Addendum to "Coulomb-diamagnetic problem in two dimensions"

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It is pointed out that the recently proposed set of conjectured solutions [Phys. Rev. A **39**, 5082 (1989)] for this problem predicts an unacceptable behavior of the energy function in the perturbative regimes. A modification of the conjectured conditions provides an alternate set of solutions that permits the development of a useful approximation scheme. An appealing approximate description of the entire spectrum that retains the previously obtained desirable features is thereby realized. Only low- and high-field regions are studied. A second derivation of the threshold regime spectrum is furnished.

An attempt to solve the four-term recursion relation for the planar Coulomb diamagnetic problem on the basis of two conjectured conditions was recently reported [1]. The results embody a number of attractive features. One finds a simple closed-form energy expression that reproduces the limits in full and provides a very welcome description of the quasi-Landau regime. The normalizability of the solutions was not established in that work.

In spite of such features, these solutions turn out to be unphysical. The predicted energy is a linear function of the magnetic field—a fact in violation of the requirements of the perturbative regimes. Intimately tied to this fact is the violation of variational bounds by the ground-state energy, first noted by Pandey and Varma [2]. Clearly, the physical set of exact solutions cannot be obtained, therefore, in the manner of Ref. [1].

There is, however, another challenging and important aspect of this problem that can be addressed meaningfully within a similar conjecture-based approach, this being to obtain a unified, consistent, and accurate description of the spectrum at least for realizable magnetic fields ($\hbar\omega_c \ll R$, ω_c being the cyclotron frequency). We wish to demonstrate that a natural, economical, and admissible modification of the previous conjecture (which its abovementioned failure, in fact, motivates) allows one to realize such an approximate description for the case of low and high magnetic fields, at least. Such a description emerges through the use of a set of formal solutions that involve two unknown functions that can be estimated systematically using perturbative and variational inputs. Specifically, in the following, we shall estimate these using first-order perturbative constraints near the two limits. Having thus secured a realistic behavior of the energy function near the limits for the case of low-lying states one will find that perturbation theory breaks down naturally and desirably for the higher states such that the anticipated features of the nonperturbative quasi-Landau regime emerge automatically.

$$R''(\xi)+\frac{1}{\xi}R'(\xi)+\left[\frac{4E}{\hbar\omega_c}-2m-\frac{m^2}{\xi^2}+\frac{\alpha}{\xi}-\xi^2\right]R(\xi)=0, \tag{1}$$

or, equivalently, the recursion relation

$$s(s+p-1)a_s\ [\alpha-\beta(p+2s-2)]a_{s-1} +(\delta-2s-4+\beta^2)a_{s-2}+2\beta a_{s-3}=0 \ . \tag{2}$$

We have set

$$R(\xi)=\xi^{|m|}\left[\exp-\left[\beta\xi+\frac{\xi^2}{2}\right]\right]\sum_{s=0} a_s\xi^s \ ,$$

$$a_0\neq 0 \ , \quad p=2|m|+1 \ , \quad \delta=\frac{4E}{\hbar\omega_c}-2m-p-1 \ ,$$

$$\alpha^2=\frac{16R}{\hbar\omega_c} \ , \quad \omega_c=\frac{|e|B}{\mu c} \ , \quad R=\frac{\mu e^4}{2\hbar^2} \ .$$

The original conjecture consisted in simultaneously imposing the two conditions that separately solve the Coulomb and Landau problems. It predicted an energy function $E(\alpha)$ with an unacceptable derivative $dE/d\alpha$ that renders it untenable. A permissible modification of the imposed conditions provides a very useful way out. The modified conjecture reads: a coefficient a_k does not contribute to the coefficient a_{k+1} and the coefficient a_{2k} does not contribute to the coefficient a_{2k+2} via the recursion relation Eq. (2) up to nonsingular functions $A(p,k,\alpha)$ and $B(p,k,\alpha)$, respectively, that vanish in either limit. Here, $k(=0,1,2,\dots)$ will continue to play the role of the radial quantum number as in Ref. [1].

