation, due to the symmetry of the problem. In addition to simplifying the ensuing calculations, the transformed problem serves to clarify a fundamental distinction between situations involving symmetry-preserving and symmetry-breaking bifurcations.

We assume that the signal-free system ($\xi_k=0$ has an in-phase stable periodic solution, so that

$$x_k(t)=x_0(t)=x_0(t+T)\ , \quad k=1,\ldots,N\ . \tag{3.2}$$

The presence of a small signal causes the output to deviate from this orbit. We thus let

$$x_k(t)=x_0(t)+\eta_k(t)\ , \tag{3.3}$$

substitute this into Eq. (3.1), and linearize the differential equations about the periodic solution. This yields evolution equations for the deviations η_k,

$$\dot{\eta}_k=\sum_j(\partial_jF_k)\eta_j+\xi_k\ , \tag{3.4}$$

where ∂_jF_k is shorthand for $\partial F_k/\partial x_j$ evaluated on the unperturbed orbit $x_k=x_0(t)$. Now, since F_k depends only on x_k and Z, we have for $j\neq k$

$$\partial_jF_k=\frac{\partial F_k}{\partial Z}\frac{\partial Z}{\partial x_j}=\frac{1}{N}\partial_ZF_k\ ,$$

which is independent of j. Thus Eq. (3.4) becomes

$$\dot{\eta}_k=(\partial_kF_k)\eta_k+\frac{1}{N}(\partial_ZF_k)\sum_{j(\neq k)}\eta_j+\xi_k$$

or, equivalently,

$$\dot{\eta}_k = \left[(\partial_k F_k) - \frac{1}{N}(\partial_Z F_k) \right] \eta_k + \frac{1}{N}(\partial_Z F_k) \sum_j \eta_j + \xi_k , \tag{3.5}$$

where the summation is no longer restricted. Now, since all of the partial derivatives are evaluated on the in-phase orbit, they are the same for each index k. This allows us to simplify the problem enormously via the transformation

$$\zeta_k = \eta_k - \eta_{k+1} , \quad k = 1, \ldots , N-1 \tag{3.6a}$$

$$H = \frac{1}{N} \sum_{j=1}^{N} \eta_j . \tag{3.6b}$$

Applying this linear transformation to Eqs. (3.5) yields the set of N *uncoupled* equations

$$\dot{H} = \left[(\partial_k F_k) + \frac{N-1}{N}(\partial_Z F_k) \right] H + \frac{1}{N} \sum_j \xi_j , \tag{3.7a}$$

$$\zeta_k = \left[(\partial_k F_k) - \frac{1}{N}(\partial_Z F_k) \right] \zeta_k + \xi_k - \xi_{k+1} , \quad k = 1, \ldots , N-1 . \tag{3.7b}$$

Each of these N equations represents an n-dimensional system of linear inhomogeneous equations with periodic coefficients. Thus each may be solved (at least formally) via the methods of Floquet theory [32,33]. In effect, the original array problem has been broken into pieces, each of which can be analyzed along the lines of Refs. [2–4]. On the other hand, this is primarily a computational victory, since we still need to transform back to the original variables of our problem, in order to assess the results. In particular, we want to monitor the output of a single array element, say x_N, and the bulk response of the array, given by $Z = (1/N) \sum_{j=1}^{N} x_k$. From Eqs. (3.6a) and (3.6b), we can deduce the required inverse transformation,

$$x_N = x_0 + H - \frac{1}{N} \sum_{k=1}^{N-1} (k \zeta_k) \tag{3.8a}$$

and

$$Z = x_0 + H . \tag{3.8b}$$

Already we can understand something important about our problem by considering the structure of Eqs. (3.7). Each equation has n Floquet exponents, which determine both the stability properties of the in-phase solution and also the scaling properties of the response to the input perturbations [1–4]. As noted earlier, the periodic coefficients in Eq. (3.7b) are independent of k, so that each of these $N-1$ equations has precisely the same set of n Floquet exponents $\{\lambda_1, \ldots , \lambda_n\}$. Meanwhile, Eq. (3.7a) has a different set of exponents $\{\Lambda_1, \ldots , \Lambda_n\}$. Stability of the underlying in-phase orbit requires that none of these exponents have positive real parts. As the control parameter is varied, the exponents move around in the complex plane: a bifurcation is signaled when one or more exponents cross into the right half-plane.

