Higher-order effects on pair creation by relativistic heavy-ion beams

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Assuming the approximation of pair independence, the number of pairs produced by a classical, imperturbable electromagnetic source is shown to be described by a Poisson distribution. The perturbational calculation of pairs by colliding heavy-ion beams is treated by straightforward summation of the diagrammatically defined series. The extended calculation, in which all orders of interaction between colliding ions and the electron-positron system are included, is also demonstrated to result in a Poisson-number distribution. Applications to current heavy-ion colliders are made.

I. INTRODUCTION

The production of electron-positron pairs by the electromagnetic fields created as charged particles pass one another at close distances has been studied over a long time [1–5]. In recent years, the development of heavy-ion colliders in which very energetic and fully stripped high-Z ions are to circulate in countercurrent beams makes new demands on estimates of these processes. The large ion charges and energies mean, on the theoretical side, that the effective interaction strengths have been moved out of the weak interaction regime, and, on the practical side, that we are dealing with important beam loss and background problems. As an example, the Brookhaven National Laboratory Relativistic Heavy Ion Collider (RHIC) [6] intends to collide ions perhaps as heavy as uranium ($Z=92$) and as energetic as 250 (Z/A) GeV/nucleon ($\gamma \sim 2\times 10^4$ in the frame of either ion, $\gamma = 2\gamma_{\rm col}^2 - 1$); both the beam loss and background problems have been most recently addressed in workshops [7,8].

The theoretical questions posed by the increased interaction strength have also come into recent discussion [7,9]. The basic problem is best posed by a review of the perturbational calculations: The perturbative cross section corresponds to the diagram shown in Fig. 1(a), in which the electron-positron field exchanges two photons, one with each ion. The diagram shows the heavy ions moving on unchanging straight line trajectories separated by an impact parameter b. The cross section has been evaluated in an analytic form by Racah [4], completely using Monte Carlo techniques [5], and via the Weizsacker-Williams approximation [1–3]. It is known that at sufficiently small impact parameters the S-matrix element corresponding to (e^+, e^-) pair production violates unitarity bounds; thus, for colliding $^{238}U^{92+}$ beams at an impact parameter $b = \hbar/m_e c = \lambda_C$ (386 fm), the probability of pair creation exceeds unity at γ values (in the frame of either ion) greater than 2×10^3. It is therefore clear that simple perturbational estimates are not adequate for the smaller values of impact parameter at RHIC energies, and higher-order damping effects must be included. The diagrams corresponding to multiple pair production that contribute such damping effects are illustrated in Figs. 1(b) and 1(c); it is to be noted that these diagrams are only a particular subset of all higher orders, a limitation that is addressed in more detail below. In addition to this warning signal from the perturbational result, there is also an underlining of the need to broaden the calculations in order to extract the number of multiple pairs; perturbation theory is a calculation of the probability of one-pair formation.

Both of these important points were addressed in a recent paper by Baur, which presented a method for sum-

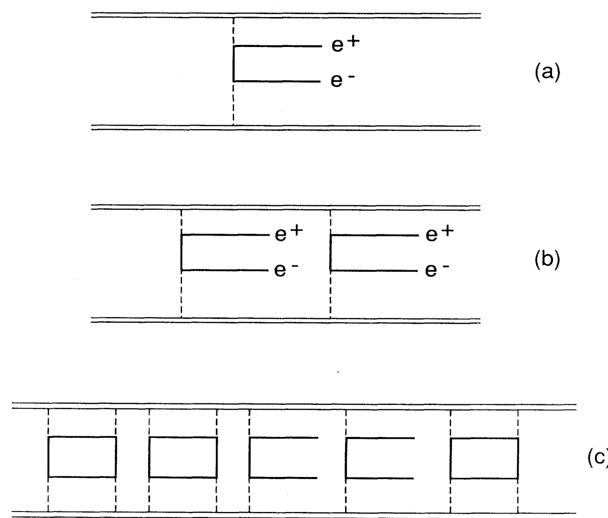

FIG. 1. Diagrams representing electron pair creation in the field of two energetic heavy ions moving in opposite directions (represented by double solid lines). (a) Single-pair creation, (b) double-pair creation, and (c) double-pair creation in higher order; the higher orders included correspond to pair creation followed by annihilation via interactions with the ions. The single lines represent electron-positron states; the dashed lines show interactions between the ion and the electron field.

ming the higher orders based on a boson mapping [7,9]. This calculation results in a Poisson distribution $P_b(N)$, for the probability of producing N pairs at an impact parameter b, thereby providing the necessary inclusion of damping and describing the multiple pair contributions. The method has been criticized [7] as failing to distinguish possible off-shell contributions and including possible artifacts of the boson mapping. In this paper we address these criticisms and show what approximations are required to achieve the results.

