Modeling of multibranched crosslike crack growth

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(Received 12 April 1991; revised manuscript received 23 May 1991)

Multibranched crosslike crack patterns formed in concentrically loaded square plates are studied in terms of fractal geometry, where the associated fractal dimension d_f is calculated for their characterization. We apply the simplest deterministic and stochastic approaches at a phenomenological level in an attempt to find generic features as guidelines for future experimental and theoretical work. The deterministic model for fracture propagation we apply, which is a variant of the discretized Laplace approach for randomly ramified fractal cracks proposed by Takayasu [Phys. Rev. Lett. **54**, 1099 (1985)] reproduces the basic ingredients of observed complex fracture patterns. The stochastic model, although not strictly a model for crack propagation, is based on diffusion-limited aggregation (DLA) for fractal growth and produces a slightly more realistic assessment of the crosslike growth of the cracks in asymmetric multibranches. Nevertheless, this simple *ad hoc* DLA version for modeling the present phenomena as well as the deterministic approach for fracture propagation give fractal dimensionality for the fracture patterns in accord with our estimations made from recent experimental data. It is found that there is a crossover of two fractal dimensions, corresponding to the core (higher d_f) and multibranched crosslike (lower d_f) regions that contains loops, which are interpreted as representing different symmetry regions within the square plates of finite size.

I. INTRODUCTION

Fracture phenomena in a wide variety of loaded brittle media have been a topic of extensive experimental research in materials science. This has resulted in an exhaustive amount of test data for the analysis of crack propagation behavior under tension in terms of grain boundaries, temperature, anisotropies, etc. Different sample shapes lead to the formation of different crack patterns, and so a large set of empirical rules exists in this field. However, in addition to this rich phenomenology, the possibility of using concentrically loaded square plates, which are supported at each corner, as a test to measure the fracture characteristics of ceramics under biaxial stress has only been recently explored [1]. It has been found that under these conditions asymmetric multibranched crosslike cracks exhibit growth directed towards the edges of the square plates, as shown in Fig.1. The practical importance of such tests that used quasi-two-dimensional (2D) square samples is that many ceramic materials are usually fabricated in this shape.

From a physical point of view it has largely been recognized that the broad class of complex crack patterns growing under a wide variety of nonequilibrium experimental conditions, such as applied shear, can be described in terms of fractal geometry, where the associated fractal dimension is the convenient tool for their characterization [2–4]. However, the level of analysis achieved for nonequilibrium processes such as diffusion-limited aggregation (DLA) [5–7], dielectric breakdown [8–10], and dendritic-crystal growth [11], which give rise to fractals, does not exist for crack formation and, in particular, for the new patterns formed by fracture stress of ceramic plates [1]. Consequently, no theoretical investigation or computer-generated simulations have yet been done that have attempted to understand the nature and physical properties, such as fractal dimension, of the complex processes generated in square-shaped materials showing asymmetric multibranched crosslike crack growth.

Studies of rapidly moving fourfold coordinated patterns without multibranching in the directions of the square lattice axes have been reported using different types of growth models. DLA clusters with crosslike (or snowflakelike) shape have been obtained when (i) "averaging" procedures were introduced [12, 13]; (ii) developing discrete Laplacian growth in which appropriate growth probabilities for the lattice sites were considered

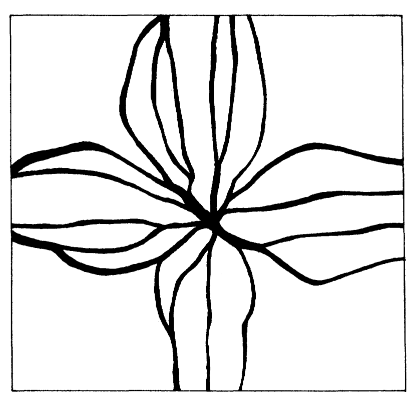

FIG. 1. Typical crack pattern generated on a square ceramic plate by concentrically applied loading within a circle of diameter 7.5 mm (from Ref. [1]).