Consequently, it is natural to distinguish between two possibilities, corresponding to whether the critical exponent(s) is one of the λ_j or one of the Λ_j, in which case we will call the bifurcation symmetry breaking or symmetry preserving, respectively. As we shall see, the response of the array is fundamentally different in these two circumstances.

[The case of some λ_j being critical may be identified as symmetry breaking because they are associated with the growth of the *relative* coordinates, as seen in Eq. (3.7a). We note that, if there is an additional symmetry in the problem beyond the permutation symmetry considered here, then the crossing of some Λ_j into the right half-plane may also correspond to a symmetry-breaking bifurcation, though not of the permutation symmetry itself. With this understanding, we will continue to use the term symmetry preserving in this instance.]

The subject of local bifurcation theory [34,35] distinguishes between different classes of bifurcations of periodic orbits, corresponding to the way in which some subset of the Floquet exponents cross into the right half-plane. The differences between these classes has important practical ramifications, for example, in determining which perturbation frequencies are most amplified [7]. Nevertheless, certain general scaling properties are shared in common for the various codimension-one bifurcations; in fact, the calculational steps for analyzing the different cases—saddle-node, transcritical, period-doubling, pitchfork, and Hopf bifurcations—are very similar. For this reason, in what follows we focus on the saddle-node bifurcation: here, a single exponent crosses into the right half-plane along the real axis. Just prior to the bifurcation point, then, we have one exponent—either λ_j or Λ_j—equal $-\epsilon$, where ϵ is a small positive quantity. All of the other exponents have real parts of order unity.

We now move on to a quantitative analysis for four special cases. The following section is devoted to periodic input signals; Sec. V covers the problem of random perturbations.

IV. HOMOGENEOUS PERIODIC SIGNAL

In this section, we assume that the input perturbation is periodic and is the same for each array element,

$$\xi_k = a \cos\omega t . \tag{4.1}$$

This is directly relevant to the study of parametric amplification, e.g., where the differential equations are the circuit laws for a Josephson-junction series array. In Fig. 1, this corresponds to the presence of an additional (small) current source in parallel to the driving source.

Combining Eqs. (3.7) and (4.1) yields

$$\dot{H} = \left[(\partial_k F_k) + \frac{N-1}{N}(\partial_Z F_k) \right] H + a \cos\omega t , \tag{4.2a}$$

$$\dot{\zeta}_k = \left[(\partial_k F_k) - \frac{1}{N}(\partial_Z F_k) \right] \zeta_k , \quad k = 1, \ldots , N-1 . \tag{4.2b}$$

That is, the equation governing the variables ζ_k is unforced. Recalling that the underlying orbit is stable, *it follows that the ζ_k necessarily decay to zero.* Consequently, the entire response is governed by the behavior of Eq. (4.2a). We consider two subcases, depending on whether the bifurcation preserves or breaks the symmetry.

A. Case 1: Symmetry-preserving bifurcation

Here there is a single real critical exponent $\Lambda_0=-\epsilon$. Thus, as described in the Appendix, the general integral expression for $H(t)$ is dominated by a single term

$$H(t)=P(t)\int_0^t Q(s)e^{-\epsilon(t-s)}a\cos(\omega s)ds \ , \tag{4.3}$$

where $P(t)$ and $Q(s)$ are periodic functions with frequency equal to the in-phase oscillation frequency $\omega_0=2\pi/T$. Introducing the Fourier series

$$P(t)=\sum_j \alpha_j e^{ij\omega_0 t} \ , \quad \alpha_{-j}=\alpha_j^*$$

$$Q(t)=\sum_j \beta_j e^{ij\omega_0 t} \ , \quad \beta_{-j}=\beta_j^*$$

$$\xi(t)=\frac{a}{2}(e^{i\omega t}+e^{-i\omega t}) \ ,$$

where the restriction on the coefficients ensures that the functions P and Q are real, the integral in Eq. (4.3) becomes

$$\sum_j \frac{a}{2}\int_0^t \beta_j e^{-\epsilon(t-s)}e^{ij\omega_0 s}(e^{i\omega s}+e^{-i\omega s})ds \ .$$

