Higher-order squeezing properties and correlation functions for squeezed number states

Paulina Marian
Laboratory of Physics, Department of Chemistry, University of Bucharest, Boulevard Carol I 13, Bucharest, 70346 Romania
(Received 26 December 1990)

Higher-order squeezing conditions for squeezed number states are derived using a normal-ordering technique for calculating the moments of the field. Intrinsic higher-order squeezing is also investigated. It is found that the normally ordered moments of the quadrature operators are oscillating functions of the squeeze parameter. Also calculated is the exact nth-order correlation function for an arbitrary squeezed number state. Finally, its behavior for large and weak squeezing is analyzed.

I. INTRODUCTION

Coherent states are conventionally defined with respect to a set of boson creation and annihilation operators $a^\dagger$ and a respectively, as the eigenstates of the annihilation operator a. Of particular interest for quantum optics are the squeezed coherent states, which are the eigenstates of a destruction operator a_s associated with the operators a and $a^\dagger$ via a Bogoliubov transformation, [1–4]

$$a_s = SaS^\dagger = a\cosh r - a^\dagger \exp(i\theta)\sinh r \tag{1a}$$

$$a_s^\dagger = Sa^\dagger S^\dagger = a^\dagger \cosh r - a\exp(-i\theta)\sinh r\ . \tag{1b}$$

In Eqs. (1), $S = S(z)$ is the squeeze operator,

$$S(z) = \exp[\tfrac{1}{2}z(a^\dagger)^2 - \tfrac{1}{2}z^* a^2]\ , \tag{2}$$

with

$$z = r\exp(i\theta)\ , \tag{3}$$

where $r = |z|$ is the squeeze parameter.

The squeezed number states are defined by the action of the squeeze operator $S(z)$ on a Fock state:

$$|m\rangle_s = S(z)|m\rangle\ . \tag{4}$$

While the number state $|m\rangle$ is determined only by its photon number, the new state (4) is phase dependent, with important squeezing properties. The photon statistics is also very interesting. These properties strongly depend on the squeeze parameter r and the value of m [5]. The squeezed number states were introduced by Yuen [2] as the eigenstates of the "quasiphoton"-number operator $N_s = a_s^\dagger a_s$. Indeed

$$N_s|m\rangle_s = Sa^\dagger a S^\dagger S|m\rangle = m|m\rangle_s\ . \tag{5}$$

The representation of these states in the Fock basis is specified by the matrix elements of the squeeze operator,

$$G_{nm} = \langle n|S(z)|m\rangle = \langle n|m\rangle_s\ , \tag{6}$$

which were calculated in Ref. [6]. An extensive study of the squeezed number states has recently been made by Kim, de Oliveira, and Knight [5]. They have found the variances of the quadrature operators, the second-order correlation function, and its behavior for large m, large r, and small r. The quasiprobability functions and the photon number distribution were also derived and discussed in great detail.

The main purpose of the present work is to investigate higher-order squeezing [7] and to derive the nth-order correlation function [8] for a squeezed number state. In Sec. II we show that the photon number distribution in such a state is, in fact, proportional to the square of a Gauss hypergeometric function. The matrix elements of the field operators in the basis of the squeezed number states are found in Sec. III, using a normal-ordering technique developed by Wilcox [9]. Higher-order moments of the quadrature operators are also calculated and a compact analytical condition for the Nth-order squeezing is derived and compared with a recent result of Gong and Aravind [10]. The normally ordered moments of the quadrature operators are then obtained and a discussion on intrinsic higher-order squeezing is given. The exact normalized nth-order correlation function is derived and discussed in Sec. IV. In particular, our results for the squeezed vacuum state are found to be in agreement with some previous ones [11]. We conclude by stressing the importance of the squeezed number states for subsequent applications.