[14, 15]; (iii) introducing a noise-reduction parameter [4]; or, more simply, (iv) assuming that the seed configuration was not isotropic in the sense of the original version of DLA [16].

Another theoretical possibility for describing the growth of fourfold interfaces has been the application of the boundary integral method to solve the Laplace equation without an underlying lattice [17, 18]. However, it is important to mention here that all these approaches for studying such specific fractal morphologies, in conjunction with some alternative simulation models of crack formation on surfaces [19–23], lack many important features present in the experiments carried out on concentrically loaded square plates, such as the appearance (for large sizes and large times) of multibranched cracks directed to the edges of the lattices [1].

It is the aim of this work to make a first attempt towards understanding qualitatively the crack patterns formed in concentrically loaded 2D lattices at a phenomenological level. We shall restrict ourselves to the comparison of the simplest deterministic and stochastic approaches to model the observation of asymmetric multibranched crosslike crack growth and from it, to study the associated fractal properties of such particular fracture patterns in defect-free media. We interpret the observed crack patterns as having a less dense, isotropic central region, and outside this region a crosslike formation of multibranches directed towards the edges of the square plate (see Fig. 1). We shall keep this basic interpretation in mind when discussing fractal dimension calculations and shall show, based on our estimations using reported crack data, that this idea is quite reasonable.

The deterministic model we propose is an extension of early ideas put forward by Takayasu [19] for randomly ramified fractal cracks in a 2D lattice. Here we also solve the discretized Laplace equation in a 2D square net but, as a difference, we generate spontaneous crack patterns when the center of the square brittle lattice is pulled out, while keeping all opposite corners fixed with null displacement. Our stochastic approach, on the other hand, although not strictly a model for crack propagation, is based on DLA for fractal growth and heuristically includes some restrictions to account for the effects of the 2D lattice corners. We shall show that this simple *ad hoc* DLA version as well as the deterministic approach for fracture propagation give fractal dimensionality for the fracture patterns in accord with our estimations made from experimental data.

The rest of this paper is organized as follows. In the following section we introduce our deterministic model of fracture growth and our stochastic approach for studying these phenomena. Based on these models we present our simulation results in Sec. III and discuss them in view of the experiments of Entwistle [1]. Section IV contains some concluding remarks.

II. FORMALISM

In order to model the recent experimental pictures of Entwistle [1] of fracture in concentrically loaded square plates, and to investigate the possible fractal nature of such crack patterns, we introduce next the deterministic model of fracture growth in defect-free media and the alternative stochastic approach.

A. Deterministic model

In the original deterministic model of fracture analyzed by Takayasu [19], a square-plane network consisting of brittle sticks that are connected stiffly at each lattice point is considered. The essential physics that follows from this model is rather simple. If a thin brittle stick that is fixed at one end is displaced at the other free end by a certain amount, it shows rigidity and a restoring force is observed. But once the displacement of the free end exceeds a critical value d_c, then the stick becomes broken, in which case the modulus of rigidity of the stick is suddenly reduced to a very small number. Once the fracture phenomenon takes place in the material, it is conceivable that the critical value of the displacement at which the modulus of rigidity diminishes would be modified as the crack pattern develops.

Incorporating these ideas, we have arrived at a new procedure to simulate the crack propagation in a network of brittle sticks under concentrically applied shear. This new procedure also assumes that the displacements at the lattice points are perpendicular to the plane, so we express the equilibrium of the forces at the (i,j)th lattice point as the discrete version of the Laplace equation that would be obeyed in a continuous model,

$$\sum_{k=1}^{4} G_k(i,j)[u_k(i,j) - u(i,j)] = 0 \,, \tag{1}$$

where $u(i,j)$ denotes the displacement of the (i,j)th lattice point, $u_k(i,j)$ is the displacement of one of the four nearest neighbors, i.e., $k = 1,\ldots,4$, and $G_k(i,j)$ is the corresponding modulus of rigidity of the stick connecting the (i,j)th lattice point to its neighbor.