Upon performing the integral, this becomes

$$\sum_j \frac{a}{2}\frac{B_i e^{i(j\omega_0+\omega)t}}{\epsilon+i(j\omega_0+\omega)}+(\omega\to-\omega) \ , \tag{4.4}$$

where we have ignored terms that are exponentially small for times $t\gg 1/\epsilon$. One sees that the response will be very large provided that the signal frequency ω is nearly equal to any integer multiple of the pump frequency ω_0. Thus it is natural to introduce the detuning δ, such that

$$\omega=m\omega_0+\delta \ , \tag{4.5}$$

where m is an integer and δ is a small quantity. We see that each sum appearing in (4.3) is dominated by a single term: the term $j=-m$ in the first expression, and the term $j=+m$ in the second. Thus, to good approximation, Eq. (4.3) reduces to

$$H(t)=\sum_j \alpha_j e^{ij\omega_0 t}\frac{a}{2}\left[\beta_{-m}\frac{e^{i\delta t}}{\epsilon+i\delta}+\text{c.c.}\right] \ , \tag{4.6}$$

where "c.c." denotes the complex conjugate of the first expression.

Together with the result that $\zeta_k=0$, we can transform back to the original coordinates of the problem. For a single oscillator [see Eq. (3.8a)]

$$x_N(t)=x_0(t)+H(t)$$

and for the total response of the array [see Eq. (3.8b)]

$$\sum_k x_k=NZ=Nx_0(t)+NH(t) \ .$$

We have, in this case, a very simple result: the total response is just N times the response of a single oscillator.

We see that an input signal at the single frequency ω gives rise to a large response at many frequencies, namely $j\omega_0\pm\delta$ for any integer j. In a power spectrum, this shows up as pairs of sharp lines around each of the signal-free spectral lines. Apart from the coefficients α_j which set the overall amplitude of each pair of lines, the response at each frequency has the same *scaling behavior* as a function of ϵ and δ. The total response likewise has this same scaling behavior.

More specifically, we can take as a measure of the response the mean-square amplitude. For a single oscillator, we have

$$\langle x_N^2\rangle=\langle x_0^2\rangle+2\langle x_0 H\rangle+\langle H^2\rangle \ . \tag{4.7}$$

Since x_0 and H have no frequency components in common, the cross term is zero. Meanwhile, from Eq. (4.6), it is a straightforward matter to compute the time average of H^2, with the result

$$\langle x_N^2\rangle=\langle x_0^2\rangle+\frac{a^2|\beta_m|^2}{2(\epsilon^2+\delta^2)}\sum_j|\alpha_j|^2 \ . \tag{4.8}$$

Of course, for the total response of the array, we have simply

$$\left\langle\left[\sum_k x_k\right]^2\right\rangle=N^2\langle x_0^2\rangle+\frac{N^2a^2|\beta_m|^2}{2(\epsilon^2+\delta^2)}\sum_j|\alpha_j|^2 \ . \tag{4.9}$$

The results (4.8) and (4.9) have a simple interpretation: each element in the array amplifies the input signal in the same way; moreover, the responses add coherently, so that the total output scales as N^2. This is just what one expects if the array acts "like one big element." As we shall now see, this is *not* the situation near a symmetry-breaking bifurcation.

B. Case 2: Symmetry-breaking bifurcation

In this case, it is the degenerate exponent λ_0 that goes to zero at the bifurcation point, while Λ_0 is of order unity. We have the following result: *no significant amplification takes place as the bifurcation point is approached.*

This observation follows from two facts. First, since the perturbation is homogeneous, the variables ζ_k necessarily decay to zero [see Eq. (4.2b)]. Second, the quantity $H(t)$ remains small because the associated Floquet exponent Λ_0 remains bounded away from zero. In particular, $H(t)$ cannot be reduced from its general form [see Appendix, Eq. (A5)]

$$H(t)=\mathbf{\Phi}_H(t)\int_0^t\mathbf{\Phi}_H^{-1}(s)\xi(s)ds \ , \tag{4.10}$$

where $\mathbf{\Phi}_H$ is the fundamental matrix associated with Eq. (4.2a). Moreover, this quantity shows no divergent behavior as the bifurcation point is approached. It is a simple matter to show that $H(t)$ has frequency components in precisely the same place as in case 1 (Sec. IV A), but its amplitude remains small regardless of the value of λ_0.