II. PHOTON NUMBER DISTRIBUTION

The matrix elements (6) can be calculated in a direct fashion using a normal-order form of the squeeze operator [3,12]

$$S(z) = (\cosh r)^{-1/2}\exp[\tfrac{1}{2}\exp(i\theta)(\tanh r)(a^\dagger)^2] \times \left[\sum_{n=0}^{\infty}\frac{(\operatorname{sech} r - 1)^n}{n!}(a^\dagger)^n a^n\right] \times\exp[-\tfrac{1}{2}\exp(-i\theta)(\tanh r)a^2]\ . \tag{7}$$

With the squeeze operator written in the form (7) we get after a simple algebra

$$G_{nm}=\begin{cases}\dfrac{(-1)^{m/2}}{\left[\dfrac{m}{2}\right]!\left[\dfrac{n}{2}\right]!}\left[\dfrac{n!m!}{\cosh r}\right]^{1/2}\exp\left[\dfrac{i(n-m)\theta}{2}\right]\left[\dfrac{\tanh r}{2}\right]^{(n+m)/2} \\ \qquad\times{}_2F_1\left[-\dfrac{m}{2},-\dfrac{n}{2};\dfrac{1}{2};-\dfrac{1}{(\sinh r)^2}\right], \quad \text{for } m,n \text{ even} \\ \dfrac{(-1)^{(m-1)/2}}{\left[\dfrac{m-1}{2}\right]!\left[\dfrac{n-1}{2}\right]!}\left[\dfrac{n!m!}{(\cosh r)^3}\right]^{1/2}\exp\left[\dfrac{i(n-m)\theta}{2}\right]\left[\dfrac{\tanh r}{2}\right]^{(n+m)/2-1} \\ \qquad\times{}_2F_1\left[-\dfrac{m-1}{2},-\dfrac{n-1}{2};\dfrac{3}{2};-\dfrac{1}{(\sinh r)^2}\right], \quad \text{for } m,n \text{ odd} \\ 0, \quad \text{otherwise}\ . \end{cases} \tag{8}$$

The Gauss hypergeometric functions ${}_2F_1$ entering the expressions (8) are polynomials [13]. Essentially $S(z)$ creates two excitations every time it acts, therefore only odd-odd and even-even elements G_{nm} are nonvanishing. The matrix elements G_{nm} were also obtained in Ref. [6] by a different method and in a slightly different form. It is worth mentioning that we have recovered the result (8) by applying the method indicated by Yuen [2]. This method is an indirect one and makes use of the Fock representation of a squeezed state [14].

The photon number distribution $P_n(|m\rangle_s)=|G_{nm}|^2$ was derived by Kim, de Oliveira, and Knight [5]. We stress that their result can be written in a compact form involving the square of a certain Gauss function. Then, by a suitable transformation of this ${}_2F_1$ function [15] it takes the simpler form given by our formulas (8), which display the symmetry property

$$P_n(|m\rangle_s)=P_m(|n\rangle_s)\ . \tag{9}$$

An extensive discussion of the photon number distribution for large squeezing and large m is also made in Ref. [5].

III. HIGHER-ORDER SQUEEZING

The matrix elements G_{nm} are complicated enough to use them for the calculation of some expectation values in a squeezed number state. To this end we now obtain from Eqs. (1) that

$$S^\dagger a^n S=[a\cosh r+a^\dagger\exp(i\theta)\sinh r]^n\ , \tag{10a}$$

$$S^\dagger(a^\dagger)^n S=[a^\dagger\cosh r+a\exp(-i\theta)\sinh r]^n\ , \tag{10b}$$

as a consequence of the unitarity of the operator $S(z)$. The matrix element of the operator $(a^\dagger)^n$ with respect to the squeezed number states $|m\rangle_s$ and $|k\rangle_s$ is

$$ {}_s\langle m|(a^\dagger)^n|k\rangle_s =\langle m|[a^\dagger\cosh r+a\exp(-i\theta)\sinh r]^n|k\rangle\ . \tag{11}$$

