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Chaos and nonisochronism in weakly coupled nonlinear oscillators

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Nonisochronism, the dependence of oscillation frequencies on amplitudes, substantially changes the dynamics of two weakly coupled nonlinear oscillators. Using averaged equations, we find that a nonisochronous passive circuit coupled with an active oscillator displays chaotic and two-frequency oscillation regimes, behaviors not found in the corresponding isochronous system. This system also shows hysteresis at the transitions from two- to one-frequency regimes, and from the one-frequency regime to the suppression of oscillations. The transitions to chaos, via period doubling and type-I intermittency, occur as the detuning of the two partial frequencies of the oscillators is varied, and we identify conditions for the onset of chaos in this system.

I. INTRODUCTION

A system of two coupled quasilinear oscillators is one of the most widely studied models of multimodal systems [1,2]. It exhibits important features of these systems such as competition between modes, hysteresis of different oscillatory regimes, and generation suppression [1-5]. However, when studying multimodal systems analytically, one usually restricts the mathematical model to the isochronous case [4], neglecting the dependence of the frequencies of oscillations upon their amplitudes. This approximation is not satisfactory for many cases [1,6], such as electrical oscillatory systems with reactive circuit elements.

In this paper, we present the basic effects that are seen in a system of two weakly nonlinear oscillators, coupled directly, including nonisochronism. Our analysis treats cases when both oscillators are active and when one of them is passive. We show that the dependence of the oscillation frequencies upon the amplitudes not only leads to the displacement of the whole state portrait of the system along the frequency axis, but also causes regimes of qualitatively different behavior to occur, such as chaotic dynamics, which are impossible for the analogous isochronous system.

II. MATHEMATICAL MODEL

We consider two linearly coupled oscillators described by the following general equations:

$$\ddot{x}_1-\mu_1(1-\nu_1x_1^2)\dot{x}_1+\omega_1^2(1-\delta_1x_1^2)x_1=Kx_2\ ,$$
$$\ddot{x}_2-\mu_2(1-\nu_2x_2^2)\dot{x}_2+\omega_2^2(1-\delta_2x_2^2)x_2=Kx_1\ . \tag{1}$$

If the two oscillators are weakly nonlinear, weakly coupled, and the difference in their partial frequencies is small compared with either frequency, solutions will be of the form

$$x_1(t)=a(t)\cos[\omega t+\phi_1(t)]\ ,$$
$$x_2(t)=b(t)\cos[\omega t+\phi_2(t)]\ , \tag{2}$$

and the averaging method introduced by Krylov and Bogoliubov [8,9] yields the following equations for the oscillator amplitudes a and b and phase difference $\psi\equiv\phi_2-\phi_1$:

$$\frac{da}{d\tau}=(\alpha_a-\gamma_a a^2)a+kb\sin\psi\ , \tag{3a}$$

$$\frac{db}{d\tau}=(\alpha_b-\gamma_b b^2)b-ka\sin\psi\ , \tag{3b}$$

$$\frac{d\psi}{d\tau}=-\Delta+\beta a^2-\kappa b^2+k\left[\frac{b}{a}-\frac{a}{b}\right]\cos\psi\ . \tag{3c}$$

Here, $\alpha_{a,b}=\mu_{1,2}/2$; $\gamma_{a,b}=(\nu_{1,2}\alpha_{a,b})/4$; $k=K/(2\omega)$, the coefficient of resonant interaction between the two oscillators; $\Delta\equiv\omega_1-\omega_2$, the detuning of the two partial frequencies; τ is the "slow" time; coefficients α_a and α_b characterize the linear and γ_a and γ_b the nonlinear dissipative features of oscillators. The coefficients β and κ represent nonisochronous features of the oscillators and are dependent upon the parameters $\delta_{1,2}$ and $\mu_{1,2}$ of Eq. (1). When $\beta=\kappa=0$, the partial frequencies of the oscillators do not depend upon their amplitudes and the system is isochronous.

Similar equations have been derived previously [1,4,7], and have been studied for various cases of identical oscillators. They may be obtained using other first-order asymptotic methods such as the two-variable expansion perturbation method [7]. The direct, nonscalar coupling that we have included in Eq. (1) is typical of electrical circuits [5,6].

