PHYSICAL REVIEW A VOLUME 44, NUMBER 5 1 SEPTEMBER 1991

Exponential decrease in phase uncertainty

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(Received 4 April 1991)

The phase probability curve of a recently proposed photon state consists of a broad background with a sharp central peak [J. H. Shapiro, S. R. Shepard, and N. Wong, Phys. Rev. Lett. **62**, 2377 (1989)]. These authors argue that the inverse peak-height phase uncertainty $\delta\varphi$ of this distribution decreases inversely as the *square* of the mean photon number $\langle m \rangle$—an improvement over either coherent or highly squeezed states. We show that the width $\Delta\varphi$ of the best-fitting Gaussian to the central peak—a measure of phase uncertainty tailored to this narrow feature—decreases *exponentially* with increasing $\langle m \rangle$. The importance of this result may be offset by the observation that the area under this peak also vanishes very rapidly.

A recent Letter [1] proposes the reciprocal likelihood, that is, the inverse of the maximum of the phase probability distribution, as a novel measure for phase uncertainty $\delta\varphi$ of a quantum state. In the course of that work, the authors of Ref. [1] have discovered a new and intriguing quantum state $|\psi_s\rangle$ that minimizes $\delta\varphi$ for a fixed average photon number $\langle m \rangle$. In this case, $\delta\varphi$ is inversely proportional to $\langle m \rangle^2$, a property unique to this state. (Recall that $\delta\varphi \propto \langle m \rangle^{-1/2}$ for a coherent state and $\delta\varphi \propto \langle m \rangle^{-1}$ for a highly phase-squeezed state.) The resulting semiclassical phase distribution [2,3] $W_\varphi[|\psi_s\rangle]$ consists of a broad background essentially independent of $\langle m \rangle$ and an extremely narrow peak $W_\varphi^{(\text{peak})}$. In the present work we point out an even more impressive property of this quantum state: The phase uncertainty $\Delta\varphi$ associated with the width of the best-fitting Gaussian to this central narrow peak decreases *exponentially* with the average number of photons.

Although this striking feature seems to suggest a vast reservoir of potential applications for this state, we issue this caveat: The peak height increases only quadratically with increasing $\langle m \rangle$. Hence, the phase probability associated with the area underneath the peak rapidly vanishes—leaving behind a broad background $W_\varphi^{(\text{back})}$.

The quantum state

$$|\psi_s\rangle \equiv \mathcal{N}(m_0) \sum_{m=0}^{m_0} (1+m)^{-1} |m\rangle \tag{1}$$

of Ref. [1] is a superposition of m_0+1 photon number eigenstates $|m\rangle$. This state enjoys the semiclassical phase distribution [2–4]

$$W_\varphi[|\psi_s\rangle] = \frac{\mathcal{N}^2(m_0)}{2\pi} \left| \sum_{m=0}^{m_0} (1+m)^{-1} e^{-im\varphi} \right|^2 , \tag{2}$$

shown in Fig. 1 by a solid curve. The normalization condition $\langle \psi_s | \psi_s \rangle = 1$ yields, for $\mathcal{N}^2$, the equation [5]

$$1 = \mathcal{N}^2 \left[\frac{\pi^2}{6} - \zeta(2; m_0+2) \right] = \mathcal{N}^2 \left[\frac{\pi^2}{6} - \frac{1}{m_0+1} + O\left[\frac{1}{m_0^2} \right] \right] . \tag{3}$$

Here, $\zeta(s,v)$ denotes the generalized Riemann ζ function [6].

In the limit $m_0 \gg 1$ (corresponding to large mean photon number), this distribution exhibits a broad background and an extremely narrow peak at $\varphi=0$, as indicated in Figs. 1 and 2. We can approximate the background alone by extending the summation in Eq. (2) to infinity, and we arrive [3] at

$$W_\varphi^{(\text{back})}[|\psi_s\rangle] = \frac{3}{\pi^3}[\ln^2(2|\sin\varphi/2|) + \tfrac{1}{4}(\pi - |\varphi|)^2] \tag{4}$$

for $-\pi \le \varphi \le \pi$, but $\varphi \ne 0$. This curve is shown in Fig. 1 by a dashed curve. In the neighborhood of $\varphi=0$, we replace the central peak, depicted in Fig. 2 by a solid line, by the best-fitting Gaussian of identical height

$$W_\varphi^{(\text{peak})}[|\psi_s\rangle] = W_{\varphi=0} \exp\left[- \left[\frac{\varphi}{\Delta\varphi} \right]^2 \right] \tag{5a}$$

that has a width

$$\Delta\varphi = (2 W_{\varphi=0} / |W''_{\varphi=0}|)^{1/2} . \tag{5b}$$

Here the prime denotes the derivative with respect to φ. When we substitute the phase distribution Eq. (2) into Eq. (5b) for the width we find [3], in the limit of $m_0 \gg 1$,

$$\Delta\varphi = \sqrt{2}[C + \ln(m_0+1)]^{1/2}(m_0+1)^{-1} + O([C+\ln(m_0+1)]^{-1/2}(m_0+1)^{-1}), \tag{6}$$

where $C \cong 0.577\,215$ is Euler's constant. We now reexpress Eq. (6) in terms of the mean number of photons