The Poisson form is, in fact, an immediate consequence of the classical, imperturbable nature of the electromagnetic source (the unchanging heavy-ion motion) taken together with the noninteraction between the produced pairs. This general statement is illustrated in the following by analysis of the two technologies that have been applied to the pair-creation problem: the various forms of perturbation theory and more recent efforts to include higher-order ion-electron interactions via a coupled-channel approach [10–12].

We begin with the first of these, perturbation theory. The analysis here is based on a direct summing of the restricted class of S-matrix diagrams that correspond to the subset illustrated in Fig. 1. This direct procedure enables us to explicitly examine the contribution of on-shell and off-shell parts of the amplitudes (in fact, the latter do not enter), and to free ourselves of any mapping problems.

II. DIRECT SUMMING

A. Summing higher-order diagrams

In this section we discuss the approximations required to enable higher-order effects to be resummed. The nature and limitations of the calculation are best explained in terms of diagrams. Figure 1(a) shows the lowest-order pair-creation diagram. The motion of the very energetic heavy ions, represented by the double solid lines, is assumed to be unperturbed by the pair-creation process. The very large momenta of the heavy ions suffer totally negligible recoils or energy losses. Within this unperturbed-ion assumption the creation of two pairs, as shown in Fig. 1(b), is understood to involve independent ion-electron interactions for both the first and second pairs. Since the energy of the pairs is of the order $m_e c^2 \ln\gamma$ while those of the heavy ions are of the order γAM_p, this assumption appears quite valid for large γ.

In addition to the creation of 1,2,. . . multiple pairs, the creation followed by destruction of pairs will be included; this is illustrated by the example in Fig. 1(c). Not included is the interaction between the electrons (positive or negative). Since we have very heavy, completely stripped ions with high charge Z, this approximation can be seen as ignoring interactions of relative order $1/Z$.

In summary, higher-order diagrams to be included in

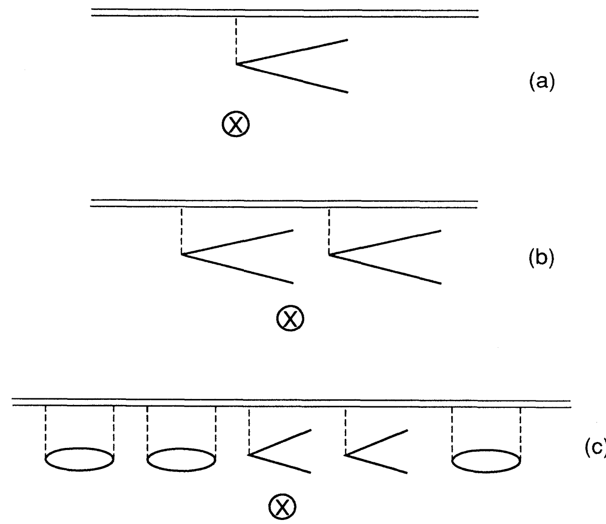

FIG. 2. Diagrams representing the same physical processes as in Fig. 1. The reference frame is fixed in one of the ions, which is taken as stationary (represented by the encircled cross). The electron lines are understood to include Coulomb interactions with the stationary ion.

this restricted summation of higher-order effects consist only of (1) any number of pair creations, but each pair taken as independent of all other real or virtual pairs, ignoring interactions or Pauli blocking; and (2) any number of pair creations, followed by annihilations. Again, it is understood that interactions or exclusions between pairs are ignored.

Since we here analyze the semiclassical case in which the two energetic heavy ions, moving with very large momenta in opposite directions, are not effectively perturbed by momentum recoils or energy losses, it is very useful to choose the reference frame on one of the ions, and take all Coulomb interactions between the electron field and the stationary ion as included in the electron state represented by the particle lines. Then, the processes of Figs. 1(a)–1(c) can now be pictured as in Figs. 2(a)–2(c).