Values of the nonzero steady amplitudes ($a=A$, $b=B$) of system (3) are determined by the resonance curve equation

$$\Delta=\beta A^2-\kappa B^2 \pm\left[\frac{B^2}{A^2}-1\right]\sqrt{(k^2A^2/B^2)-(\alpha_b-\gamma_b B^2)^2}\ , \tag{4a}$$

where

$$B^2=\frac{1}{2\gamma_b}[\alpha_b\pm\sqrt{\alpha_b^2-4\gamma_b A^2(\gamma_a A^2-\alpha_a)}]\ . \tag{4b}$$

The signs "±" indicate the two different branches of the resonance curve, which correspond to different magnitudes of the stationary phase difference

$$\psi=\psi_{1,2} =\arccos\left[\pm\left\{1-\frac{B^2}{k^2A^2}(\alpha_b-\gamma_b B^2)^2\right\}^{1/2}\right]\ . \tag{4c}$$

Stable nonzero fixed points of system (3) correspond to one-frequency regimes with the oscillations in each of the coupled oscillators having the same frequency. Two-frequency quasiperiodic oscillations correspond to limit cycles in the phase space of system (3), with the magnitudes of a, b, and ψ changing periodically with time. System (1) also has a trivial solution, $x_1=x_2=0$, which corresponds to a fixed point at the origin for system (3), $A=B=0$ with an undefined phase. Its stability can be determined by eigenvalue analysis of the linearized full four-dimensional system at the origin [1].

Typical resonance curves for isochronous and nonisochronous cases are shown in Fig. 1, with (a) and (b) exhibiting cases of active-passive coupled modes while (c) and (d) represent two active modes interacting. One can see that, for the case $\alpha_a<0$, $\alpha_b>0$, and $\beta=\kappa=0$ [Fig. 1(a)], system (3) has stable steady states in the whole range of detuning Δ. Each stable branch of a resonance curve corresponds to a generation of a signal on one of the normal frequencies of system (3). There is also a certain interval near $\Delta=0$ where both steady states are stable and the dependence of the frequency of oscillations upon frequency detuning Δ shows hysteresis. The latter phenomenon is well known and widely used for purposes of frequency stabilization, for instance, in electrical circuits.

For two coupled modes with $\alpha_a>0$ and $\alpha_b>0$, in the isochronous case [Fig. 1(c)], there exist two basic states: a synchronized state within a certain interval around $\Delta=0$ and an asynchronous state outside this interval, similar to the "no shear" case [1]. Here, within a certain range of detuning, we also have competition between the two normal frequencies of the coupled system.