It follows then that we (i) specify $\{G\}$, either by assigning *a priori* values or using random numbers, and the boundary conditions for $\{u\}$; (ii) solve Eq. (1) for $\{u\}$; and (iii) check every stick, except those that are already broken. If the breakdown condition $|u_k(i,j) - u(i,j)| > d_c$ is satisfied we then let $G_k(i,j) = \epsilon G_k(i,j)$, where ϵ is a small positive number. Here d_c may be regarded as a parameter that changes as the crack formation evolves. (iv) We stop if no new stick is broken in the present iteration, or a completed (percolation) crack pattern is reached, otherwise we go to step (ii).

In our new application of the Takayasu model to the concentrically loaded square plate, the boundary conditions are such that in an $N \times N$ net, the four corners are fixed and held at $u = 0$, and the lattice point in the middle of the net has an initial displacement u, representing the effect of the loading. A subsequent iterative solution of the set of equations for all the other displacements of the lattice points yields the crack pattern once suitable d_c's are chosen to satisfy the breakdown condition. This new model is as completely deterministic as the original proposed by Takayasu, in that, once the boundary conditions are specified, the growth pattern for fracture may develop in accordance with Eq. (1).

B. Stochastic model

The stochastic model we define on a 2D lattice is based on DLA [5]. Connected patterns are formed assuming that there are two different regions, say, one isotropic [24] containing a core region and the other anisotropic containing the multibranching, for the cluster aggregate to grow following two simple rules.

(i) Inside the first region, spherical particles diffuse through DLA. That is, they diffuse one at a time and are deposited adjacent to (occupied) lattice sites. These complex processes, which are known to present fractal behavior for large clusters, are repeated until we reach a core (or deposition) radius R_d that is situated between a central seed particle and the edges of the lattice. R_d is the radius outside of which asymmetric multibranching is assumed to start forming.

(ii) We define then the second region as being limited on the one hand by R_d and on the other by the lattice edges. In order to account for anisotropy, this second region, in turn, is divided into four equally spaced but imaginary channels (or paths) of width smaller than twice R_d. The spherical particles diffusing from the lattice ends and approaching the center core, which contains the DLA cluster, are then randomly added to the cluster by imposing some extra conditions on this process if the deposition results only inside the channels. If this is the case, the growth process is then determined as follows: one incoming spherical particle is transformed so as to occupy (up to) three nearest unoccupied lattice sites to become rod shaped with an axial orientation perpendicular to each edge of the lattice. But if the deposition results outside such imaginary channels, i.e., it occurs in the smaller square region that shares the corners of the original (and larger) 2D lattice, the process of deposition continues to be of the DLA type by simply adding spherical particles to the cluster, as also is done inside the isotropic region covered by R_d.

Although this stochastic model, which we are proposing for modeling the present phenomena, may be seen as heuristic, we shall show in the following section that it enables us to reproduce asymmetric multibranched crosslike structures that simultaneously grow and avoid approaching the lattice corners.

III. RESULTS AND DISCUSSION

As previously mentioned, our aim is to model the growth of asymmetric multibranched crosslike cracks and investigate their fractal dimensionalities. A typical crack pattern exhibiting such features is reproduced in Fig. 1 which has been obtained using a 7.5 mm-diam loading circle on a 103-mm square alumina plate (1 mm thick, 2.5% porosity, and 2-10 μm grain size) supported at each corner [1]. In these experiments the fracture origin is found to be close to the (concentrically applied) loading circle, so it is not unreasonable to consider, within our deterministic model, that cracks will be generated when the center of the square brittle lattice is pulled out while keeping all opposite corners fixed, whereas within our stochastic approach the pattern will grow starting from the seed particle placed at the intersection of the plate diagonals.

We interpret the experimental crack growth patterns of Fig. 1 as follows. There exists a core region which looks isotropic and extends out to a radius R_d. Then what we call anisotropy sets in, and the overall observed pattern shows a crosslike behavior with multibranches directed towards the edge of the square plate. As can be seen in this figure, the branches outside the region R_d sometimes intersect to form loops, which further justifies the assumption about anisotropy made in our deterministic and stochastic models.