The interaction between a moving ion (velocity $v\hat{\mathbf{z}}$, $v\sim c$, and energy γAM_p) and the electron field ψ is the usual Lorentz-transformed Coulomb form,

$$V(t)=\frac{\gamma\alpha Z}{[(\boldsymbol{\rho}-\mathbf{b})^2+\gamma^2(z-vt)^2]^{1/2}}\psi^{+}(1-v\alpha_z)\psi\ , \qquad (2.1)$$

where $(\boldsymbol{\rho},z)$ are the electron coordinates measured from the stationary ion and $\mathbf{b}$ represents the transverse separation of the ions. α is the usual fine-structure constant, and α_z is the Dirac matrix.

The S matrix can be written in the usual time-ordered form:

$$S=1+\frac{1}{i\hbar}\int_{-\infty}^{\infty}V(t)dt+\frac{1}{(i\hbar)^2}\int_{-\infty}^{\infty}V(t_2)dt_2\int_{-\infty}^{t_2}V(t_1)dt_1+\frac{1}{(i\hbar)^3}\int_{-\infty}^{\infty}V(t_3)dt_3\int_{-\infty}^{t_3}V(t_2)dt_2\int_{-\infty}^{t_2}V(t_1)dt_1+\cdots\ . \qquad (2.2)$$

The interaction operator $V(t)$ can be written in symbolic form as

$$V(t)=\int d\omega\, a(\omega)e^{-i\omega t}H(i,j,\omega)e^{i(\varepsilon_i^+ +\varepsilon_j^-)t}b_i^\dagger a_j^\dagger + \text{H.c.}$$
$$=\sum_{i,j} V_{ij}^{(+)}(t)b_i^\dagger a_j^\dagger + \sum_{i,j} V_{ij}^{(-)}(t)a_i b_j \tag{2.3}$$

and for the creation of the particular pair, $\bar{i}\,\bar{j}$,

$$\mathcal{M}_{\bar{i}\bar{j}}^{(+)}=\left\langle \bar{i}\,\bar{j}\left|\frac{1}{\hbar}\int_{-\infty}^{\infty}V(t)dt\right|0\right\rangle=\frac{2\pi}{\hbar}\int d\omega\, a(\omega)H(ij;\omega)\delta(\omega-\varepsilon_{\bar{i}}^+-\varepsilon_{\bar{j}}^-)\ , \tag{2.4}$$

where $b_i^\dagger$ and $a_j^\dagger$ are the creation operators for positron i and electron j with energies ε_i^+ and ε_j^-, respectively. Since we are interested only in the cross sections for pair production and not specifically in the angular or momentum distribution we can ignore the details of differentiating between incoming and outgoing wave boundary conditions. Instead, it is to be understood that we limit ourselves to probabilities that arise from summing over the whole energy shell.

To illustrate the structure of the resumming technique we first concentrate on the third-order contribution in the S-matrix expansion (2.2) for the production of a pair $\bar{i}\,\bar{j}$; the detail serves to explain the treatment of all higher orders. In the third order, one of the three V factors corresponds to the creation of the $\bar{i}\,\bar{j}$ pair, the other two to the creation—followed by destruction of all possible pairs, ij. We specifically assume, following the overall format, that we are to drop the interference between the $\bar{i}\,\bar{j}$ pair and the sequence of ij pairs, as we have dropped the interactions between pairs. Then the V factor corresponding to the $\bar{i}\,\bar{j}$ pair creation *commutes with the other V factors;* this in turn allows the three terms (the $\bar{i}\,\bar{j}$ factor can be first, middle, or last) to be very usefully combined. Thus we have

$$\begin{aligned}S_{\bar{i}\bar{j}}^{(3)}=\frac{1}{(i\hbar)^3}\sum_{i,j}\Bigg[&\int_{-\infty}^{\infty}V_{ij}^{(-)}(t_3)dt_3\int_{-\infty}^{t_3}V_{ij}^{(+)}(t_2)dt_2\int_{-\infty}^{t_2}V_{\bar{i}\bar{j}}^{(+)}(t_1)dt_1\\&+\int_{-\infty}^{\infty}V_{ij}^{(-)}(t_3)dt_3\int_{-\infty}^{t_3}V_{\bar{i}\bar{j}}^{(+)}(t_2)dt_2\int_{-\infty}^{t_2}V_{ij}^{(+)}(t_1)dt_1\\&+\int_{-\infty}^{\infty}V_{\bar{i}\bar{j}}^{(+)}(t_3)dt_3\int_{-\infty}^{t_3}V_{ij}^{(-)}(t_2)dt_2\int_{-\infty}^{t_2}V_{ij}^{(+)}(t_1)dt_1\Bigg]\ .\end{aligned} \tag{2.5}$$