One can see that nonisochronism produces not only

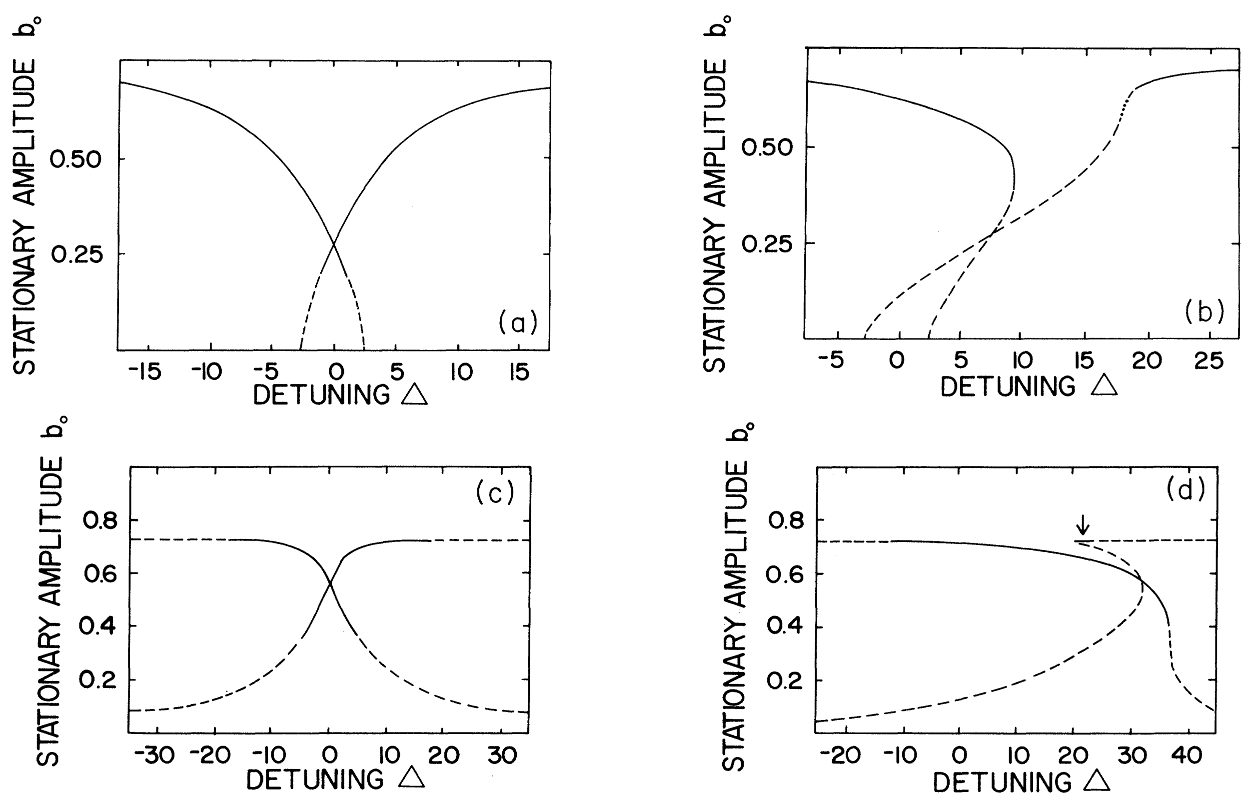

FIG. 1. Resonance curves of system (3) with $\gamma_a=0.3$, $\gamma_b=0.3$, and $k=6.5$. (a) Isochronous pair of coupled active and passive oscillators ($\alpha_a=-1$, $\alpha_b=1.5$); (b) same coupled active and passive oscillators as (a) but for the nonisochronous case ($\beta=10$, $\kappa=0.3$); (c) isochronous pair of coupled active oscillators ($\alpha_a=0.5$, $\alpha_b=1.5$); (d) same coupled active oscillators as (c), but for the nonisochronous case ($\beta=10$, $\kappa=0.3$). The solid (dashed) lines show stable (unstable) fixed points.

geometrical displacement of the resonance curves along the Δ axis by $(\beta A^2-\kappa B^2)$, but it changes the stability of the fixed points. The regions of instability grow as κ or β increases. For two active modes, this means that one of the branches of the resonance curve starts losing stability, and, beyond a certain magnitude of nonisochronism, can become totally unstable [Fig. 1(d)].

The bifurcation portrait of the system in the parameter plane (Δ,α_a), for both the isochronous and nonisochronous cases, is given in Fig. 2. If $\beta=\kappa=0$ [Fig. 2(a)], one has three regions in the parameter plane where the system has qualitatively different behavior: the solid region, where dissipation in the passive circuit $(\alpha_a<0)$ suppresses oscillations in the active one (no oscillations at all), a single-frequency region (hatched area) where oscillations of both modes (in both circuits) are synchronized on one of the normal frequencies, either F_1 or F_2, and a region of two-frequency oscillations (open area) where each circuit is oscillating at its own frequency.

It has long been believed that these are the only regimes possible for the system of two coupled oscillators, since, when one of the oscillators is passive in an isochronous system, oscillations in both oscillators are either suppressed or they are synchronized. However, our studies show that, for a nonisochronous system, such conclusions cannot be drawn. Nonisochronism impacts the stability of steady states of the system to such a degree that regions arise where none of the branches of the resonance curve is stable [Fig. 1(b)]. Since there is no synchronization in this region, there may exist two- or multiple-frequency oscillations. This region is shown in Fig. 2(b) as the open area and marked T_1^2. Our studies show that this region is restricted by limits

$$-\alpha_b \lesssim \alpha_a < 0 \tag{5}$$

and is situated between regions in which two stable states compete [double hatched areas in Fig. 2(b)]. The region of asynchronous oscillations usually extends below the line

$$\Delta=(\alpha_a+\alpha_b)(\beta-\kappa)/(\gamma_a+\gamma_b) \tag{6}$$

shown as a dash-dotted line in Fig. 2(b). This line, Eq. (6), corresponds to the centered axis $\Delta=0$ in the isochronous case because, at that line, the values of the steady amplitudes of both modes are equal ($A=B$).