It is also important to indicate that there are a few differences between our modified version of the Takayasu model for crack growth [9] and the actual experimental conditions. In the experiments reported by Entwistle [1] the square plates were loaded within a circle of finite radius, whereas we have here assumed that loading takes place on a single lattice point. Furthermore the plates supported at four corners exhibit complex movements due to elastic strains, change of slope at the point of support, and plate deflection.

In our deterministic simulations we have neglected all such effects and considered the square net to be fixed at the corners where the displacements were set to $u = 0$, as part of the boundary conditions. Since our aim in the present work is to chiefly obtain a qualitative understanding of the observed crack patterns, we have not attempted to simulate the case of a load applied to a finite region in the square lattice. In order to search for crosslike behavior in the formation of crack patterns we first assume that all the moduli of rigidity $G_k(i,j)$ have the same value B, except for those sticks that are on the edges of the square lattice, in which case their value is set to A. This is the main difference between our deterministic model and that of the original Takayasu model [19]. Percolation patterns are thus obtained deterministically in our case with the minimal set of input parameters. The resulting crack pattern, as displayed in Fig. 2(a), is completely symmetric due to the simplicity in the choice of the input (A=20 and $B = 2$), yet exhibits the salient features of the experiment. We note that the crosslike behavior with multibranching and an isotropic core region, as previously discussed, is already present in this model. To this end it is important to mention that at least two different moduli of rigidity are required to observe the crosslike pattern formation with the condition $A >> B$. If all $G_k(i,j)$'s are chosen to be equal, i.e., $A = B$, we find that the growth pattern for the cracks shows a marked difference, namely, the sticks tend to break, starting from the center towards the corners along the diagonals of the square lattice. As in the original Takayasu model [19], cracks grow from the place where tension is applied to the points of null displacement, i.e., $u(i,j) = 0$.

On the other hand, when the moduli of rigidity are allowed to be random at the first iteration, we obtain a slightly more realistic crack pattern, as shown in Fig. 2(b) and as compared to Fig. 1. In this figure, we have taken

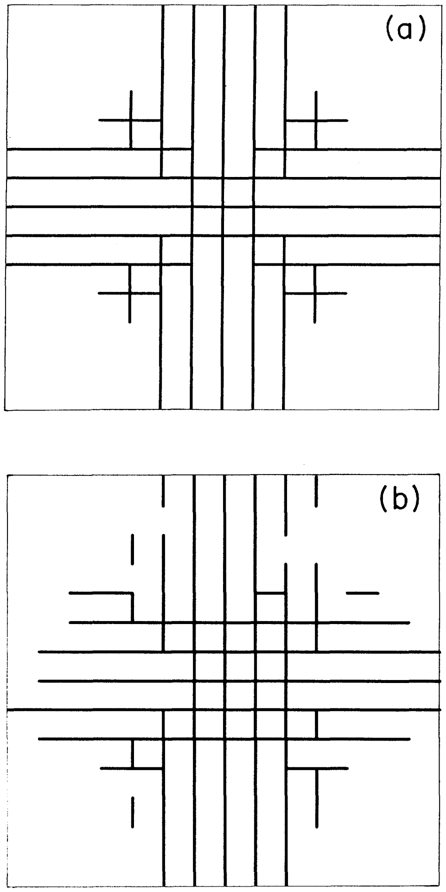

FIG. 2. Final fracture pattern in the present deterministic model with (a) constant moduli of rigidity A and B, and (b) random moduli of rigidity A and B. In both cases the condition $A>>B$ is satisfied.