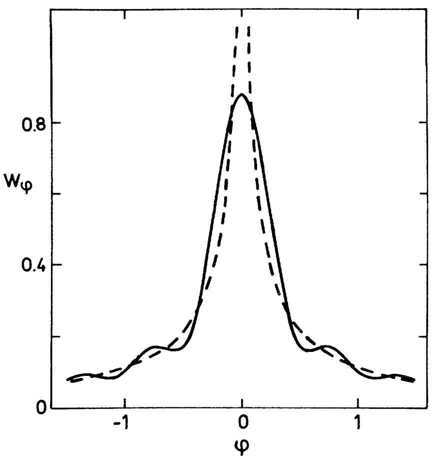

FIG. 1. The phase distribution $W_\varphi[|\psi_s\rangle]$, Eq. (2), of the state $|\psi_s\rangle$, Eq. (1), is displayed here by a solid curve for a sum cutoff of $m_0=10$, corresponding to a mean photon number $\langle m\rangle\cong 1$. This distribution exhibits a maximum at the phase $\varphi=0$ and broad oscillatory wings. We approximate these wings by the background distribution $W_\varphi^{(\mathrm{back})}[|\psi_s\rangle]$ of Eq. (4), shown here by a dashed curve. This approximate distribution results from extending the summation in Eq. (2) to infinity, and is hence singular at $\varphi=0$. The replacement of the finite Fourier sum in Eq. (2) by an infinite one "wipes out" the oscillations in the wings.

$\langle m+1\rangle$, rather than the cutoff m_0, by using [5,6]

$$\langle m+1\rangle=\mathcal{N}^2(m_0)\sum_{m=0}^{m_0}\frac{1}{m+1}$$
$$=\mathcal{N}^2(m_0)\left[C+\ln(m_0+1)+\frac{1}{2(m_0+1)}+O\left[\frac{1}{m_0^2}\right]\right]$$
$$=\frac{6}{\pi^2}[C+\ln(m_0+1)]+O\left[\frac{\ln(m_0+1)}{m_0+1}\right]. \quad (7a)$$

That is,

$$m_0+1\cong\gamma^{-1}\exp\left[\frac{\pi^2}{6}\langle m+1\rangle\right], \quad (7b)$$

where we have defined $\gamma=\exp(C)$. In determining the remainder, we have applied, in the last step of Eq. (7a), the asymptotic expression for $\mathcal{N}$, Eq. (3). With the help of Eqs. (7a) and (7b), Eq. (6) reads

$$\Delta\varphi=\frac{\gamma\pi}{\sqrt{3}}\langle m+1\rangle^{1/2}e^{-(\pi^2/6)\langle m+1\rangle}+O(\langle m+1\rangle^{-1/2}e^{-(\pi^2/6)\langle m+1\rangle}), \quad (8)$$

which shows that the Gaussian-approximated width $\Delta\varphi$ of the central peak decreases *exponentially* with the average number of photons. Applying the well-known Rayleigh's criterion of peak resolution from optics to the Gaussian, a shift of the central peak away from $\varphi=0$ could be resolved if the amount of the shift was greater than or equal to the exponentially small $\Delta\varphi$.

However, the phase probability or area "caught" underneath the peak is [7]