We are left to calculate a matrix element in the Fock basis. By introducing the operators

$$b=a\exp(-i\theta)\sinh r,\quad b'=a^\dagger\cosh r\ , \tag{12}$$

which satisfy the commutation relation

$$[b,b']=\exp(-i\theta)\sinh r\cosh r\equiv c\ , \tag{13}$$

we can use a normal-ordering formula given by Wilcox [16],

$$(b+b')^n=\sum_{k=0}^{\{n/2\}}\sum_{s=0}^{n-2k}(\tfrac{1}{2}c)^k\frac{n!(b')^s b^{n-2k-s}}{k!s!(n-2k-s)!}\ . \tag{14}$$

After simple algebra we get

$$ {}_s\langle m|(a^\dagger)^n|k\rangle_s=\exp[i(p-n)\theta]\left[\frac{k!}{m!}\right]^{1/2}\frac{n!}{p!(n-2p)!} \times(\sinh r)^n(2\tanh r)^{-p}{}_2F_1(-p,-m;n-2p+1;2)\delta_{m+n,k+2p},\quad p \text{ integer}\ . \tag{15}$$

In what follows we denote by $\langle A\rangle_{\rm SN}$ the expectation value of an operator A in the squeezed number states $|m\rangle_s$.

For $k=m$, Eq. (15) yields

$$\langle (a^\dagger)^n \rangle_{SN} = \begin{cases} \exp\left[-\frac{in\theta}{2}\right](n-1)!!(\sinh r \cosh r)^{n/2}\, {}_2F_1\left[-m,-\frac{n}{2};1;2\right], & n \text{ even}, \\ 0, \quad n \text{ odd}. \end{cases} \tag{16}$$

By specializing our result (16) to $n=2$ we recover a known formula [5]

$$\langle (a^\dagger)^2 \rangle_{SN} = \exp(-i\theta)(2m+1)\sinh r \cosh r \; . \tag{17}$$

An equivalent form of Eq. (15) can be readily obtained by use of some transformation rules for the Gauss hypergeometric function [15],

$$\langle m | S^\dagger (a^\dagger)^n S | k \rangle = \frac{\exp[i(p-n)\theta](p+k)! 2^{-p}}{p!(n-p)!(m!k!)^{1/2}} (\sinh r)^n (\tanh r)^{-p}\, {}_2F_1(-p,-m;-k-p;-1)\delta_{m+n,k+2p}, \quad p \text{ integer} \; . \tag{18}$$

This formula proves to be useful in the calculation of the nth-order correlation function.

The higher-order moments of the field are involved in the generalization of the squeezing concept. According to Hong and Mandel [7], a field is squeezed to any even order N if

$$\langle (\Delta X_j)^N \rangle < (N-1)!! \; , \tag{19}$$

where $\Delta X_j = X_j - \langle X_j \rangle$ and X_j is one of the quadrature operators defined as

$$X_1 = a + a^\dagger, \quad X_2 = -i(a - a^\dagger) \; . \tag{20}$$

These operators satisfy the commutation relation

$$[X_1, X_2] = 2i \; . \tag{21}$$

If the Nth-order squeezing condition (19) holds for the quadrature X_1 the expectation value $\langle (\Delta X_1)^N \rangle$ is less than it is for a coherent state. There is no classical description for a state that exhibits higher-order squeezing. In a squeezed number state we obtain from (16)

$$\langle X_1 \rangle_{SN} = 0 \tag{22}$$

so the condition for Nth-order squeezing reads

$$\langle (a+a^\dagger)^N \rangle_{SN} < (N-1)!!, \quad N \text{ even} \; . \tag{23}$$

From Eqs. (10) one gets

$$\langle (a+a^\dagger)^N \rangle_{SN} = \langle m | (\alpha a^\dagger + \alpha^* a)^N | m \rangle \tag{24}$$

with

$$\alpha = \cosh r + \exp(i\theta)\sinh r \; . \tag{25}$$

We apply (14) to have finally

$$\langle (\Delta X_1)^N \rangle_{SN} = |\alpha|^N (N-1)!!\, {}_2F_1\left[-\frac{N}{2},-m;1;2\right] \; . \tag{26}$$