the moduli of rigidity, for the bonds within the lattice (excluding the edges), to be $B = 1+\xi_1$, and for the bonds on the surrounding edges of the lattice, $A = 20(1+\xi_2)$, satisfying the condition $A \gg B$. Here ξ_i $(i = 1, 2)$ are random numbers uniformly distributed in (0,1). The simulation for the pattern illustrated in Fig. 2(b) was performed on a 15×15 lattice with $\epsilon = 0.01$, initial perpendicular displacement of the lattice site in the middle $u = 3$, and the critical value for displacement $d_{c1} = 0.75$ necessary to break the four neighboring sticks to the middle lattice point [25]. In the next iteration d_{c2} is taken to be smaller, i.e., 0.003, to obtain the final fracture pattern. Both d_{c1} and d_{c2} were determined by inspection. It is interesting to notice that we would require different values of the critical displacement d_c in each iteration in order to get temporal crack propagation. This is to be compared with the original Takayasu model [19] in which a single d_c is used throughout a given simulation. However, if $\epsilon \gg 1$, we recover the Takayasu dynamics with a single d_c, which does not correspond to fracture but to dielectric breakdown [9]. That the value of d_c changes depending on the configuration generated in the previous iteration may be interpreted as some material-dependent phenomenon.

Comparison of Figs. 2(a) and 2(b) reveals that the asymmetry of the directed growth of the multibranched crosslike fracture is due to the randomness introduced in the otherwise completely deterministic model. To this end, we point out that when the discretized Lamé equation of the theory of elasticity is solved on a lattice with appropriate boundary conditions, the resulting crack growth pattern shows crosslike behavior that is fully symmetric [21–23].

We now focus on the results of the stochastic approach. Motivated by the findings in our deterministic model concerning the isotropy in the central region [24], and the crosslike crack propagation, we develop the stochastic model to generate a similar class of growth patterns with emphasis on asymmetry. Some examples of the structures generated by computer studies of our DLA-based stochastic model are shown in Figs. 3(a) and 3(b).

These figures show clusters containing 2740 and 3360 spherical particles, respectively, grown in a 200×200 square lattice. Model particles are added, from one to three at a time to a particular DLA cluster that is contained inside a core region via random-walk trajectories

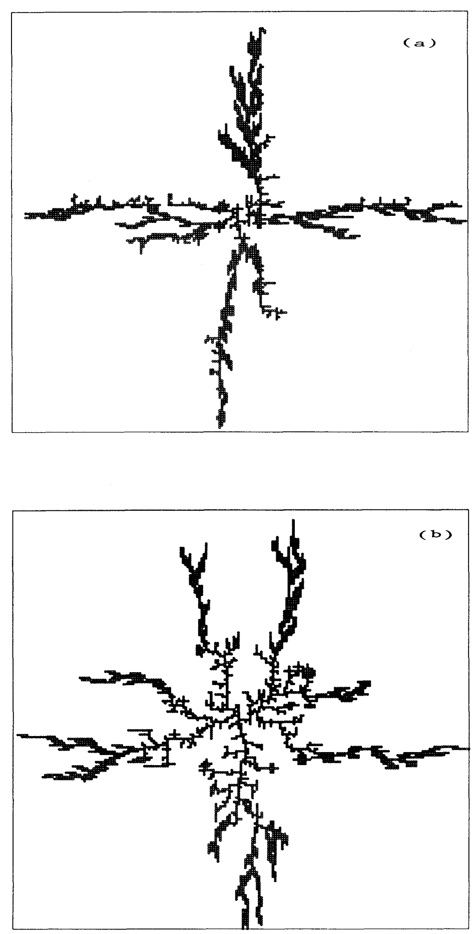

FIG. 3. Crack growth in a 200×200 lattice using the present stochastic DLA-based model with a core region containing (a) 10^2 sites and (b) 10^3 sites.

that originate from the region compressed between the growing DLA cluster and the lattice edges. Such oriented particles cross imaginary channels throughout this external region in the lattice. The core regions in Figs. 3(a) and 3(b) contain, respectively, 10^2 and 10^3 occupied lattice sites and the imaginary channel widths are about 12.8% and 53.5% of the linear size of the 2D lattice.