By rearranging and relabeling the dummy integration variables these three terms can be rewritten as

$$\begin{aligned}S_{\bar{i}\bar{j}}^{(3)}&=\frac{1}{(i\hbar)^3}\sum_{i,j}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t_3)dt_3\int_{-\infty}^{t_3}V_{ij}^{(+)}(t_2)dt_2\left[\int_{-\infty}^{t_2}+\int_{t_2}^{t_3}+\int_{t_3}^{\infty}\right]V_{\bar{i}\bar{j}}^{(+)}(t_1)dt_1\\&=\sum_{i,j}\left[\frac{1}{(i\hbar)^2}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right]\left[\frac{1}{i\hbar}\int_{-\infty}^{+\infty}V_{\bar{i}\bar{j}}^{(+)}(t)dt\right]\ .\end{aligned} \tag{2.6}$$

It is important to note that the $V_{\bar{i}\bar{j}}^{(+)}$ refers to the creation of a particular pair, $\bar{i}\,\bar{j}$, while the combined other factor is a sum over the creation and destruction of all possible pairs; we shall return to the necessary analysis of this latter part. The $(n+1)$th order can be shown in the same way to result in a contribution to the creation of an $\bar{i}\,\bar{j}$ pair. In this way we get, for the $(n+1)$th order,

$$S_{\bar{i}\bar{j}}^{(n+1)}=\frac{1}{n!}\left[\frac{1}{(i\hbar)^2}\sum_{i,j}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right]^n\left[\frac{1}{i\hbar}\int_{-\infty}^{\infty}V_{\bar{i}\bar{j}}^{(+)}(t)dt\right]\ , \tag{2.7}$$

and the whole series can be immediately summed as

$$S_{\bar{i}\bar{j}}=\left[\frac{1}{i\hbar}\int_{-\infty}^{\infty}V_{\bar{i}\bar{j}}^{(+)}(t)dt\right]\exp\left[\frac{-1}{\hbar^2}\sum_{i,j}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right]\ . \tag{2.8}$$

Similarly, the S-matrix element for the creation of particular pairs $\bar{i}\,\bar{j},\bar{\bar{i}}\,\bar{\bar{j}},\ldots$ gives, in just the same way,

$$S_{\bar{i}\bar{j},\bar{\bar{i}}\bar{\bar{j}}}=\left[\frac{1}{i\hbar}\int_{-\infty}^{\infty}V_{\bar{i}\bar{j}}^{(+)}(t)dt\right]\left[\frac{1}{i\hbar}\int_{-\infty}^{\infty}V_{\bar{\bar{i}}\bar{\bar{j}}}^{(+)}(t)dt\right]\cdots\exp\left[\frac{-1}{\hbar^2}\sum_{i,j}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right]\ . \tag{2.9}$$

One of the reasons for explicitly carrying through the simple combinatorics required to arrive at Eq. (2.6) is to emphasize that there is *no* limitation on the time overlaps or time orderings of the separate pairs (within our overall assumption of no interactions between pairs).

From Eq. (2.8) we can write immediately the probability of forming the particular pair $\bar{i}\,\bar{j}$:

$$P_b(\bar{i}\,\bar{j})=\left|\frac{1}{\hbar}\int_{-\infty}^{\infty}V_{\bar{i}\bar{j}}^{(+)}(t)dt\right|^2\exp\left[\mathrm{Re}\left[\frac{-2}{\hbar^2}\sum_{i,j}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right]\right], \tag{2.10}$$

or more compactly

$$P_b(\bar{i}\,\bar{j})=|\mathcal{M}_{\bar{i}\bar{j}}^{(+)}|^2e^E ,$$

$$E\equiv\mathrm{Re}\left[\frac{-2}{\hbar^2}\sum_{i,j}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right].$$

Equation (2.10) may be readily extended for the particular $\bar{i}\,\bar{j},\bar{\bar{i}}\,\bar{\bar{j}},\ldots$ pairs.