Nth-order squeezing is possible for some phase angle for which

$$|\alpha|^N\, {}_2F_1\left[-\frac{N}{2},-m;1;2\right] < 1 \; . \tag{27}$$

In the case $\theta=\pi$, which is the phase choice in Ref. [5] we get the condition

$$r > \frac{1}{N} \ln \left| {}_2F_1\left[-\frac{N}{2},-m;1;2\right] \right| \; . \tag{28}$$

The squeezing in the different orders sets in at different squeeze parameters $r_{\min}^{(N)}$. A condition for Nth-order squeezing in a squeezed number state was recently obtained by Gong and Aravind [10]. Our result is in full agreement with their condition which by a simple algebra can be cast into the compact form (28).

In Table I we present a numerical evaluation of the minimum squeeze parameter $r_{\min}^{(N)}$ for higher-order squeezing in the cases of $N=2,4,6,8$ for squeezed number states with $m=1,2,5,10$. It is interesting to note that $r_{\min}^{(N)}$ decreases when N increases, at a fixed m, namely, for $r_{\min}^{(N)} < r < r_{\min}^{(2)}$ the squeezed number states exhibit Nth-

TABLE I. The minimum squeeze parameter $r_{\min}^{(N)}$ for higher-order squeezing in the cases $N=2,4,6,8$ for squeezed number states with $m=1,2,5,10$.

m \ N	2	4	6	8
1	0.549	0.402	0.324	0.274
2	0.805	0.641	0.536	0.464
5	1.199	1.027	0.907	0.815
10	1.522	1.349	1.225	1.129

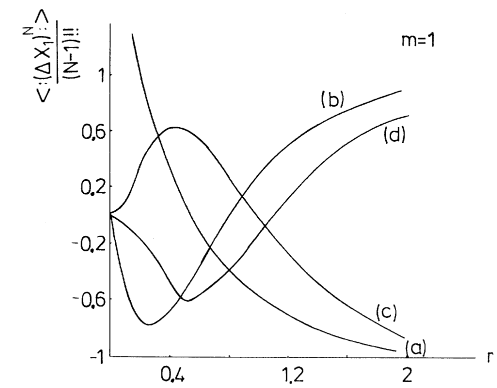

FIG. 1. The dependence of $\langle :(\Delta X_1)^N: \rangle_{SN}/(N-1)!!$ on the squeeze parameter r for $N=2$ [curve (a)]; $N=4$ [curve (b)], $N=6$ [curve (c)], and $N=8$ [curve (d)] in the squeezed number state with $m=1$.

order squeezing but not second-order squeezing [17]. Consequently we ask if the squeezing is intrinsically of higher order (in the sense of Ref. [7]). To answer this question we have to calculate the normally ordered moments, $\langle :(\Delta X_1)^N: \rangle_{SN}$, and to examine their sign.

For a coherent state all the normally ordered moments vanish. The condition for intrinsic higher-order squeezing is

$$\langle :(\Delta X_1)^N: \rangle < 0 \; . \tag{29}$$

Following Hong and Mandel [18] we have found the normally ordered moments as a finite expansion of the higher-order moments

$$\langle :(\Delta X_1)^N: \rangle = N! \sum_{l=0}^{\frac{N}{2}} \frac{(-1)^l \langle (\Delta X_1)^{N-2l} \rangle}{l!(N-2l)!2^l} \; . \tag{30}$$

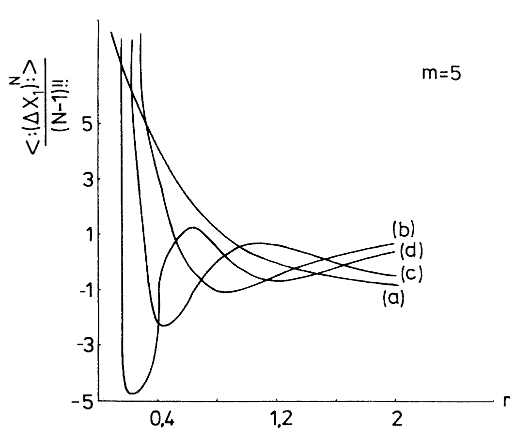

FIG. 2. As in Fig. 1 for $m=5$.