Using the present algorithm for the stochastic simulation, it now becomes possible to generate asymmetric multibranched crosslike patterns in reasonable accord with observations reproduced in Fig. 1. We find that the several branches appearing in the growth processes almost simultaneously reach the lattice edges. For the larger core radius considered, as seen in Fig. 3(b), we produce a larger amount of asymmetric needlelike arms, which split up to form a ramified but oriented pattern, than in the case of starting with a less dense cluster than that shown in Fig. 3(a). Contributions to the random growth process from the spherical particles in the region outside the central core and the imaginary channels become extremely small. This fact, which occurs within the region containing the lattice corners, may to some extent be a way of taking into account the presence of the corner supports [1]. The multibranched crosslike patterns stop growing after (at least) one arm touches the boundary edges.

We examine next the fractal properties of the simulated crack patterns displayed in Figs. 2(a)–3(b) as well as those from the experimental data shown in Fig. 1. To determine the decay of density correlation functions in our deterministic model, the number of broken bonds in a box of side h centered around the origin of the 2D lattice are plotted versus h in a log-log plot (not shown). Lines with slopes larger than unity and smaller than the space dimension are obtained, which indicate the fractal nature of these patterns. We find that the deterministic simulation patterns, carried out in a 15×15 lattice, yield fractal dimensionality $d_f \approx 1.86$ for the case of constant moduli of rigidity and it further approaches the space dimension for random moduli of rigidity. In this calculation we are also able to identify a smaller theoretical d_f corresponding to the anisotropic region (still far from the lattice edges). Larger scale simulations of our deterministic model are needed to distinguish clearly between these different fractal dimensions. We point out that owing to the underlying symmetry in the boundary conditions, even in the case of random moduli of rigidity, the number of broken bonds in the edges (or x and y) directions are approximately equal [cf., Figs. 2(a) and 2(b)].

Within the present DLA-based stochastic model, we count the number of particles, $N(r)$, inside a circle of increasing radius r (in lattice units) around the seed particle and plot it as a function of r in a log-log plot, as depicted in Fig. 4. For this purpose those particles that have been transformed to occupy up to three nearest (unoccupied) lattice sites to become rod shaped, having an orientation perpendicular to the lattice edges, are counted as single spheres if they are included in the circle. For comparison, we also show in this figure the average results of a very good fit, by a straight line, for a 2D DLA cluster containing 2740 particles with a well-defined fractal dimension: $d_f = 1.71$ [5] grown in a 200×200 lattice.

Figure 4 clearly illustrates the crossover between the two regions—containing the core and multibranching, respectively—that we have assumed to describe the crack growth in concentrically loaded square plates. Inside the isotropic region, i.e., $r < R_d$, we obtain a slope of $d_f^{(\mathrm{iso})} \approx 1.71$, as expected for our data, because we have imposed a DLA type of behavior for the pattern to start the growth propagation. It is surprising, however, to see that within the anisotropic region, i.e., $r > R_d$, there is a "smooth" drop of the linear behavior that is present for $r < R_d$ and that gives rise to a second fractal dimension value smaller than $d_f^{(\mathrm{iso})}$. The second slopes of these patterns are $d_f^{(\mathrm{aniso})} \approx 1.41$ for the smaller ($R_d = 12.81$) core cluster and $d_f^{(\mathrm{aniso})} \approx 1.25$ for the larger ($R_d = 53.46$) core cluster studied. In our calculations of d_f we have not included the size effects due to the linear size of the lattice ($r \rightarrow 10^2$); a trivial behavior that can be seen in Fig. 4. It is also interesting to note that for each pattern obtained with a specific core radius R_d there is a unique asymmetric multibranched crosslike pattern that contains loops. Hence, within this simple approach, to each generated pattern there corresponds a different value of $d_f^{(\mathrm{aniso})}$.