The probability of forming any one pair is just

$$P_b(1)=\sum_{\bar{i},\bar{j}}P_b(\bar{i}\,\bar{j})=\sum_{\bar{i},\bar{j}}\left|\mathcal{M}_{\bar{i}\bar{j}}^{(+)}\right|^2e^E , \tag{2.11}$$

while that for forming two pairs is seen from Eq. (2.9) to be

$$P_b(2)=\frac{1}{2!}\sum_{\bar{i},\bar{j}}\left|\mathcal{M}_{\bar{i}\bar{j}}^{(+)}\right|^2\sum_{\bar{\bar{i}},\bar{\bar{j}}}\left|\mathcal{M}_{\bar{\bar{i}}\bar{\bar{j}}}^{(+)}\right|^2e^E \tag{2.12}$$

$$=\frac{1}{2!}\left[\sum_{\bar{i},\bar{j}}\left|\mathcal{M}_{\bar{i}\bar{j}}^{(+)}\right|^2\right]^2e^E . \tag{2.13}$$

In general, for N pairs we can write (dropping bar notation)

$$P_b(N)=\frac{1}{N!}\left[\sum_{i,j}\left|\mathcal{M}_{ij}^{(+)}\right|^2\right]^Ne^E , \tag{2.14}$$

where once again we ignore the multiple appearance of any given pair in two or more factors.

B. Evaluation of exponent

Written out more fully, the exponent E is

$$E=\mathrm{Re}\left[\frac{-2}{\hbar^2}\sum_{i,j}\int_{-\infty}^{\infty}dt''\int_{-\infty}^{t''}dt'\int d\nu\int d\omega\, a^*(\nu)a(\omega)e^{i(\nu-\varepsilon_i^+-\varepsilon_j^-)t''}e^{i(-\omega+\varepsilon_i^++\varepsilon_j^-)t'}H^*(ij,\nu)H(ij,\omega)\langle 0|a_jb_ib_j^\dagger a_i^\dagger|0\rangle\right], \tag{2.15}$$

where the usual damping factors, i.e., $e^{\pm\delta t}$, are to be understood in the time integrations. On carrying out the time integrations we arrive at the principal-value integral;

$$2\pi^2\delta(\nu-\varepsilon_i^+-\varepsilon_j^-)\delta(\omega-\varepsilon_i^+-\varepsilon_j^-)-\pi i\delta(\nu-\omega)\mathrm{P}\frac{1}{(\omega-\varepsilon_i^+-\varepsilon_j^-)-i\delta} . \tag{2.16}$$

In this way we find

$$E=-4\pi^2\sum_{i,j}\left|\frac{1}{\hbar}\int d\omega a(\omega)H(ij,\omega)\delta(\omega-\varepsilon_i^+-\varepsilon_j^-)\right|^2 \tag{2.17}$$

$$=-\sum_{i,j}\left|\mathcal{M}_{ij}^{(+)}\right|^2 . \tag{2.18}$$

The off-mass shell portion of E does not enter because only the real part of E contributes. Combining (2.18) and (2.14) proves that $P_b(N)$ is a Poisson distribution of probabilities. Recalling that the first-order perturbational result for the one-pair probability is $\mathcal{P}(b)$,

$$\mathcal{P}(b)=\sum_{i,j}\left|\mathcal{M}_{ij}^{(+)}\right|^2 , \tag{2.19}$$

the N-pair probability can be written in terms of the lowest-order result:

$$P_b(N)=\frac{\mathcal{P}(b)^N\exp[-\mathcal{P}(b)]}{N!} . \tag{2.20}$$

From this equation we obtain the probability of an event (1,2,. . . pairs) as

$$\sum_{N=1}^{\infty}P_b(N)=1-P_b(0)=1-\exp[-\mathcal{P}(b)] . \tag{2.21}$$

The fact that we can write higher orders in terms of the lowest order is a direct consequence of the pair independence ansatz made throughout the derivation together with the classical, nonperturbed nature of the sources.