In the case of a squeezed number state, making use of our result (26) and a known summation formula for Gauss hypergeometric functions [19], Eq. (30) gives

$$\langle :(\Delta X_1)^N: \rangle_{SN} = (-1)^{N/2}(N-1)!!(1-|\alpha|^2)^{N/2} \times {}_2F_1\left[-\frac{N}{2}, -m; 1; -\frac{2|\alpha|^2}{1-|\alpha|^2}\right] , \tag{31}$$

where α is given by Eq. (25). For $r=0$ in Eq. (31) the normally ordered moments characteristic to a number state $|m\rangle$ are obtained

$$\langle m|:(\Delta X_1)^N:|m\rangle = \frac{2^{N/2}(N-1)!!m!}{\left[\frac{N}{2}\right]!\left[m-\frac{N}{2}\right]!} , \tag{32}$$

a result given also by a direct calculation.

The normally ordered moments (31) are interesting because they are oscillating functions depending on the squeeze parameter, in contrast with the similar ones for a squeezed coherent state [20] whose sign is entirely determined by the parity of $N/2$. We have plotted the dependence of $\langle :(\Delta X_1)^N: \rangle_{SN}/(N-1)!!$ on the squeeze parameter r, for $N=2,4,6,8$ and $m=1$ (Fig. 1), and the same for $m=5$ (Fig. 2). It is clear from these figures that intrinsic squeezing is possible for every N in some ranges of r. However, for large squeeze parameter r the normally ordered moments have the same limit as in the case of a squeezed coherent state [20]

$$\lim_{r\to\infty} \langle :(\Delta X_1)^N: \rangle_{SN} = (-1)^{N/2}(N-1)!! \tag{33}$$

For large squeezing there is an Nth-order intrinsic squeezing for all values of N for which $N/2$ is odd.

IV. HIGHER-ORDER CORRELATION FUNCTIONS

It is well known that in a single-mode multiphoton absorption process the rate of change of the average photon number in the field strongly depends on the statistical properties of the light [21],

$$\frac{\partial}{\partial t}\langle a^\dagger a \rangle = -2n\mathcal{J}^{(n)}\langle (a^\dagger)^n a^n \rangle \; . \tag{34}$$

$\mathcal{J}^{(n)}$ is the coefficient for n-photon absorption process.

On the other hand the statistical properties of the light passing through the absorber change in time due to the nonlinear process itself. This time evolution is considerably influenced by the initial statistics of the light beam. Therefore calculation of the nth-order correlation function for a squeezed number state is important for the evaluation of the efficiency of a multiphoton process using a beam prepared in such a state. In order to give a compact expression for the expectation value $\langle (a^\dagger)^n a^n \rangle_{SN}$ we make use of the expansion

$$\langle (a^\dagger)^n a^n \rangle_{SN} = \sum_k |\langle m|S^\dagger (a^\dagger)^n S|k\rangle|^2 \; . \tag{35}$$

After inserting Eq. (18) and some minor rearrangements the nth-order correlation function is written as

$$\langle (a^\dagger)^n a^n \rangle_{SN} = \frac{(n!)^2}{m!}(\sinh r)^{2n} \sum_{p=o}^{n} \left[\frac{(m+n-p)!}{p!(n-p)!} \right]^2 \frac{(2\tanh r)^{-2p}}{(m+n-2p)!}[{}_2F_1(-p,-m;-m-n+p;-1)]^2 . \tag{36}$$