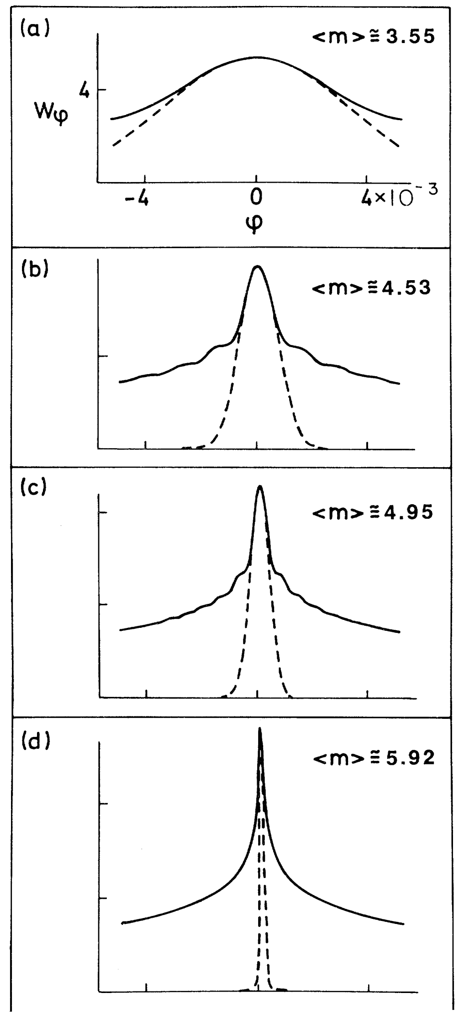

FIG. 2. The phase distribution $W_\varphi[|\psi_s\rangle]$, Eq. (2), of the state $|\psi_s\rangle$ is displayed here by solid lines. This distribution shows a maximum at $\varphi=0$ which develops a remarkably sharp peak when the mean photon number $\langle m\rangle$ increases consecutively from $\langle m\rangle\cong 3.55$ via $\langle m\rangle\cong 4.53$ and $\langle m\rangle\cong 4.95$ to $\langle m\rangle\cong 5.92$—depicted in (a), (b), (c), and (d), respectively. The upper cutoff m_0, used in the numerical evaluation of the sum, is related to the mean photon number $\langle m\rangle$ by $C+\ln(m_0+1)\cong(\pi^2/6)[\langle m\rangle+1]$. The values used were $m_0=10^3$, 5×10^3, 10^4, and 5×10^4 for (a), (b), (c), and (d), respectively. The best-fitting Gaussian distribution, Eq. (5a), of width $\Delta\varphi$, Eq. (5b), is shown here by dashed lines. This Gaussian represents an excellent approximation to this narrow peak and shows an exponential dependence of $\Delta\varphi$ on $\langle m\rangle$, Eq. (8). This narrow spike rests on a background which seems to be independent of m_0, provided $m_0\gg 1$. However, the area of this Gaussian, and hence the probability trapped under the peak, also vanishes exponentially with increasing mean photon number, Eq. (10). This result could have unfavorable implications concerning the usefulness of this state.

$$A_\varphi^{(\text{peak})}=\int_{-\infty}^{\infty} d\varphi\, W_\varphi^{(\text{peak})}=\sqrt{\pi}W_{\varphi=0}\,\Delta\varphi \tag{9}$$

or

$$A_\varphi^{(\text{peak})}=\frac{\pi^{5/2}\gamma}{12\sqrt{3}}\langle m+1\rangle^{5/2}e^{-(\pi^2/6)\langle m+1\rangle} +O(\langle m+1\rangle^{3/2}e^{-(\pi^2/6)\langle m+1\rangle})\ . \tag{10}$$

Hence, the area decreases rapidly to zero with increasing average photon number. Equation (9) makes the origin of this decrease apparent: The width of the Gaussian decreases exponentially with $\langle m+1\rangle$, Eq. (8), whereas the height

$$W_{\varphi=0}=(2\pi\mathcal{N}^2)^{-1}\left[\mathcal{N}^2\sum_{m=0}^{m_0}(1+m)^{-1}\right]^2 =(2\pi\mathcal{N}^2)^{-1}\langle m+1\rangle^2\cong\frac{\pi}{12}\langle m+1\rangle^2 \tag{11}$$

increases only *quadratically*. Here we have also made use of Eq. (7a).

The rapidly decreasing area underneath the peak for increasing $\langle m\rangle$ has important implications on the option of the reciprocal likelihood [8] as a measure of phase uncertainty $\delta\varphi$ for the state $|\psi_s\rangle$. We can visualize the quantity $\delta\varphi$ in the most elementary way for a probability distribution with a dominant central maximum and no broad wings. We crudely approximate this peak by a rectangle of identical height. We choose the width $\delta\varphi$ of the rectangle so as to have an area underneath the mock-up distribution identical to that under the original peak: The area of this rectangle, $W_{\varphi=0}\delta\varphi$, equals the area underneath the peak. When almost all of the probability concentrates in this dominant peak, we can approximate its area by its normalization condition—that is, by unity—and we find $\delta\varphi$ equals the unit area underneath the peak divided by $W_{\varphi=\varphi_{\max}}$, which equals $(W_{\varphi=\varphi_{\max}})^{-1}$.