In order to give also an estimate of the fractal properties of the experimental crack patterns displayed in Fig. 1 and compare them with our findings in Fig. 4, we use the above-mentioned procedure to compute directly d_f from the log-log plot of Fig. 5. We find that there are two slopes, indeed: one corresponding to the small-r region (marked by an arrow in Fig. 5), and the second one cor-

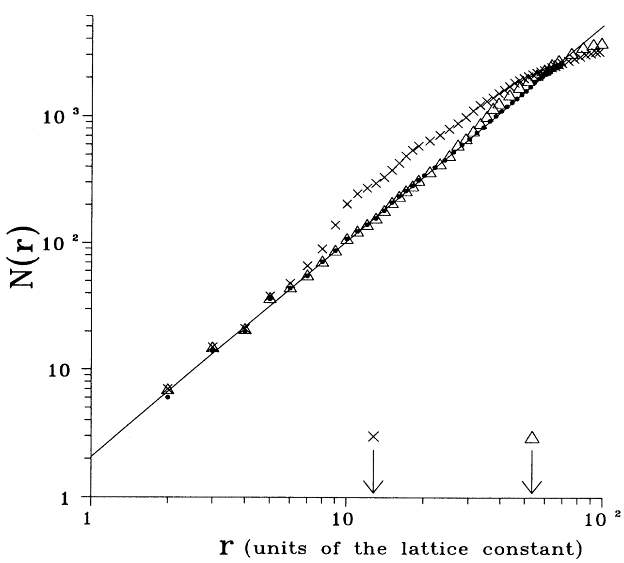

FIG. 4. Number of spherical particles, $N(r)$, in a circle of radius r covering the patterns in (×) Fig. 3(a) and (△) Fig. 3(b). The arrows mark the corresponding core radius R_d for the isotropic region (see text). For comparison, we also plot (•) the results for a DLA cluster of 2740 particles, where the slope of the line is $d_f \approx 1.71$. In all cases the system size is 200×200.

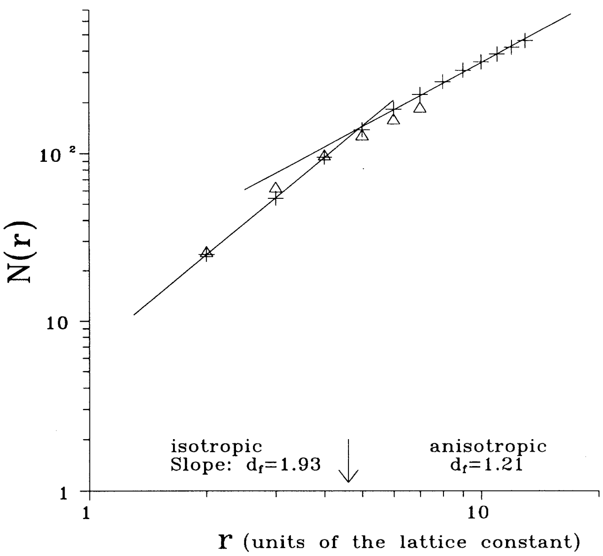

FIG. 5. Present estimations of the fractal dimension (d_f) values of the experimental crack growth patterns of Ref. [1], shown in Fig. 1. Different symbols represent different statistics. Arrow indicates an approximated boundary for the isotropic and anisotropic regions (see text).

responding to the large-r region, with fractal dimensions $d_f^{(1)} \approx 1.93$ and $d_f^{(2)} \approx 1.21$, respectively. Accordingly, the second d_f value for the smaller isotropic core system considered extends for a decade in the r axis. A smaller second d_f value is an indication of the multibranched crosslike spatial evolution of the patterns after some loop structures have been generated in Fig. 1. This interesting finding encourages us to give a further justification of our assumption that inside the square plates there may be two regions of different symmetry—say, one isotropic (say, of the DLA type [24]) and the other anisotropic (or fully oriented)—each of which presents a different fractal dimension ($d_f^{(\mathrm{iso})} > d_f^{(\mathrm{aniso})}$). In this respect, it is worthwhile to add that a crossover between two growth regimes in crack formation has also been determined by the high-stress limit of the Lamé equation, provided the probability of cracking increases with stress [21]. However, such studies have not been extended to include multibranching directed to the edges of the lattice, as was done here. Finally, it is also important to mention that the present second crossover values for d_f, obtained within our stochastic model, differ from theoretical predictions for single-branched crosslike patterns $d_f = \frac{5}{3}$ [12, 26] and approach, for instance, the values estimated from Laplacian growth (η) model with $\eta = 2$ [4] that are smaller than d_f in DLA.

