Field-equation approximations and amplification in high-gain lasers: Numerical results

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It has recently been shown that a field-equation time-derivative approximation that is commonly used in studies of laser-oscillator dynamics is not necessary and can lead to significant errors for some lasers. A related space-derivative approximation is widely used in studies of steady-state laser amplifiers. A more rigorous amplifier formalism is developed here, and the results are exact solutions of Maxwell's equations. The improved model predicts a spatial instability, single mirror oscillation, and other interesting field behavior.

I. INTRODUCTION

The acronym laser refers to light amplification, and one of the oldest problems in laser studies concerns the propagation of electromagnetic waves in light amplifiers. Such amplifiers are the essential active ingredients in laser oscillators, and they are also widely used to enhance optical signals external to laser cavities. Most treatments of laser amplifiers are based on rate-equation models that describe the intensity or photon density and the atomic or molecular populations. Among the earliest rate-equation studies of laser amplifiers were the detailed analyses by Wright and Schulz-DuBois [1] and by Rigrod [2]. Other treatments have included more complicated energy-level structures and geometries, but most such treatments are also basically rate-equation formulations.

Laser amplifiers can also be analyzed using semiclassical models. In principle, these models are able to represent electromagnetic fields having arbitrary polarizations and arbitrarily fast variations in time and space. In practice, however, most semiclassical treatments also incorporate approximations which may significantly restrict their range of applicability. The purpose of this study is to test a particular approximation that is employed in many treatments of laser amplifiers. In analyzing the electromagnetic aspects of light-matter interactions in lasers and other systems, Maxwell's equations are often combined to form a second-order wave equation. This wave equation is then reduced to first order with derivative approximations based on the familiar assumption that the wave envelope varies negligibly within a time of one optical cycle or a distance of one wavelength. These approximations have been widely used since the development of the first semiclassical Maxwell-Schrödinger laser models, and an important early example was the analysis by Lamb [3]. The resulting wave equation is sometimes referred to as the reduced wave equation [4], and the approximations themselves are sometimes called the slowly-varying-amplitude [5] or slowly-varying-envelope [6] approximations. With modern high-gain laser media, it is worthwhile to explore the possible limitations of these approximations.

Recent studies have examined closely the effects of the slowly-varying-amplitude derivative approximation on the dynamical behavior of laser oscillators [7,8]. It was found that this approximation may lead to significant errors when one considers spontaneous pulsations and other dynamical effects in high-gain lasers [8]. In particular, it was found that for some laser decay rates the approximate equations may substantially misrepresent the stability conditions and pulsation wave forms of a spontaneously pulsing laser. The more exact treatment also revealed that in high-gain wideband lasers the electric and magnetic fields do not maintain a fixed-phase relationship to each other. In the present research the companion approximation that the fields vary negligibly in a distance of one wavelength is examined in detail. A set of first-order ordinary differential equations is developed for treating the propagation of plane electromagnetic waves in media having arbitrary gain or loss per wavelength. These equations are solved numerically for the simplest case of a one-dimensional homogeneously broadened laser amplifier. The concept of the position-dependent local wavelength is introduced, and one finds that the local wavelength may vary substantially as the wave propagates. The results also reveal a spatial instability of the propagating fields. This instability causes, for example, a spatially oscillating growth of any perturbation away from a constant-intensity loss-limited wave. Viewed differently, one finds that in a high-gain amplifier there may be a strong reflection of a propagating wave. Numerical solutions are emphasized in this research, and the possibility of obtaining analytic solutions and stability criteria is explored in the following companion study [9].

A general semiclassical model is developed in Sec. II for a steady-state laser amplifier having arbitrary levels of homogeneous and inhomogeneous line broadening, and a procedure is described for avoiding the slowly-varying-amplitude derivative approximation in the field equations while still allowing relatively simple numerical solutions. In Sec. III the model is further developed for the special case of a homogeneously broadened laser, and in Sec. IV a variety of numerical solutions are obtained for the most basic class of laser amplifiers. Avoiding the approximation reveals that in high-gain wideband amplifiers there may be strong instabilities and reflections of the propagating waves; and, as in the oscillator case, the electric and magnetic fields do not maintain a fixed-phase relationship to each other. A procedure for isolating the amplitudes and intensities of the plus and minus wave components is described in Sec. V, and this procedure is illustrated with further numerical examples.