For the state $|\psi_s\rangle$, Eq. (1), the expression for $W_{\varphi=0}$, Eq. (11), immediately yields the result of Ref. [1];

$$\delta\varphi\cong\frac{12}{\pi}\langle m+1\rangle^{-2}\ .$$

However, we recall that the phase probability curve $W_\varphi[|\psi_s\rangle]$ consists of a broad background $W_\varphi^{(\text{back})}$ that contains almost all of the probability, whereas the area underneath the central peak decreases rapidly with the average number of photons $\langle m\rangle$, as implied by Eq. (10). Hence, we cannot approximate this area by unity but have to use the explicit expression for $A_\varphi^{(\text{peak})}$, Eq. (10), that yields

$$\delta\varphi=A_\varphi^{(\text{peak})}/W_{\varphi=0} =\frac{\pi^{3/2}\gamma}{\sqrt{3}}\langle m+1\rangle^{1/2}\exp\left[-\frac{\pi^2}{6}\langle m+1\rangle\right]=\pi^{1/2}\Delta\varphi\ ,$$

which, apart from a factor of $\pi^{1/2}$, is identical to the quantity $\Delta\varphi$, Eq. (8), defined as the width of the best-fitting Gaussian.

In the limit of large photon numbers, that is, when $m_0\gg1$, the broad background, $W_\varphi^{(\text{back})}$, Eq. (4), governs the phase uncertainty. We illuminate this from a different angle by calculating the *periodic* phase uncertainty measure

$$D^2\varphi\equiv\langle\sin^2\varphi\rangle_\varphi=\int_{-\pi}^{\pi}d\varphi\, W_\varphi\sin^2\varphi\ . \tag{12}$$

This is a reasonable measure for distributions with a dominant maximum at $\varphi=0$. Why? Three arguments offer themselves: (1) For a state of random phase, that is, $W_\varphi=(2\pi)^{-1}$, we find $D^2\varphi=\frac{1}{2}$, (2) for a state of well-defined phase, $W_\varphi=\delta(\varphi)$, we arrive at $D^2\varphi=0$, and (3) for a narrow distribution with maximum at $\varphi=0$ we linearize the sine function and $D^2\varphi$ approximates the second moment of W_φ; that is,

$$D^2\varphi\cong\int_{-\infty}^{\infty}d\varphi\,\varphi^2W_\varphi\ .$$

So motivated, we now substitute the phase distribution $W_\varphi[|\psi_s\rangle]$, Eq. (2), into Eq. (12) and arrive after minor algebra [3] at

$$D^2\varphi=\frac{1}{2}\left[1-\frac{9}{2\pi^2}\right]+\frac{9}{4\pi^2}\left[\frac{2}{3}[m_0^{-1}+(m_0+1)^{-1}]-\frac{6}{\pi^2}\zeta(2;m_0+2)\right]\left[1-\frac{6}{\pi^2}\zeta(2;m_0+2)\right]^{-1}$$
$$\cong\frac{1}{2}\left[1-\frac{9}{2\pi^2}\right]+\frac{9}{2\pi^2}\left[\frac{2}{3}-\frac{3}{\pi^2}\right](m_0+1)^{-1}+O((m_0+1)^{-2})\ . \tag{13a}$$

Equation (7b) allows us to express this moment in terms of the average number of photons; that is,

$$D^2\varphi\cong\frac{1}{2}\left[1-\frac{9}{2\pi^2}\right] +\frac{9}{2\pi^2}\left[\frac{2}{3}-\frac{3}{\pi^2}\right]\gamma\exp\left[-\frac{\pi^2}{6}\langle m+1\rangle\right]\ . \tag{13b}$$

Hence, the periodic phase uncertainty measure $D^2\varphi$ decays *exponentially* [9] with increasing photon number and approaches the $\langle m\rangle$-independent, constant value of $[1-9/(2\pi^2)]/2\cong0.27$. We identify this contribution as the periodic phase uncertainty measure

$$D^2\varphi^{(\text{back})}\equiv\langle\sin^2\varphi\rangle_\varphi^{(\text{back})} =\int_{-\pi}^{\pi}d\varphi\, W_\varphi^{(\text{back})}\sin^2\varphi =\frac{1}{2}\left[1-\frac{9}{2\pi^2}\right] \tag{14}$$

of the background distribution, $W_\varphi^{(\text{back})}$, Eq. (4). The improvement over a homogeneous phase distribution, by roughly a factor of 2 (a constant 0.27 rather than 0.5), reflects the localization of the phase distribution around $\phi=0$, indicated in the figures.