In the area of asynchronization, two-frequency oscillations T_1^2 are prevalent corresponding to a limit cycle in the phase space of the system. However, for a certain region of the parameter values, as detuning Δ is increased, the limit cycle undergoes a cascade of period-doubling bifurcations to chaos [Fig. 3(a)]. As Δ is increased further, the phase portrait of the system goes through a sequence of reverse bifurcations and a window of order appears, characterized by a four-cycle of unusual form [Fig. 3(b)]. With additional increases in Δ, we observe an intricate sequence of transformations from chaotic attractors to three and four cycles and vice versa. Most of these transformations happen via period doubling, but sometimes this period doubling is accompanied by intermittency.

Intermittency is illustrated in Fig. 4, which shows the fourth iterate return map (b_k,b_{k+1}) where $b_k\equiv b(kT)$ are values of the amplitude b through the period T of the preceding four cycle. Long series of quasiregular motion are clearly visible. The distribution function of the length of regular phases exhibits two peaks suggesting that the intermittency is of type I [10].

Finally, after the last reverse bifurcation, a period-one cycle remains. Its dimension gradually diminishes as Δ is increased, until it collapses to a stable fixed point indicating that the oscillations of the two modes have become synchronized. Thus, the chaotic region is situated between regions of two- and one-frequency oscillations.

Further comparisons of bifurcation portraits of the system in isochronous and nonisochronous cases show that nonisochronism substantially diminishes the region in which the two normal frequencies $F_{1,2}$ compete. As β or κ increases, both branches of the resonance curve [Eq. (4)] lose stability in such a way that the region of two-frequency oscillations divides the bistability area $F_{1,2}$ into two unconnected triangles [double hatched regions in Fig. 2(b)]. One of them lies primarily in a part of the parameter plane where both oscillators are active. The upper part of this curved triangle, indicated by the arrow in Fig. 2(b), represents the previously reported

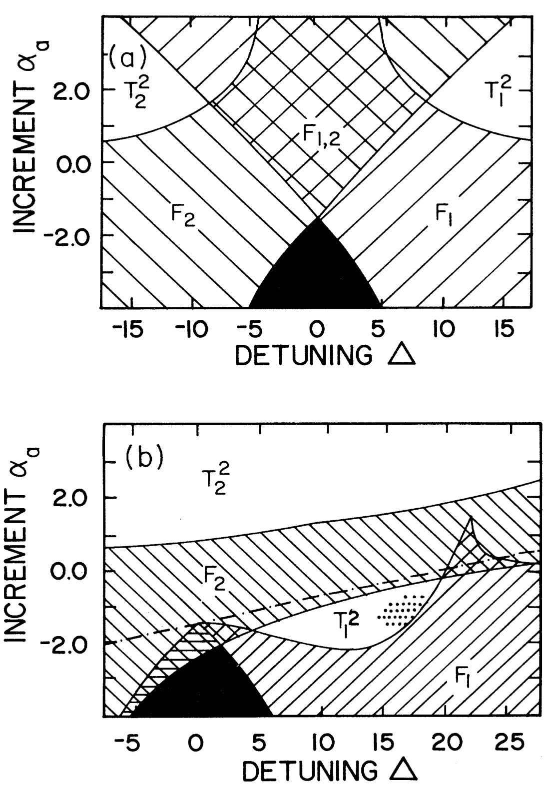

FIG. 2. Bifurcation portrait of system (3) for parameter values $\alpha_b=1.5$, $\gamma_a=0.3$, $\gamma_b=0.3$, $k=6.5$ for the (a) isochronous and (b) nonisochronous ($\beta=10$, $\kappa=0.3$) cases. The oscillation suppression areas (solid), single-frequency regions F_1 and F_2 (hatched), and two-frequency regions T_1^2 and T_2^2 are illustrated. The chaotic region is dotted.