II. GENERAL MODEL

A semiclassical laser model is usually understood to be one in which the atomic or molecular variables are governed by Schrödinger's equation while the electromagnetic fields are solutions of Maxwell's equations. The possible avoidance of a standard derivative approximation of the electromagnetic field equations will be illustrated here for a semiclassical model that governs an important class of laser amplifiers. As in Ref. [8], it will be assumed that the laser medium has scalar permittivity, permeability, and conductivity, and that the electromagnetic field is a plane wave polarized in the x direction and propagating in the z direction. Thus the vector field equations reduce to the scalar set

$$\frac{\partial E(z,t)}{\partial z}=-\mu_1\frac{\partial H(z,t)}{\partial t}\ , \tag{1}$$

$$\frac{\partial H(z,t)}{\partial z}=-\epsilon_1\frac{\partial E(z,t)}{\partial t}-\frac{\partial P(z,t)}{\partial t}-\sigma E(z,t)\ , \tag{2}$$

where $E(z,t)$ and $H(z,t)$ represent the x component of the electric field and the y component of the magnetic field, respectively; σ is the conductivity; the permittivity and permeability independent of the polarization $P(z,t)$ and the magnetization $M(z,t)$ of the lasing atoms or molecules are represented by ϵ_1 and μ_1, respectively; and for the case of interest here the extra magnetization $M(z,t)$ is equal to zero. Before proceeding to a more exact treatment, the usual approximate incorporation of these equations into a semiclassical amplifier model will be briefly described.

Maxwell's equation are usually combined immediately into a second-order wave equation. If one differentiates Eq. (1) with respect to z and Eq. (2) with respect to t, one obtains the familiar result

$$\frac{\partial^2 E(z,t)}{\partial z^2}-\mu_1\sigma\frac{\partial E(z,t)}{\partial t}-\mu_1\epsilon_1\frac{\partial^2 E(z,t)}{\partial t^2}=\mu_1\frac{\partial^2 P(z,t)}{\partial t^2}\ . \tag{3}$$

In a semiclassical model for a general inhomogeneously broadened laser, the polarization driving this equation can be related back to the off-diagonal density-matrix elements by

$$P(z,t)=\int_0^\infty\int_{-\infty}^\infty \mu\rho_{ab}(v,\omega_\alpha,z,t)dv\,d\omega_\alpha+\text{c.c.}\ , \tag{4}$$

where μ is the dipole moment of the transition, and the notation c.c. means the complex conjugate of the preceding terms. The laser medium is assumed to have both Doppler and non-Doppler broadening mechanisms, with v being the z component of the velocity and ω_α the positive center frequency of the laser transition for members of an atomic or molecular class α. Taken together with the density-matrix form of Schrödinger's equation, Eqs. (3) and (4) provide a complete set from which the time and space dependences of the electric field and of the atomic or molecular parameters can be determined, subject to any boundary conditions.

The simplest solutions for the model that has just been described are those that apply to steady-state uniformly pumped one-directional laser amplifiers, in which both the electric field and the polarization are traveling waves. Thus the rapid time and space variations in the model can be factored out by means of the substitutions

$$E(z,t)=\tfrac{1}{2}E'(z)\exp(ikz-i\omega t)+\text{c.c.}\ , \tag{5}$$

$$\rho_{ab}(v,\omega_\alpha,z,t)=P'(v,\omega_\alpha,z)\exp(ikz-i\omega t)/2\mu\ . \tag{6}$$

Equations (3)–(6) may be combined to obtain the new time-independent wave equation for the complex field amplitude $E'(t)$:

$$c_1^2\frac{d^2E'(z)}{dz^2}+2i\Omega c_1\frac{dE'(z)}{dz}+\left[\omega^2-\Omega^2+i\frac{\sigma}{\epsilon_1}\omega\right]E'(z)=-\frac{\omega^2}{\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P'(v,\omega_\alpha,z)dv\,d\omega_\alpha\ , \tag{7}$$

where the new velocity $c_1=(\mu_1\epsilon_1)^{-1/2}$ and frequency $\Omega=k(\mu_1\epsilon_1)^{-1/2}$ have been introduced, and the conductivity σ is meant to represent all amplifier losses. It is now usual to drop some of the higher derivative terms by arguing that the field envelope varies negligibly in a wavelength. The terms that remain may be written

$$c_1\frac{dE'(z)}{dz}=-\frac{\gamma_c\omega}{\Omega}E'(z)+i\frac{(\omega^2-\Omega^2)}{2\Omega}E'(z)+i\frac{\omega^2}{2\epsilon_1\Omega}\int_0^\infty\int_{-\infty}^\infty P'(v,\omega_\alpha,z)dv\,d\omega_\alpha\ , \tag{8}$$

where the losses are represented by the field decay rate $\gamma_c=\sigma/2\epsilon_1$. If $E'(z)$ is real, the frequency Ω can now be recognized as the optical frequency at which the dispersion or real part of the complex polarization amplitude $P'(v,\omega_\alpha,z)$ is equal to zero. A major purpose of this study is to examine the validity and possible limitations of the derivative approximation that has just been employed in reducing Eq. (7) to Eq. (8).