We conclude that to achieve significant amplification of a homogeneous signal, it is not sufficient to tune the system close to an arbitrary bifurcation point; rather, it is essential that the bifurcation be of the symmetry-preserving type.

V. INHOMOGENEOUS RANDOM SIGNAL

We now consider the situation where each element is subject to a different input perturbation. In the Josephson-junction problem, this arises naturally when considering the effects of thermal noise currents generated by the resistance of each Josephson junction. Consequently, we take the inputs appearing in Eqs. (3.7) to be independent δ-correlated random functions with intensity κ:

$$\langle \xi_k(t) \rangle = 0 \ , \tag{5.1}$$

$$\langle \xi_j(t)\xi_k(s) \rangle = \kappa\delta_{jk}\delta(t-s) \ , \tag{5.2}$$

where δ_{jk} is the Kronecker delta, $\delta(t-s)$ is the Dirac delta function, and the angular brackets denote an ensemble average. Our goal is to calculate the mean-square fluctuation $\langle x_N^2 \rangle$ for a single element and also $\langle (\sum_k x_k)^2 \rangle$ for the entire array. As in Sec. IV, the results depend crucially on whether the bifurcation preserves or breaks the symmetry.

A. Case 1: Symmetry-preserving bifurcation

The exponent $\Lambda_0 = -\epsilon$ is small, while all the other $nN-1$ exponents are of order unity. In contrast to Sec. IV there are now contributions from both ζ_k and H. We begin with $H(t)$, whose general form is given by Eq. (A6) in the Appendix. Near the bifurcation point this expression is dominated by a single term: combining Eqs. (3.7a) and (A7) yields

$$H(t) = P(t) \int_0^t Q(s) e^{-\epsilon(t-s)} \frac{1}{N} \sum_k \xi_k(s) ds \ . \tag{5.3}$$

The mean-square fluctuation is

$$\begin{aligned}\langle H^2(t) \rangle &= \frac{1}{N^2} \sum_{j,k} P^2(t) e^{-2\epsilon t} \\ &\quad \times \int_0^t \int_0^t e^{\epsilon t'} e^{\epsilon t''} Q(t') Q(t'') \\ &\quad \times \langle \xi_j(t')\xi_k(t'') \rangle dt' dt'' \\ &= \frac{\kappa}{N} P^2(t) \int_0^t e^{-2\epsilon(t-t')} Q^2(t') dt' \ .\end{aligned}$$

For small ϵ, the exponential decays slowly compared with the period of $Q(t)$, so this expression reduces to

$$\langle H^2(t) \rangle = \frac{\kappa}{2N\epsilon} P^2(t) \{ Q^2 \} \ ,$$

where the curly brackets denote a time average over one period. The mean-square fluctuation is itself time periodic, owing to the periodic nature of the underlying unperturbed orbit x_0. Averaging over this orbit yields a time-independent quantity

$$\{ \langle H^2 \rangle \} = \frac{\kappa}{2N\epsilon} \{ P^2 \} \{ Q^2 \} \ . \tag{5.4}$$

We turn next to the quantity ζ_k, governed by Eq. (3.7b). We have the general result [see Eq. (A6)]

$$\zeta_k(t) = \Phi_k(t) \int_0^t \Phi_k^{-1}(s) [\xi_k(s) - \xi_{k+1}(s)] ds \ . \tag{5.5}$$

Now, although this quantity is not zero (as in the homogeneous case), it undergoes no significant increase as the bifurcation point is approached, since its size is independent of the critical exponent Λ_0. Consequently, ζ_k remains negligibly small. In terms of the original coordinates, we thus have

$$x_N = x_0 + H \ , \tag{5.6}$$

so that

$$\{ \langle x_N^2 \rangle \} = \{ x_0^2 \} + \frac{\kappa}{2N\epsilon} \{ P^2 \} \{ Q^2 \} \ . \tag{5.7}$$

[There is no cross term $\langle x_0 H \rangle$ since the fluctuations in H have zero mean, as can be seen directly from Eqs. (5.1) and (5.3).]

Although one cannot see it directly from this calculation of the mean-square fluctuation, the two contributions in Eq. (5.7) are easily separable in experiments by looking at the power spectrum, since the piece $\{x_0^2\}$ corresponds to the sharp lines induced by the unperturbed orbit, while the second piece shows up as a broadband contribution, and is purely noise induced (going to zero as κ goes to zero). We see that the latter has an interesting scaling structure, with a magnitude depending inversely on the product ϵN. Consequently, the large "noisy precursors" encountered in single-oscillator systems [2] are substantially reduced *per oscillator,* when the oscillators are "embedded" in large arrays.