IV. CONCLUDING REMARKS

As pointed out by Mandelbrot [2], "One reason to estimate fractal dimensions d_f and $f(\alpha)s$ (i.e., its generalization to self-similar multifractals [27]), is to do Physics. Another reason is to compare 'messy' data with theory." We believe this to be appropriate for the large set of empirical rules and test data, which exists for crack propagation behavior obtained under tension, because of its practical importance. Of particular interest are the approximate fractal dimension estimations we have made for the recent experiments by Entwistle [1] that were carried out on concentrically loaded square plates, which we have shown in Fig. 5.

In this work we have taken the first steps towards understanding the recently observed asymmetric multibranched crosslike crack growth in a 2D lattice in terms of fractal geometry, where the associated d_f is the convenient tool for their characterization. Fractal dimensionality in such fracture patterns presents a crossover of two d_f, which we have interpreted as representing different symmetry regions within the square plates of finite size. Of course, this new and interesting problem can be studied with a number of numerical methods that are more sophisticated than the ones applied in our paper and that should probably give the best results (as compared to experiments) [19–23].

Nevertheless, by using the simplest deterministic model we may apply for crack propagation—which is a generalization of the discretized Laplace approach for randomly ramified fractal cracks with appropiate boundary conditions, as proposed by Takayasu [19]—we argue that the physics behind this phenomenon is in the crossover of two fractal dimensions, one corresponding to a core (higher d_f) region and the other to multibranched crosslike (lower d_f) regions. On the other hand, although a stochastic DLA-based model is not essentially a model for crack propagation, we have shown that in its present version, it can produce patterns resembling the experimental crack picture in asymmetric multibranches when some restrictions are heuristically included to account for the effects of the lattice corners. But, even more important, this simple *ad hoc* DLA version leads to modeling the same physical phenomena when discussing d_f for multibranched crosslike crack patterns. As also mentioned in the text, a crossover between two growth regimes in crack formation has also been determined by the high-stress limit of the Lamé equation [21]. However, the latter, "more" complicated numerical studies have not been extended yet to include multibranching directed to the edges of the 2D lattice, as seen in experiments.

By invoking the simple deterministic model for fracture propagation, we found that using identical moduli of rigidity, i.e., $A = B$, crack propagation is along the diagonals of the square lattice. For the same boundary conditions, i.e., null displacement at the lattice corners, having $A \gg B$ drastically changes the symmetry to give crosslike growth patterns, making them look closer to experiments; besides, the DLA-based algorithm for simulation allows us to generate asymmetric multibranched crosslike patterns that contain loops, which are in slightly more reasonable agreement with observations. The several branches appearing in the crack growth almost simultaneously reach the lattice edges.

We have assumed that bonds break under external shear and not under compression or drying. It is known that the mechanisms leading to fracture are material dependent and this has not been considered within the present work. Within our stochastic approach, material dependence may somehow be related to the adopted core size, and in the case of the deterministic model, appropriate selection of the moduli of rigidity for the sticks may be tried. Although any firm conclusion on this cannot be established here, we believe that our approaches to the study of multibranched crosslike crack patterns can be useful to find generic features as guidelines for future experimental and theoretical research concerning the existence of a crossover between two growth regimes during crack propagation in concentrically loaded square plates.

ACKNOWLEDGMENTS

We would like to acknowledge M. Ferconi for valuable discussions. We also thank Professor Abdus Salam, the International Atomic Energy Agency, and the United Nations Educational, Scientific and Cultural Organizations for their hospitality at the International Centre for Theoretical Physics in Trieste.

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