C. Comments on resummed series

From Eq. (2.20), the N-pair cross section $\sigma_{N\ \mathrm{pair}}$ is obtained by integrating the N-pair probability over the impact parameter b, i.e.,

$$\sigma_{N\ \mathrm{pair}}=\int d^2b\, P_b(N),\quad N=1,2,\ldots \tag{2.22}$$

and the cross section for any number of pairs is given by (2.21) as

$$\sigma_{\mathrm{pair}}=\int d^2b\{1-\exp[-\mathcal{P}(b)]\}=\sum_{N=1}\sigma_{N\ \mathrm{pair}}\ . \tag{2.23}$$

The average number of pairs,

$$\sum_N NP_b(N)=\mathcal{P}(b)\ , \tag{2.24}$$

is just equal to the perturbational value for the one-pair probability. Similarly the weighted cross-section sum gives

$$\sum_N N\sigma_{N\ \mathrm{pair}}=\sigma_c\ , \tag{2.25}$$

where σ_c is the perturbative cross section discussed in the Introduction.

From these equations it is obvious $\sigma_c>\sigma_{\mathrm{pair}}$ or $\mathcal{P}(b)>P_b(N)$, $N=1,2,\ldots$; i.e., according to this summation procedure the first-order perturbative result always overestimates the probability for producing a pair at a given impact parameter.

D. More general analysis

The above discussion is based on a detailed examination of matrix elements, and was designed to immediately connect with traditional perturbational calculations. However, a much simpler proof follows on noting that

$$V_{ij}^{(-)}(t)=V_{ij}^{(+)*}(t)\ , \tag{2.26}$$

so that within the expression for the exponent term E, we can separate into real and imaginary parts:

$$\begin{aligned}\frac{1}{\hbar^2}\int_{-\infty}^{\infty}V_{ij}^{(-)}(t'')dt''\int_{-\infty}^{t''}V_{ij}(t')dt'=&\frac{1}{2\hbar^2}\left[\int_{-\infty}^{\infty}V_{ij}^{(+)*}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'+\int_{-\infty}^{\infty}V_{ij}^{(+)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)*}(t')dt'\right]\\&+\frac{1}{2\hbar^2}\left[\int_{-\infty}^{\infty}V_{ij}^{(+)*}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)}(t')dt'\right.\\&\left.-\int_{-\infty}^{\infty}V_{ij}^{(+)}(t'')dt''\int_{-\infty}^{t''}V_{ij}^{(+)*}(t')dt'\right]\ .\end{aligned} \tag{2.27}$$

The first term, which by its construction is real, is just

$$\frac{1}{2\hbar^2}\int_{-\infty}^{\infty}V_{ij}^{(+)*}(t'')dt''\int_{-\infty}^{\infty}V_{ij}^{(+)}(t')dt'=\tfrac{1}{2}\left|\mathcal{M}_{ij}^{(+)}\right|^2 .$$

Then, since only the real part enters into the expression for E, we directly obtain (2.18), without the need to know the details of the matrix elements.

III. APPLICATION TO COUPLED-CHANNEL APPROACH

Coupled-channel calculations [10–12] aim at the inclusion of interactions between the electron system and the projectile ion to all orders—within as large a function sub-space as the available computers permit. In this section we will again demonstrate the relevance of the Poisson distribution for the nonperturbative procedures inherent in these coupled-channel methods.

The basic procedure is the solution of the Dirac equation for electrons in the classical time-dependent field of the two potentials, again with neglect of the interelectron interactions and any source perturbations. The electron-ion forces are kept to all orders. Then the problem reduces to the solution of the time-dependent Dirac equation

$$H(t)\Psi=\frac{\hbar}{i}\frac{\partial\Psi}{\partial t}\ , \tag{3.1}$$

where $H(t)$ is a sum of one-electron Hamiltonians. The wave function $\Psi(t)$ for such a Hamiltonian is given as an antisymmetrized product of one-particle functions,

$$\Psi=\mathcal{A}\prod_\alpha\Psi_\alpha\ , \tag{3.2}$$

where each Ψ_α is the solution to its corresponding one-particle Hamiltonian and obeys the further condition that

$$\lim_{t\to-\infty}\Psi_\alpha(t)\to\varphi_\alpha\ , \tag{3.3}$$

where φ_α is an unperturbed, steady-state function that represents the initial electron state. For our purposes here the φ_α are the negative-energy solutions, and the full $\mathcal{A}\prod_\alpha\varphi_\alpha$ represents the original, unperturbed, and filled Dirac vacuum. The Ψ_α can be written in terms of the basis made up of the unperturbed negative- and positive-energy solutions:

$$\Psi_\alpha = \sum_{\varepsilon(\beta)<0} A_{\alpha\beta}(t)\varphi_\beta(t) + \sum_{\varepsilon(i)>0} A_{\alpha i}(t)\varphi_i(t)$$

where

$$\sum_\beta |A_{\alpha\beta}|^2 + \sum_i |A_{\alpha i}|^2 = 1 , \qquad (3.4)$$

$$A_{\alpha\beta}(-\infty) = \delta_{\alpha\beta}, \quad A_{\alpha i}(-\infty) = 0 .$$