Meanwhile, the fluctuation observed in the bulk variable $\sum_k x_k$ are just N^2 times the result for a single oscillator, as can be seen by comparing Eqs. (3.8b) and (5.6):

$$\begin{aligned}\left\{ \left\langle \left[\sum_k x_k \right]^2 \right\rangle \right\} &= N^2 \{ \langle Z^2 \rangle \} \\ &= N^2 \{ x_0^2 \} + \frac{N\kappa}{2\epsilon} \{ P^2 \} \{ Q^2 \} \ ,\end{aligned} \tag{5.8}$$

so that the noise-induced contribution grows linearly with array size N. This makes sense on physical grounds: since the input noise sources are uncorrelated, the total effect is given by an incoherent sum of the individual sources

B. Case 2: Symmetry-breaking bifurcation

We now take the small quantity to be $\lambda_0 = -\epsilon$, with Λ_0 of order 1. In this case, the quantity H remains small, and we neglect it compared to the large contribution due to ζ_k, which we now calculate.

The dominant contribution to Eq. (5.5) is [see Eq. (A7)]

$$\zeta_k(t)=p(t)\int_0^t q(s)e^{-\epsilon(t-s)}[\xi_k(s)-\xi_{k+1}(s)]ds \ . \tag{5.9}$$

In view of Eq. (5.1) we see immediately that

$$\langle \zeta_k \rangle = 0 \ . \tag{5.10}$$

The second moments are given by

$$\langle \zeta_j(t)\zeta_k(t)\rangle = p^2(t)e^{-2\epsilon t}\int_0^t\int_0^t e^{\epsilon(t'+t'')}q(t')q(t'') \times \Xi_{jk}(t',t'')dt'dt'' \ , \tag{5.11}$$

where

$$\Xi_{jk}(t',t'')=\langle [\xi_j(t')-\xi_{j+1}(t')][\xi_k(t'')-\xi_{k+1}(t'')]\rangle \ .$$

Now, from Eq. (5.2), which states that the ξ_k are uncorrelated sources, we see that $\Xi_{jk}(t',t'')=0$ unless $|j-k|\le 1$. In fact, since the sources all have equal strength κ, one has $\Xi_{jk}(t',t'')=2\kappa\delta(t'-t'')$ if $j=k$, while $\Xi_{jk}(t',t'')=-\kappa\delta(t'-t'')$ if $j=k\pm1$. Consequently, we introduce the quantity Ω_{jk} given by

$$\Omega_{jk}=\begin{cases} +2 & \text{for } j=k \\ -1 & \text{for } j=k\pm 1 \\ 0 & \text{otherwise} \ . \end{cases} \tag{5.12}$$

Then, Eq. (5.11) becomes

$$\langle \zeta_j(t)\zeta_k(t)\rangle = \kappa\Omega_{jk}p^2(t)\int_0^t e^{-2\epsilon(t-t')}q^2(t')dt' \ .$$

For sufficiently small ϵ, this becomes simply

$$\{\langle \zeta_j(t)\zeta_k(t)\rangle\} = \Omega_{jk}\frac{\kappa}{2\epsilon}\{p^2\}\{q^2\} \ , \tag{5.13}$$

where we have taken the time average, as before.

We can now find the mean-square fluctuation $\langle x_N^2\rangle$ for a single element. From Eq. (3.8a), and treating H as negligible, we have

$$\langle x_N^2\rangle = \left\langle 2\left[x_0-\frac{1}{N}\sum_{k=1}^{N-1}(k\zeta_k)\right]^2\right\rangle$$

$$= x_0^2-\frac{1}{N^2}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} jk\langle \zeta_k\zeta_k\rangle \ ,$$

where in writing the preceding expression we have used Eq. (5.10). Using Eq. (5.13), this becomes

$$\{\langle x_N^2\rangle\} = \{x_0^2\}-\frac{1}{N^2}\frac{\kappa}{2\epsilon}\{p^2\}\{q^2\}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\Omega_{jk}jk \ .$$

The double sum is easily shown to be equal to $N(N-1)$, so that

$$\{\langle x_N^2\rangle\} = \{x_0^2\}-\frac{N-1}{N}\frac{\kappa}{2\epsilon}\{p^2\}\{q^2\} \ . \tag{5.14}$$

Thus, for any true array (that is, for $N>1$), there are large fluctuations which become prominent as the bifurcation point is approached ($\epsilon\to 0$). These appear as broadband contributions to the power spectrum: indeed, they are the familiar noisy precursors observed in single-oscillator systems [2,7]. Moreover, the dependence of Eq. (5.11) on the array size N is very weak, being significant only for small N.