In the case $n=1$ we obtain the mean photon number

$$\langle a^\dagger a \rangle_{SN} = m + (2m+1)(\sinh r)^2 . \tag{37}$$

Another parameter which is relevant to the photon statistics is the second-order degree of coherence defined as

$$g^{(2)}(0) = \frac{\langle (a^\dagger)^2 a^2 \rangle}{\langle a^\dagger a \rangle^2} . \tag{38}$$

For $n=2$ in Eq. (36) we get

$$g^{(2)}(0) = 1 + \frac{1}{\langle a^\dagger a \rangle^2}[-m + (2m^2+1)(\sinh r)^2 + 2(m^2+m+1)(\sinh r)^4] \tag{39}$$

The particular results (37) and (39) were also obtained in Ref. [5].

A particularly simple form has the nth-order correlation function for the squeezed vacuum state. We place $m=0$ in Eq. (36) and get

$$\langle (a^\dagger)^n a^n \rangle_{SV} = n!(\sinh r)^{2n} {}_2F_1 \left[-\frac{n}{2}, -\frac{n-1}{2}; 1; (\tanh r)^{-2} \right] \tag{40}$$

and

$$g^{(n)}(0) = n!\, {}_2F_1 \left[-\frac{n}{2}, -\frac{n-1}{2}; 1; (\tanh r)^{-2} \right] . \tag{41}$$

The nth-order correlation function for a squeezed vacuum state is derived in a different way by Janszky and Yushin [11]. After some algebraic transformations their result can be written in the form (40).

Only for weak squeezing, $(\sinh r)^2 \lesssim m/(2m^2+1)$, and $m>0$ the field has sub-Poissonian statistics. We discuss now the form of Eq. (36) for *strong* squeezing, $(\sinh r)^2 \gg 1$,

$$\langle (a^\dagger)^n a^n \rangle_{SN} \cong \frac{(n!)^2 (\sinh r)^{2n}}{m!} \sum_{l=0}^{n} \left[\frac{(m+n-l)!}{l!(n-l)!} \right]^2 \frac{2^{-2l}}{(m+n-2l)!}[{}_2F_1(-m,-l;-m-n+l;-1)]^2 . \tag{42}$$

The normalized nth-order correlation function $g^{(n)}(0) = \langle (a^\dagger)^n a^n \rangle_{SN} / (\langle A^\dagger a \rangle_{SN})^n$ does not depend on the squeeze parameter in the case of strong squeezing. Indeed

$$(\langle a^\dagger a \rangle_{SN})^n \simeq (2m+1)^n (\sinh r)^{2n} . \tag{43}$$

We give the values of the normalized nth-order correlation function in some particular squeezed number states. In the squeezed vacuum state, $m=0$, we recover a main result of Ref. [11]

$$g^{(n)}(0) = (2n-1)!! \tag{44}$$

For $m=1$ Eqs. (42) and (43) give

$$g^{(n)}(0) = (2n-1)!! \frac{2n+1}{3^n} \tag{45}$$

while for $m=2$ one finds

$$g^{(n)}(0) = (2n-1)!! \frac{2n^2+2n+1}{5^n} . \tag{46}$$

The normalized nth-order correlation function is maximum in the squeezed vacuum state. This state is superchaotic for every squeeze parameter r as can be stated from Eq. (39) for $m=0$.

A compact formula for the nth-order correlation function can be found in the case of *weak* squeezing, $(\sinh r)^2 \ll 1$. For $m \geq n$ the main contribution in Eq. (36) is given by

$$\langle (a^\dagger)^n a^n \rangle_{SN} \simeq \frac{m!}{(m-n)!} + \frac{m!}{(m-n)!} n (\sinh r)^2 \left[1 + \frac{n}{4} \frac{(2m+3-n)^2}{(m-n+1)(m-n+2)} \right] . \tag{47}$$