With A and B so constrained the limits are recovered in full, as before [1]. We have no means to determine these functions exactly anymore. However, we can nail them down suitably by imposing some perturbative and variational requirements that are readily available for low-lying states. The worthiness of the approach will be judged through its performance for other states. Following Ref. [1], we now have, straightforwardly,

$$\beta=\frac{\alpha}{p+2k}+A(p,k,\alpha) \ , \tag{3}$$

$$\delta+\beta^2=4k+B(p,k,\alpha) \ , \tag{4}$$

and hence

$$\frac{4E}{\hbar\omega_c}=4k+2m+p+1 -\frac{\alpha^2}{(p+2k)^2}-\frac{2\alpha A}{(p+2k)}-A^2+B \ . \tag{5}$$

It is useful to parametrize A and B as

$$A=\frac{g}{\alpha}\exp(-f/\alpha^2)\ , \quad f>0 \tag{6}$$

$$B=\alpha h\exp(-\alpha q)\ , \quad q>0\ . \tag{7}$$

The functions f,g,h,q can be estimated using the known constraints in the neighborhood of the limits, the incorporation of which is, in fact, binding in a correct theory.

It is an inescapable consequence of strict variational considerations that, for the low-lying states, the energy in the limit $\omega_c\to 0$ may not contain a term linear in ω_c except the Zeeman term. The same is also demanded by perturbation theory. This is guaranteed by selecting

$$g=(2k+p)(4k+p+1)/2\ . \tag{8}$$

One must then have

$$\exp(-f/\alpha^2)=1+O(\omega_c)\ , \quad \omega_c\to 0\ .$$

Indeed, this requirement cannot only be met, but complete accord with the first-order perturbation theory is also ensured at the same time by choosing

$$f=\frac{2k+p}{2g}(g^2+\langle\xi^2\rangle_c)\ , \tag{9}$$

where

$$\langle\xi^2\rangle_c=\frac{(2k+p)^2}{8}[5(2k+p)^2-12m^2+7] \tag{10}$$

is the expectation value of ξ^2 in the Coulombic states. As required $f>0$.

Further orders of perturbation theory can be incorporated if necessary. This accord guarantees that the energy of any low-lying state follows the desirable path quite accurately as a function of ω_c, away from the Coulomb limit. The functional form of f ensures that as we move up the spectrum the perturbation theory will break down for suitably high levels well below the threshold.

The function B is constructed similarly by examining the Landau limit $\alpha\to 0$. Again using variational and perturbation-theory considerations one finds

$$h=-\left\langle\frac{1}{\xi}\right\rangle_L\ , \quad q=-\frac{1}{(p+2k)^2h}\ , \quad q>0 \tag{11}$$

where $\langle 1/\xi\rangle_L$ is the expectation value of $1/\xi$ in the Landau states. This ensures the proper continuation of $E(\alpha)$ away from the Landau limit. The functions f,g,h,q so fixed, already provide close accord with good variational bounds also, for low-lying states near the limits.

We now come to the discussion of our main result, namely, the energy spectrum given by Eq. (5). For the physically interesting case $\alpha\gg 1$, the deep levels now follow perturbation theory since the function B is negligible. As we move up the spectrum, the function A begins to decrease and the energy gradually develops a linear ω_c dependence leading to the eventual breakdown of Coulombic perturbation theory.

Moving further up we reach the nonperturbative threshold regime which is the most challenging part of the spectrum. To discuss this we introduce the principal quantum number $n=k+|m|$. For $\alpha\gg 1$, n is large in the threshold region. Consider the case $n\gg|m|$ and m fixed. For $E\simeq 0$ we obtain from Eq. (5) the following expression for energy:

$$E\simeq\frac{\hbar\omega_c n}{2}-\frac{R}{n^2}\ . \tag{12}$$

This gives straightforwardly, for the spacings in the quasi-Landau regime,

$$\frac{1}{\hbar\omega_c}\frac{\partial E}{\partial n}\Bigg|_{\substack{E=0\\ m\ll n}}\simeq\frac{1}{2}+\frac{2R}{n^3\hbar\omega_c} \tag{13}$$

with

$$n^3-2\left[\frac{E}{\hbar\omega_c}\right]n^2-2\frac{R}{\hbar\omega_c}=0\ . \tag{14}$$

This shows that the spacing at $E=0$ is very nearly $\frac{3}{2}\hbar\omega_c$ [3]. The other features of the spacing remain the same as in Ref. [1].

Continuing upwards in energy the function A decreases exponentially. The entire Landau term is restored whereupon the Landau-regime perturbation theory controlled by the function B begins. Thus, for any α whatsoever, the high enough levels of positive energy remain perturbative. This is consistent with the Wentzel-Kramers-Brillouin (WKB) requirement.