On the other hand, the bulk fluctuations are given by

$$\left\langle \left[\sum_k x_0\right]^2\right\rangle = N^2x_0^2+N^2\langle H^2\rangle \ .$$

Now, although the broadband piece grows as N^2, the quantity H remains negligibly small. Consequently, no significant broadband contribution is observed as the bifurcation is approached.

Recapping, we have this surprising result: although each of the elements suffers increasingly large-scale fluctuations as the bifurcation point is approached, these fluctuations "destructively interfere" completely, so that the total array displays no large-scale fluctuations.

VI. DISCUSSION

Our study of the amplification properties of globally coupled arrays leads us to a fairly complete understanding of the problem. In many respects, the results are straightforward generalizations of the corresponding single-oscillator theory [7], though the array problem presents a greater number of possible subcases. The essential lesson is that the high-gain regime of the array coincides with onset of dynamical bifurcations. One new aspect of the array problem is the distinction between bifurcations that preserve the coherence of the elements (which we called symmetry-preserving bifurcations) and bifurcations that destroy this coherence (symmetry-breaking bifurcations). We find, quite generally, that maximum amplification is achieved near bifurcations of the symmetry-preserving type. (Ironically, this is the "uninteresting" case when considering the mathematics of bifurcation theory, since in effect the symmetry plays no role in the analysis [31].)

This basic picture is not the whole story: we found a couple of surprises, which occur in the vicinity of symmetry-breaking bifurcations. The first is that there is virtually no amplification of a *homogeneous* periodic signal, even very close to the symmetry-breaking bifurcation point. Still more striking is the result for inhomogeneous random signals: one expects each element to exhibit increasingly wild fluctuations as the symmetry-breaking bifurcation point is approached, but the bulk response across the entire array remains steady.

Another interesting result is that the large noisy precursors encountered in single oscillator systems [2] are substantially reduced per oscillator, when the oscillators are embedded in large arrays. This is perhaps not too surprising from a physical perspective, since the oscillators are coupled to each other, so that one might expect there to be some "averaging" effect due to the presence of the other oscillators.

In general, these effects should be very easy to see in experiments. (As a rule, people have found near-resonant amplification effects for single-oscillator systems to be readily observable.) Ideally, one would like to have a system in which one can monitor both the total array output and also the output of at least one of the individual array elements. In view of this, and also because the global coupling is readily achieved in such systems, we expect that nonlinear electric circuits would provide a fertile testing ground for these ideas. The multimode laser is another attractive possibility, though as a practical matter it may be difficult there to investigate the effects of inhomogeneous perturbations if the *inherent* fluctuations from spontaneous emission (which are typically small) are below detection limits.

Of obvious practical importance is the degree to which our results are relevant for arrays consisting of nonidentical elements. On a purely technical level, this symmetry was crucial, insofar as it allowed the transformation which rendered the calculations tractable. However, we can make some broad statements about the effect of "externally" removing the symmetry, based on general properties of bifurcating dynamical systems [35]. Typically, introducing some small variation between elements dramatically alters the dynamics in the vicinity of symmetry-breaking bifurcations, while having relatively little effect otherwise. Consequently, the results obtained for symmetry-preserving bifurcations—which are the most important regimes for high-gain parametric amplification—should persist, while the *completely* destructive interference predicted for case 2 of Sec. V B may well be unobservable. Of course, these statements are only qualitative, any quantitative progress on this important issue requires detailed calculations of the type carried out here, though without benefit of the diagonalizing transformation.