The first term in (47) is characteristic to a Fock state $|m\rangle$. In the case $m<n$ we approximate $(\tanh r)^{-2}\simeq(\sinh r)^{-2}$ and get

$$\langle (a^{\dagger})^{n}a^{n}\rangle_{\rm SN}\simeq\begin{cases}\dfrac{(n!)^2}{m!\{[(n-m)/2]!\}^2}\left[\dfrac{\sinh r}{2}\right]^{n-m}, & \text{for } (n-m) \text{ even}\\[2ex] \dfrac{[(n+1)!]^2}{m!\{[(m-n+1)/2]!\}^2}\left[\dfrac{\sinh r}{2}\right]^{n-m+1}, & \text{for } (n-m) \text{ odd}\ .\end{cases} \tag{48}$$

For the squeezed vacuum state, Eq. (48) becomes

$$\langle (a^{\dagger})^{n}a^{n}\rangle_{\rm SV}\simeq\begin{cases}[(n-1)!!]^2(\sinh r)^{n}, & n \text{ even}\\ (n!!)^2(\sinh r)^{n+1}, & n \text{ odd}\ .\end{cases} \tag{49}$$

The expectation value (49) is in agreement with the previous one in Ref. [11].

V. CONCLUSIONS

The photon number states are nonclassical. To produce experimentally such states some methods have already been proposed [22]. Using the number states as an input field in a parametric amplifier a squeezed number state can, in principle, be obtained. Therefore we have examined in this paper some higher-order squeezing and correlation properties of the squeezed number states.

These states exhibit squeezing in every even order N for $r>r_{\rm min}^{(N)}$. We have found a simple expression of $r_{\rm min}^{(N)}$ for any m. We have also studied the normally ordered moments $\langle :(\Delta X_1)^N:\rangle_{\rm SN}$ and found an oscillatory dependence on the squeeze parameter r, the number of oscillations being determined by m and N. The nth-order correlation function was also calculated in closed form and discussed in the case of strong and weak squeezing. The calculation shows a great efficiency of the squeezed number states in multiphoton processes.

[1] D. Stoler, Phys. Rev. D **1**, 3217 (1970); **4**, 1925 (1971).
[2] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).
[3] J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).
[4] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[5] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Phys. Rev. A **40**, 2494 (1989).
[6] M. Venkata Satyanarayana, Phys. Rev. D **32**, 400 (1985).
[7] C. K. Hong and L. Mandel, Phys. Rev. A **32**, 974 (1985).
[8] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[9] R. M. Wilcox, J. Math. Phys. **8**, 962 (1967).
[10] J. J. Gong and P. K. Aravind, in *Coherence and Quantum Optics VI,* edited by J. H. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1989), p. 425.
[11] J. Janszky and Y. Yushin, Phys. Rev. A **36**, 1 288 (1987).
[12] R. A. Fisher, M. M. Nieto, and V. D. Sandberg, Phys. Rev. D **29**, 1107 (1984).
[13] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1, Chap. 2.
[14] H. P. Yuen, Ref. [2], Eq. (3.23).
[15] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions,* Ref. [13] [Sec. 2.10, Eqs. (1)–(6)].
[16] R. M. Wilcox, Ref. [9], Eq. (10.43).
[17] In Sec. VII of Ref. [7] a discussion on the physical realizability of states which exhibit higher-order squeezing but not second-order squeezing is given. The squeezed number states are such states, but only for $r_{\rm min}^{(N)}<r<r_{\rm min}^{(2)}$.
[18] C. K. Hong and L. Mandel, Ref. [7], Eq. (8).
[19] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions,* Ref. [13] [Sec. 2.5, Eq. (9) for $\zeta=0$].
[20] C. K. Hong and L. Mandel, Ref. [7], Eq. (30).
[21] G. S. Agarwal, Phys. Rev. A **1**, 1445 (1970) [Eqs. (4.13)–(4.15)].
[22] H. P. Yuen, Phys. Rev. Lett. **56**, 2176 (1986); B. Yurke and D. Stoler, Phys. Rev. A **36**, 1955 (1987).