Further frequency approximations are also commonly introduced. If the lasing frequency ω is close to the nondispersed frequency Ω, then the term $(\omega^2-\Omega^2)/2\Omega$ can be approximated by $\omega-\Omega$, and the ratio ω/Ω can be replaced by unity. If the lasing frequency is also close to a characteristic frequency of the laser transition ω_0, then the remaining ω multiplying the polarization integrals may be replaced by ω_0. With these approximations, Eq. (8) reduces to

$$c_1\frac{dE'(z)}{dz}=-\gamma_cE'(z)+i(\omega-\Omega)E'(z)+i\frac{\omega_0}{2\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P'(v,\omega_\alpha,z)dv\,d\omega_\alpha\ . \tag{9}$$

Special cases of this result appear in most models of cw laser amplifiers. The limitations on the validity of the frequency approximations for the steady-state [10] and dynamic [8] behavior of laser oscillators have recently been considered in detail, and it will be shown below that these approximations can readily be avoided in laser amplifier studies.

To proceed further, one needs to adopt a specific form for the density-matrix equations. A useful set includes the following [10]:

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]\rho_{ab}(v,\omega_\alpha,z,t) = -(i\omega_\alpha+\gamma)\rho_{ab}(v,\omega_\alpha,z,t) - \frac{i\mu}{\hbar}E(z,t)[\rho_{aa}(v,\omega_\alpha,z,t)-\rho_{bb}(v,\omega_\alpha,z,t)] , \quad (10)$$

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]\rho_{aa}(v,\omega_\alpha,z,t) = \lambda_a(v,\omega_\alpha,z,t)-\gamma_a\rho_{aa}(v,\omega_\alpha,z,t) + \left[\frac{i\mu}{\hbar}E(z,t)\rho_{ba}(v,\omega_\alpha,z,t)+\text{c.c.}\right] , \quad (11)$$

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]\rho_{bb}(v,\omega_\alpha,z,t) = \lambda_b(v,\omega_\alpha,z,t)-\gamma_b\rho_{bb}(v,\omega_\alpha,z,t) + \gamma_{ab}\rho_{aa}(v,\omega_\alpha,z,t) - \left[\frac{i\mu}{\hbar}E(z,t)\rho_{ba}(v,\omega_\alpha,z,t)+\text{c.c.},\right] , \quad (12)$$

$$\rho_{ba}(v,\omega_\alpha,z,t)=\rho^*_{ab}(v,\omega_\alpha,z,t) , \quad (13)$$

where the subscripts a and b denote the upper and lower laser levels, respectively, γ_a and γ_b are the total decay rates for these levels, γ_{ab} is the rate of direct decays from level a to level b, γ is the decay rate for the off-diagonal elements, and λ_a and λ_b are the pumping rates.

With Eqs. (5) and (6) for the field and off-diagonal density-matrix element and use of the rotating-wave approximation, the density-matrix equations given above as Eqs. (10)–(13) reduce to the new set

$$v\frac{\partial P'(v,\omega_\alpha,z)}{\partial z} = i(\omega-\omega_\alpha-kv)P'(v,\omega_\alpha,z)-\gamma P'(v,\omega_\alpha,z) - \frac{i\mu^2}{\hbar}E'(z)[\rho_{aa}(v,\omega_\alpha,z)-\rho_{bb}(v,\omega_\alpha,z)] , \quad (14)$$

$$v\frac{\partial \rho_{aa}(v,\omega_\alpha,z)}{\partial z} = \lambda_a(v,\omega_\alpha,z)-\gamma_a\rho_{aa}(v,\omega_\alpha,z) + \frac{i}{4\hbar}[E'(z)P'^*(v,\omega_\alpha,z)-E'^*(z)P'(v,\omega_\alpha,z)] , \quad (15)$$

$$v\frac{\partial \rho_{bb}(v,\omega_\alpha,z)}{\partial z} = \lambda_b(v,\omega_\alpha,z)-\gamma_b\rho_{bb}(v,\omega_\alpha,z)+\gamma_{ab}\rho_{aa}(v,\omega_\alpha,z) - \frac{i}{4\hbar}[E'(z)P'^*(v,\omega_\alpha,z)-E'^*(z)P'(v,\omega_\alpha,z)] , \quad (16)$$

where it is also assumed that the pump and population density functions are independent of time.