Orthonormalization for all times follows directly from the defining equations. The fully antisymmetrized state may then be compactly represented via creation and annihilation operators as

$$\Psi_\alpha^\dagger(t) = \sum_\beta A_{\alpha\beta}(t) C_\beta^\dagger + \sum_i A_{\alpha i}(t) C_i^\dagger ,$$
$$\Psi^\dagger(t) = \Psi_\alpha^\dagger \Psi_\beta^\dagger \cdots |0\rangle . \qquad (3.5)$$

It is simple to verify that $\langle \Psi(t)\Psi^\dagger(t)\rangle = 1$ for all times.

The probability of finding an electron in a positive-energy state i at $t = +\infty$ is then

$$\langle \Psi(\infty) C_i^\dagger C_i \Psi^\dagger(\infty)\rangle$$
$$= \langle 0| \cdots \Psi_\beta(\infty)\Psi_\alpha(\infty) C_i^\dagger C_i \Psi_\alpha^\dagger(\infty)\Psi_\beta^\dagger(\infty) \cdots |0\rangle = p_i ,$$

$$p_i = \sum_\alpha \left| A_{\alpha i}(\infty) \right|^2 , \qquad (3.6)$$

and the probability of finding just one such positive-energy state occupied is

$$p_i \prod_{j\,(\neq i)} (1-p_j) . \qquad (3.7)$$

Then

$$P_b(1) = \sum_i p_i \prod_{j\,(\neq i)} (1-p_j) \qquad (3.8)$$

is just the one-and-only-one-pair probability. Thus we find

$$P_b(1) = \left[\sum_i p_i \right] \exp \left[\sum_{j\,(\neq i)} \ln(1-p_j) \right]$$
$$\cong \left[\sum_i p_i \right] \exp \left[-\sum_j p_j \right] , \qquad (3.9)$$

ignoring higher powers of the infinitesimal p's. Proceeding to two electrons in the positive domain and dropping infinitesimals result in

$$P_b(2) = \sum_{i,j} p_i p_j \exp \left[-\sum_k p_k \right]$$
$$= \frac{1}{2!} \left[\sum_i p_i \right]^2 \exp \left[-\sum_k p_k \right] ; \qquad (3.10)$$

and finally,

$$P_b(N) = \frac{1}{N!} \left[\sum_i p_i \right]^N \exp \left[-\sum_k p_k \right] . \qquad (3.11)$$

Of course, the actual calculation of the central ingredient, $\sum_i p_i$, is not to be obtained by such generalities. The Poisson form follows from the general assumptions of independence of pairs and imperturbability of the source, whereas the magnitude to be inserted in the Poisson form does not.

IV. APPLICATION TO HEAVY-ION COLLIDERS

In this section we apply the result of Sec. II to two planned heavy-ion colliders, RHIC at Brookhaven National Laboratory and the Large Hadron Collider (LHC) at CERN. We assume the form for $\mathcal{P}(b)$ is given by the lowest-order perturbative result derived long ago and later modified to include Coulomb correlations to the positron wave functions [2]. This gives

$$\mathcal{P}(b) \simeq \frac{14}{9\pi^2}(Z^2\alpha^2)^2 \left[\frac{\lambda\!\!\!^{-}_C}{b} \right]^2 \ln^2 \left[\frac{\gamma\delta\lambda\!\!\!^{-}_C}{2b} \right] + \Delta(Z) , \qquad (4.1)$$

where $\gamma\delta/2 \geq b/\lambda\!\!\!^{-}_C \geq 1$, and γ is the ion energy in units of its rest mass as measured in the frame of the other ion ($\gamma = 2\gamma_{\rm col}^2 - 1$); δ is a constant that takes the value 0.681 and $\Delta(Z)$ is a modification of the original formula due to Coulomb distortion that we will ignore here.