To summarize, the entire spectrum divides into three characteristic regimes [4], namely, the two perturbative ones and the nonperturbative quasi-Landau domain with a typically equally spaced spectrum described by the remarkably simple energy formula given by Eq. (12).

We now come to an additional approximate description of the quasi-Landau regime. For this we introduce the parameter $\varepsilon\equiv\hbar\omega_c/2R$. Consider $\varepsilon\ll 1$ and focus attention on levels near the threshold so that the principal quantum number $n\gg 1$. Rewriting the recursion relation Eq. (2) in Coulombic units we have

$$s(s+p-1)a_s+[2-\beta(p-2s+2)]a_{s-1}+\left[\left[\frac{\delta}{2}-s+2\right]\varepsilon+\beta^2\right]a_{s-2}+\beta\varepsilon a_{s-3}=0\ . \tag{15}$$

In the spirit of our approximation the last term on the right-hand side of Eq. (5) can be ignored and so also can an order ε contribution to β. The solutions are then immediately seen to be polynomials [5] of degree k $(k=0,1,2,\dots)$ such that

$$\beta=\frac{2}{p+2k}\ \text{and}\ \delta=2k-2\beta^2/\varepsilon\ . \tag{16}$$

This gives

$$E_{nm}=\frac{\hbar\omega_c}{2}(n+m)-\frac{R}{n^2}\ , \quad n=k+|m| \tag{17}$$

and

$$\frac{1}{\hbar\omega_c}\frac{\partial E}{\partial n}\Bigg|_{\substack{E=0\\ m}}=\frac{3}{2}\ , \quad n\gg|m|\ . \tag{18}$$

This directly substantiates our previous result near $E=0$. Interestingly, a second pacing of $\frac{1}{2}\hbar\omega_c$ is also indicated by Eq. (17).

Going back to Eq. (15) one notices easily that it admits a set of formal solutions which correspond precisely to the energies given by Eq. (17), obtained by demanding that a given coefficient a_k does not contribute to the next two coefficients a_{k+1} and a_{k+2}. This leads to the two conditions in Eq. (16) and hence to Eqs. (17) and (18). In view of the above results one can say with justifiable confidence that the solutions in the threshold region indeed are obtained essentially in this manner. This is obviously true for $\varepsilon \ll 1$, at least. Coupled with the fact that the deep and the high levels are well described by perturbation theory, we have a description of the entire problem again.

The modified conjecture integrates all the three regimes into a set of two basic conditions. A little reflection will show that in the threshold regime it reduces *de facto* to the conditions noted in the previous paragraph and which already have been made plausible.

Two remarks are in order. First, we would like to point out that the energy expression of Eq. (5) represents the simplest dimensionally permissible structure that is consistent with the limiting and perturbative constraints and capable of accommodating the entire spectrum accurately. That such a structure emerges naturally from the Schrödinger equation via the conjecture is gratifying.

Second, it must be stressed that a specific problem whose set of unique features have no known parallels in all of quantum mechanics has been addressed here. Hence, in no way is a similar conjecture-based approach suggested to be a recipe for other multiple-term recursion relation problems in general.

To summarize, the fact that the conjectured solutions correctly contain the limits, are in accord with the attendant low-order perturbation theories, and predict a verifiably satisfactory description of the nonperturbative threshold regime is a clear testimony to the effect that the modified conjecture is able to extract the essential underlying physics of the problem, at least for realizable fields.

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[1] S. C. Chhajlany, V. N. Malnev, and N. Kumar, Phys. Rev. A **39**, 5082 (1989).

[2] R. K. Pandey and V. S. Varma, Phys. Rev. A **42**, 6928 (1990).

[3] Using Eqs. (5)–(11) one finds the spacing at $E=0$ for $m=0$ to be about $1.485\hbar\omega_c$.

[4] See, for example, A. R. P. Rau, Nature **325**, 577 (1987).

[5] The approximate polynomial solutions for $\varepsilon \ll 1$ obtained in this manner must be contrasted with the other polynomial solutions of Eq. (15) that are obtained for an infinite set of precisely tuned values of ε. Such tuned solutions, which correspond to positive energies only, have been noted in Ref. [2] above. These solutions are unphysical. See S. C. Chhajlany and V. N. Malnev, Phys. Rev. A **43**, 581 (1991) for details.