Next, it is helpful to separate the field and polarization into their real and imaginary parts in the forms $E'(z)=E_r(z)+iE_i(z)$ and $P'(v,\omega_\alpha,z) = P_r(v,\omega_\alpha,z) + iP_i(v,\omega_\alpha,z)$. With these substitutions Eqs. (14)–(16) become

$$v\frac{\partial P_r(v,\omega_\alpha,z)}{\partial z} = -(\omega-\omega_\alpha-kv)P_i(v,\omega_\alpha,z)-\gamma P_r(v,\omega_\alpha,z)+\frac{\mu^2}{\hbar}E_i(z)D(v,\omega_\alpha,z) , \quad (17)$$

$$v\frac{\partial P_i(v,\omega_\alpha,z)}{\partial z} = (\omega-\omega_\alpha-kv)P_r(v,\omega_\alpha,z)-\gamma P_i(v,\omega_\alpha,z)-\frac{\mu^2}{\hbar}E_r(z)D(v,\omega_\alpha,z) , \quad (18)$$

$$v\frac{\partial D(v,\omega_\alpha,z)}{\partial z} = \lambda_a(v,\omega_\alpha,z)-\lambda_b(v,\omega_\alpha,z)-\frac{\gamma_a+\gamma_{ab}+\gamma_b}{2}D(v,\omega_\alpha,z) - \frac{\gamma_a+\gamma_{ab}-\gamma_b}{2}M(v,\omega_\alpha,z)+\frac{1}{\hbar}[E_r(z)P_i(v,\omega_\alpha,z)-E_i(z)P_r(v,\omega_\alpha,z)] , \quad (19)$$

$$v\frac{\partial M(v,\omega_\alpha,z)}{\partial z} = \lambda_a(v,\omega_\alpha,z)+\lambda_b(v,\omega_\alpha,z)-\frac{\gamma_a-\gamma_{ab}-\gamma_b}{2}D(v,\omega_\alpha,z)-\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2}M(v,\omega_\alpha,z) , \quad (20)$$

where the population difference $D(v,\omega_\alpha,z) =\rho_{aa}(v,\omega_\alpha,z)-\rho_{bb}(v,\omega_\alpha,z)$ and sum $M(v,\omega_\alpha,z) =\rho_{aa}(v,\omega_\alpha,z)+\rho_{bb}(v,\omega_\alpha,z)$ have also been introduced. From Eq. (9) the real and imaginary parts of the field are governed by

$$c_1\frac{dE_r(z)}{dz} = -\gamma_c E_r(z)-(\omega-\Omega)E_i(z) - \frac{\omega_0}{2\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P_i(v,\omega_\alpha,z)dv\,d\omega_\alpha , \quad (21)$$

$$c_1\frac{dE_i(z)}{dz}=-\gamma_c E_i(z)+(\omega-\Omega)E_r(z)+\frac{\omega_0}{2\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P_r(v,\omega_\alpha,z)dv\,d\omega_\alpha\ . \quad (22)$$

Equations (17)–(22) summarize a semiclassical model for steady-state laser amplifiers. Our purpose here is to test whether the derivative approximation inherent in Eqs. (21) and (22) might be introducing significant errors in the predicted amplifier behavior. To answer this question, one may set up a similar model which includes neither the derivative approximation nor the frequency approximation described above. The first possibility that one might consider for removing the derivative approximation is to simply replace Eqs. (21) and (22) with the real and imaginary parts of Eq. (7). However, this equation has second-order derivatives in the field variables, and we find that it is more efficient and more informative to avoid forming the second-order wave equation in the first place. Thus one can instead work from Maxwell's equations and retain the significance that the variables correspond to the electric and magnetic fields.