In Figs. 3 and 4 we show the N-pair probability distributions $P_b(N)$ derived from the Poisson form as a function of impact parameter b. Also shown in these figures

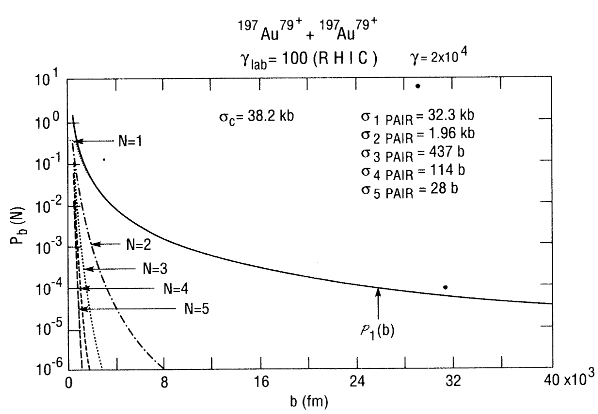

FIG. 3. Probability distribution for N-pair production as a function of impact parameter b. The curves are plotted for ^{179}Au^{79+} beams at the highest RHIC energies. $\mathcal{P}(b)$ is the lowest-order perturbative result corresponding to Eq. (4.1) in the text omitting $\Delta(Z)$. $P_b(N)$ is the result of the Poisson form for N-pair production using $\mathcal{P}(b)$ as input. The perturbative cross section σ_c and the N-pair cross sections $\sigma_{N\ \rm pair}$ are also shown. Note that the value of γ as measured in an ion frame is related to laboratory value by $\gamma = 2\gamma_{\rm col}^2 - 1$. Note, that for a collider, $\gamma_{\rm lab} = \gamma_{\rm c.m.} = \gamma_{\rm col}$, which for RHIC is 100.

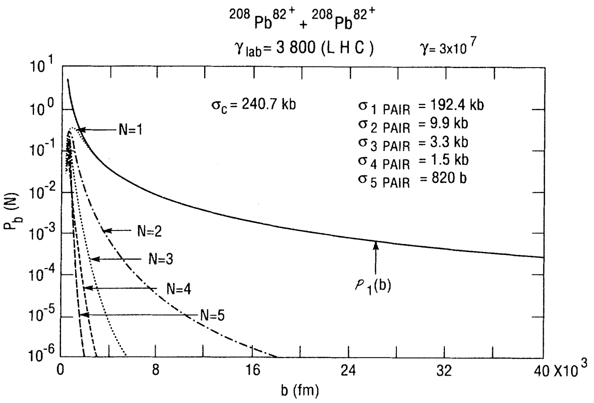

FIG. 4. Same as Fig. 3, but for $^{208}Pb^{82+}$ beams at the highest LHC energies. For the LHC $\gamma_{lab}=\gamma_{col}=3800$.

are the original lowest-order perturbative results that correspond to Eq. (4.1). The curves are plotted for the heaviest available nuclei at the respective facilities, and at top beam energy.

It can be seen that one-pair production completely dominates the picture at both RHIC and LHC energies. Also, the difference between σ_c and $\sigma_{1\ \mathrm{pair}}$ can be attributed to the large values of probability at impact parameters very close to the Compton wavelength λ_C. In addition, the two-, three-, four- and five-pair probability distributions are seen to decrease very rapidly for impact parameter values larger than λ_C.

Utilizing Eq. (2.24), the average number of pairs per interaction of $^{179}Au^{79+}$ beams at impact parameter λ_C at the highest RHIC energies is 1.4. For $^{208}Pb^{82+}$ beams at LHC the corresponding average number of pairs is 5.2. With a collider luminosity value [6] of 2×10^{26} $\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$, this corresponds to a pair production rate of 10^7/sec at RHIC and 2.5×10^8/sec at LHC.

It is important to reemphasize that the above estimates for N-pair production are based on the Poisson distribution for $P_b(N)$ and the lowest-order perturbation expression for $\mathcal{P}(b)$. More recent work [10–12] based on coupled-channel techniques includes the projectile-lepton interactions to all orders, and predicts a value for $\sum_i p_i$ much larger than $\mathcal{P}(b)$. These nonperturbative contributions continue to be a subject under active investigation.

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