On the theoretical side, we note that the present analysis presented is based on a linearization of the governing dynamical equations, and although this is expected to accurately capture many prominent effects, undoubtedly there are additional interesting effects to be gleaned from a fully nonlinear analysis. Our guess is that a normal form analysis along the lines of Ref. [6] is suitable for the case of symmetry-preserving bifurcations (where the active phase-space dimension remains small), but that the case of symmetry-breaking bifurcations—in which many phase-space dimensions become important simultaneously—may not be readily solved by such an approach.

Finally, we turn to the practical implications of this work for the design of real parametric amplifier arrays. In the original work on single-element systems, the lesson [1–7] was that for a given nonlinear oscillator, one should map out the curves in parameter space corresponding to various types of bifurcations. For the array case, we see that it is most important to determine in addition whether the bifurcations are symmetry preserving or symmetry breaking: only the former are expected to give desirable performance, in which the power gain scales as N^2. For example, the kind of numerical work which mapped out the parameter space for various Josephson-junction arrays [26,27] could be usefully supplemented by labeling whether the instabilities cataloged preserve or break the permutation symmetry of the dynamics.

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APPENDIX

We collect here the results from Floquet theory which are relevant to our problem. A fuller exposition of these and further results may be found elsewhere [32,33].

Consider the system of linear, inhomogeneous, first-order ordinary differential equations

$$\dot{\boldsymbol{\eta}} = \mathbf{A}(t)\boldsymbol{\eta} + \mathbf{f} \ , \tag{A1}$$

where $\boldsymbol{\eta}$ and $\mathbf{f}$ are n-dimensional real vectors,and $\mathbf{A}(t)$ is an $n \times n$ real matrix with period T. Then there are solutions $\boldsymbol{\Psi}^k$ of the associated homogeneous problem of the form

$$\boldsymbol{\Psi}^k = e^{\rho_k t}\boldsymbol{\chi}^k \tag{A2}$$

where $\boldsymbol{\chi}^k(t+T) = \boldsymbol{\chi}^k(t)$, and ρ_k is a constant. In general, both $\boldsymbol{\chi}^k$ and ρ_k may be complex. For the situations studied in this paper, all of the ρ_k lie in the left half-plane. From a set of n linearly independent vectors $\boldsymbol{\Psi}^k$, one can form a fundamental matrix $\boldsymbol{\Phi}$,

$$\boldsymbol{\Phi} = (\boldsymbol{\Psi}^1, \boldsymbol{\Psi}^2, \ldots, \boldsymbol{\Psi}^n) \ , \tag{A3}$$

so that $\boldsymbol{\Psi}^k$ is the kth column of $\boldsymbol{\Phi}$. The general solution of Eq. (A1) is then

$$\boldsymbol{\eta}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0)\boldsymbol{\eta}(0) + \boldsymbol{\Phi}(t)\int_0^t \boldsymbol{\Phi}^{-1}(s) f(s) ds \ . \tag{A4}$$

With the initial condition $\boldsymbol{\eta}(t) = \mathbf{0}$, this reduces to

$$\boldsymbol{\eta}(t) = \boldsymbol{\Phi}(t)\int_0^t \boldsymbol{\Phi}^{-1}(s) f(s) ds \ . \tag{A5}$$

When one of the ρ_k is small, expression (A5) can be simplified. First, express Eq. (A5) in component notation:

$$\eta_j(t) = \sum_{k,l} \Phi_{jk}(t)\int_0^t \Phi_{kl}^{-1}(s) f_l(s) ds \ , \tag{A6}$$

so that each function η_j is the sum of $2n$ terms. If all of the ρ_k have (negative) real parts of order unity or greater, one can show that η_j remains small provided that the forcing function is small. However, if one or more of the ρ_k approach the imaginary axis, the response η_j can diverge. In the situation studied in this paper, all but one of the ρ_k have negative real parts of order unity, while one of the $\rho_k = -\epsilon$, with ϵ a small positive (real) number. In this case, Eq. (A6) is dominated by a single term:

$$\eta_j(t) \approx \Phi_{j0}(t)\int_0^t \Phi_{00}^{-1}(s) f_0(s) ds \ .$$

This can also be written, in view of Eq. (A2), as

$$\eta_j(t) = e^{-\epsilon t} P(t)\int_0^t e^{+\epsilon s} Q(s) f_0(s) ds \ , \tag{A7}$$

where P and Q are functions of period T. In fact, the term retained is of order ϵ^{-1}, and so becomes increasingly dominant as $\epsilon \rightarrow 0$.

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