As a first step in reducing the field equations, one must also factor out the rapid time and space variations in the magnetic field, and a useful substitution is

$$H(z,t)=\tfrac{1}{2}(\epsilon_1/\mu_1)^{1/2}H'(z)\exp(ikz-i\omega t)+\text{c.c.} \quad (23)$$

If Eqs. (4) to (6) and (23) are substituted into Eqs. (1) and (2), one obtains

$$c_1\frac{dH'(z)}{dz}=-2\gamma_c E'(z)+i[\omega E'(z)-\Omega H'(z)]+i\frac{\omega}{\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P'(v,\omega_\alpha,z)dv\,d\omega_\alpha\ , \quad (24)$$

$$c_1\frac{dE'(z)}{dz}=i[\omega H'(z)-\Omega E'(z)]\ . \quad (25)$$

Equations (24) and (25) may be separated into their real and imaginary parts:

$$c_1\frac{dH_r(z)}{dz}=-2\gamma_c E_r(z)-[\omega E_i(z)-\Omega H_i(z)]-\frac{\omega}{\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P_i(v,\omega_\alpha,z)dv\,d\omega_\alpha\ , \quad (26)$$

$$c_1\frac{dH_i(z)}{dz}=-2\gamma_c E_i(z)+[\omega E_r(z)-\Omega H_r(z)]+\frac{\omega}{\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P_r(v,\omega_\alpha,z)dv\,d\omega_\alpha\ , \quad (27)$$

$$c_1\frac{dE_r(z)}{dz}=-[\omega H_i(z)-\Omega E_i(z)]\ , \quad (28)$$

$$c_1\frac{dE_i(z)}{dz}=[\omega H_r(z)-\Omega E_r(z)]\ . \quad (29)$$

The revised model consisting of Eqs. (17)–(20) and (26)–(29) may be applied to a variety of problems concerning steady-state laser amplifiers. This equation set can be integrated using, for example, a Runge-Kutta method for the space derivatives and Simpson's rule for the velocity and frequency integrals. It should be emphasized that this model avoids the derivative and frequency approximations indicated above without adding substantially to the difficulty of the numerical solutions.

III. HOMOGENEOUS BROADENING

The purpose of the foregoing analysis has been to establish a formalism for treating the spatial evolution of electromagnetic fields in a general class of steady-state mixed-broadened laser amplifiers. However, it is not necessary to solve the most general laser configurations to understand the implications of the more exact model. In this discussion we will specialize the model to the simplest special cases. First of all, it will be assumed that the velocity of the atoms or molecules is small enough that all Doppler effects may be neglected. With $v=0$, Eqs. (17)–(20) can be written

$$0=-(\omega-\omega_\alpha)P_i(\omega_\alpha,z)-\gamma P_r(\omega_\alpha,z)+\frac{\mu^2}{\hbar}E_i(z)D(\omega_\alpha,z)\ , \quad (30)$$

$$0=(\omega-\omega_\alpha)P_r(\omega_\alpha,z)-\gamma P_i(\omega_\alpha,z)-\frac{\mu^2}{\hbar}E_r(z)D(\omega_\alpha,z)\ , \quad (31)$$

$$0=\lambda_a(\omega_\alpha,z)-\lambda_b(\omega_\alpha,z)-\frac{\gamma_a+\gamma_{ab}+\gamma_b}{2}D(\omega_\alpha,z)-\frac{\gamma_a+\gamma_{ab}-\gamma_b}{2}M(\omega_\alpha,z)+\frac{1}{\hbar}[E_r(z)P_i(\omega_\alpha,z)-E_i(z)P_r(\omega_\alpha,z)]\ , \quad (32)$$

$$0=\lambda_a(\omega_\alpha,z)+\lambda_b(\omega_\alpha,z)-\frac{\gamma_a-\gamma_{ab}-\gamma_b}{2}D(\omega_\alpha,z)-\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2}M(\omega_\alpha,z)\ , \quad (33)$$

where a δ function in the velocity has been factored out of all of the velocity-dependent functions, and the velocity integrals in Eqs. (21), (22), (26), and (27) can be eliminated.