We conclude by emphasizing again that the width of the central peak of the phase distribution of a novel quantum state of Shapiro and co-workers decreases *exponentially* with the average number of photons in this state. However, its phase probability rapidly disappears from the peak—rendering the utility of such a state open to question.

ACKNOWLEDGMENTS

We would like to thank R. E. Slusher and B. Yurke for drawing this problem to our attention. In particular, we thank V. Akulin, R. Bruch, C. M. Caves, R. Y. Chiao, R. Hellwarth, and Y. Yamamoto for many fruitful discussions. One of us (R.J.H.) expresses his thanks to the Alexander von Humboldt Foundation for a stipend, and another of us (J.P.D.) acknowledges H. Walther and the Max-Planck-Institut für Quantenoptik for hospitality and support and also the National Research Council for support.

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[7] Note that the Gaussian fit of the peak, $W_\varphi^{(\text{peak})}$, Eq. (5), is different from the properly normalized Gaussian $G(\varphi)\equiv\pi^{-1/2}(\Delta\varphi)^{-1}\exp[-(\varphi/\Delta\varphi)^2]$. Whereas $W_\varphi^{(\text{peak})}$ is tailored to have a height identical to $W_{\varphi=0}[|\psi_s\rangle]$, the height of $G(\varphi)$, that is, $\pi^{-1/2}(\Delta\varphi)^{-1}$, adjusts itself to the width of the Gaussian, as to keep the area normalized. The decay of $A_\varphi^{(\text{peak})}$ with increasing $\langle m\rangle$ is a consequence of the adjustment of its height to $W_{\varphi=0}$, rather than an adjustment of its area to maintain unity. Moreover, for this reason the width of the Gaussian is not identical to its second moment, as discussed in Ref. [9].

[8] Simulations of a sequence of phase measurements based on the scheme of Ref. [1] have been reported by C. M. Caves, A. S. Lane, and S. L. Braunstein, Abstract for NATO Workshop on Quantum Measurements in Optics, Cortina, 1991 (unpublished); also in *Proceedings of the Workshop on Squeezed States and Uncertainty Relations,* edited by D. Han, Y. S. Kim, and W. W. Zachary (NASA, Greenbelt, MD, 1991).

[9] This decrease does not originate from the second moment associated with the narrow peak, Eq. (5a), $\langle\sin^2\varphi\rangle^{(\text{peak})}\equiv\int_{-\pi}^{\pi}d\varphi\,W_\varphi^{(\text{peak})}\sin^2\varphi\cong\int_0^\infty d\varphi\,\varphi^2W_\varphi=(\sqrt{\pi}/2)W_{\varphi=0}\Delta\varphi^3$, which, according to Eq. (6), reads $\langle\sin^2\varphi\rangle^{(\text{peak})}\propto(m_0+1)^{-3}$. Note that, in deriving Eq. (13a), we have already neglected contributions of this order. The $(m_0+1)^{-1}$ decay, Eq. (13a), results from the decay of the oscillations in the wings, apparent in Figs. 2(b), 2(c), and 2(d); that is, from the approach of the phase distribution at finite m_0 towards the background distribution where $m_0\to\infty$. The presence of these oscillations can be interpreted as a consequence of using a truncated Fourier series to express the distribution. The oscillations are artifacts of the sharpness of the cutoff at m_0, which can most easily be seen by writing the distribution $W[|\psi_s\rangle]$, Eq. (2), in terms of the Lerch transcendent Φ function, $\Phi(z,s,v)=\sum_{k=0}^{\infty}(v+n)^{-s}z^n$ as $W[|\psi_s\rangle]=(\mathcal{N}^2/2\pi)|\Phi(e^{i\varphi},1,1)-\exp[i(m_0+1)\varphi]\Phi(e^{i\varphi},2,m_0+1)|^2$, that has a removable singularity at $\varphi=0$. (This closed-form expression could offer an analytical alternative to the troublesome numerical calculations needed by Caves *et al.* [8] in computing the Fischer information associated with this distribution.) The number of wiggles is completely governed by the factor $\exp[i(m_0+1)\varphi]$ in front of the second Φ function and hence there are m_0+1 oscillations, periodic with a period of $2\pi(m_0+1)^{-1}$, that go to zero with increasing m_0. When $m_0\to\infty$, the amplitude of the oscillations given by the second Φ function asymptotically approaches zero, as $\Phi(e^{-i\varphi},1,m_0+2)=(1-e^{i\varphi})^{-1}/(m_0+1)+O(m_0^{-2})$. This leaves us with only the first Φ function, which just results in $W_\varphi^{(\text{back})}$, Eq. (4).