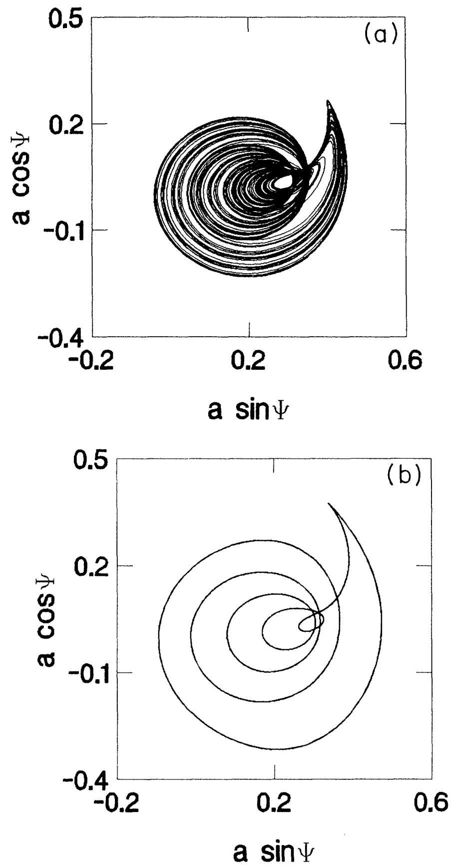

FIG. 3. The phase portrait for system (3) with parameter values $\alpha_a=-1.3$, $\alpha_b=1.5$, $\gamma_a=0.3$, $\gamma_b=0.3$, $\beta=10$, $\kappa=0.3$, $k=6.5$ (a) with frequency detuning $\Delta=17.0$ and (b) $\Delta=17.5$.

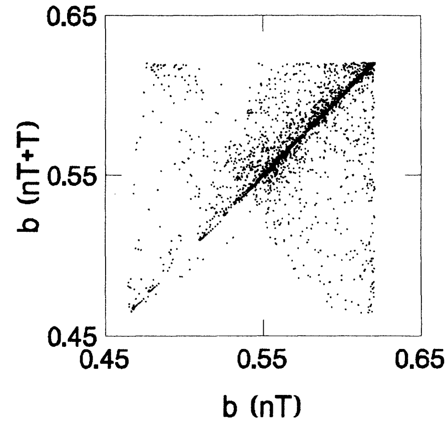

FIG. 4. Fourth iterate return map for the amplitude b, with $\alpha_a=-1.3$, $\alpha_b=1.3$, $\gamma_a=0.3$, $\gamma_b=0.3$, $\beta=10$, $\kappa=0.3$, $k=6.5$, and $\Delta=17.65$, showing patterns of regular behavior (along the diagonal) interrupted by chaos.

phenomenon of an island steady state [5], totally overlapped by another steady state of different frequency so that the system cannot be brought to the island state by changing only the frequency detuning, Δ.

Another part of the region where oscillations of both frequencies $F_{1,2}$ compete occurs near another area where the system shows hysteresis between oscillation suppression and oscillation generation. The existence of this latter area is a consequence of the fact that the stability of the trivial steady state $A=0$ and $B=0$, in the linear approximation, does not depend upon terms in the system containing the parameters of nonisochronism βa^2 and κb^2, but instead these terms change the stability of those nontrivial steady states that, in the isochronous case, are unstable when $A=0$ and $B=0$ are stable. If either β or κ is not equal to zero, the regions of stability of trivial and nontrivial steady states overlap and form an area where the two regimes compete [the double hatched area with the horizontal grid in Fig. 2(b)]. In this region we observe a kind of hard excitation of oscillations, in spite of the fact that our nonlinearity in Eqs. (1) is cubic and not of fifth order.

We should also mention that a hysteresis between oscillations with different numbers of independent frequencies is characteristic of nonisochronous systems. A similar result was reported by Aronson *et al.* [1], in their study of scalarly coupled oscillators with shear. For two coupled active modes, nonisochronism leads to the formation of a wide zone along one or two sides of the synchronization area where one-frequency F_2 and two-frequency T_2^2 oscillations compete. A discussion of the analytical interpretation of this phenomenon is published, by the authors, in a separate paper [3].

To summarize the results of this study, we show that nonisochronism of a coupled oscillatory system dramatically changes its fundamental behavior. In particular, coupled active and passive oscillators may exhibit not only one-frequency generation or a generation gap, but may also show two-frequency and chaotic dynamics. The observed transitions to chaos occur through period doubling or via type-I intermittency. An additional characteristic of nonisochronous systems is the presence of hysteresis between two- and one-frequency regimes and between one-frequency and suppressed oscillations.

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