Equations (30) and (31) can be solved for the polarizations, and the results are

$$P_r(\omega_\alpha,z)=\frac{\mu^2 D(\omega_\alpha,z)}{\gamma\hbar}\frac{[(\omega-\omega_\alpha)/\gamma]E_r(z)+E_i(z)}{1+[(\omega-\omega_\alpha)/\gamma]^2}\ , \quad (34)$$

$$P_i(\omega_\alpha,z)=\frac{\mu^2 D(\omega_\alpha,z)}{\gamma\hbar}\frac{[(\omega-\omega_\alpha)/\gamma]^2E_i(z)-E_r(z)}{1+[(\omega-\omega_\alpha)/\gamma]^2}\ . \quad (35)$$

With these substitutions, Eq. (32) becomes

$$0=\lambda_a(\omega_\alpha,z)-\lambda_b(\omega_\alpha,z)-\frac{\gamma_a+\gamma_{ab}+\gamma_b}{2}D(\omega_\alpha,z)-\frac{\gamma_a+\gamma_{ab}-\gamma_b}{2}M(\omega_\alpha,z)-\frac{\mu^2 D(\omega_\alpha,z)}{\gamma\hbar^2}\frac{E_r^2(z)+E_i^2(z)}{1+[(\omega-\omega_\alpha)/\gamma]^2} . \quad (36)$$

If $M(\omega_\alpha,z)$ is eliminated between Eqs. (33) and (36), one finds that the population difference can be written

$$D(\omega_\alpha,z)=\frac{D_0(\omega_\alpha,z)}{1+[A_r^2(z)+A_i^2(z)][1+(\omega-\omega_\alpha)^2/\gamma^2]^{-1}} , \quad (37)$$

where we have introduced the unsaturated population difference

$$D_0(\omega_\alpha,z)=(1-\gamma_{ab}/\gamma_b)\lambda_a(\omega_\alpha,z)/\gamma_a-\lambda_b(\omega_\alpha,z)/\gamma_b , \quad (38)$$

and the normalized field amplitudes

$$A_r(z)=\left[\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2\gamma\gamma_a\gamma_b}\right]^{1/2}\frac{\mu E_r(z)}{\hbar} , \quad (39)$$

$$A_i(z)=\left[\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2\gamma\gamma_a\gamma_b}\right]^{1/2}\frac{\mu E_i(z)}{\hbar} . \quad (40)$$

With the population difference from Eq. (37), the polarization components of Eqs. (34) and (35) become

$$P_r(\omega_\alpha,z)=\frac{\mu}{\gamma}\left[\frac{2\gamma\gamma_a\gamma_b}{\gamma_a-\gamma_{ab}+\gamma_b}\right]^{1/2}\times\frac{[(\omega-\omega_\alpha)/\gamma]A_r(z)+A_i(z)}{1+[(\omega-\omega_\alpha)/\gamma]^2+A_r^2(z)+A_i^2(z)}\times D_0(\omega_\alpha,z) , \quad (41)$$

$$P_i(\omega_\alpha,z)=\frac{\mu}{\gamma}\left[\frac{2\gamma\gamma_a\gamma_b}{\gamma_a-\gamma_{ab}+\gamma_b}\right]^{1/2}\times\frac{[(\omega-\omega_\alpha)/\gamma]A_i(z)-A_r(z)}{1+[(\omega-\omega_\alpha)/\gamma]^2+A_r^2(z)+A_i^2(z)}\times D_0(\omega_\alpha,z) . \quad (42)$$

It is now useful to introduce the following normalized polarization components and population difference:

$$P_r(U,z)=\frac{\gamma^2\omega_0}{2k\epsilon_1\gamma_c}\frac{\mu}{\hbar}\left[\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2\gamma\gamma_a\gamma_b}\right]^{1/2}P_r(\omega_\alpha,z) , \quad (43)$$

$$P_i(U,z)=\frac{\gamma^2\omega_0}{2k\epsilon_1\gamma_c}\frac{\mu}{\hbar}\left[\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2\gamma\gamma_a\gamma_b}\right]^{1/2}P_i(\omega_\alpha,z) , \quad (44)$$

$$D_0(U,z)=\frac{\gamma\omega_0\mu^2}{2k\epsilon_1\gamma_c\hbar}D_0(\omega_\alpha,z) , \quad (45)$$

where the normalized center frequency offset is $U=(\omega_\alpha-\omega_0)/\gamma$. With these changes of variables, Eqs. (41) and (42) simplify to

$$P_r(U,z)=\frac{(y-U)A_r(z)+A_i(z)}{1+(y-U)^2+A_r^2(z)+A_i^2(z)}D_0(U,z) , \quad (46)$$

$$P_i(U,z)=\frac{(y-U)A_i(z)-A_r(z)}{1+(y-U)^2+A_r^2(z)+A_i^2(z)}D_0(U,z) , \quad (47)$$

where the parameter $y=(\omega-\omega_0)/\gamma$ is the normalized lasing frequency. When Eqs. (46) and (47) are substituted into Eqs. (21) and (22), one obtains the approximate amplifier model

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\left[A_r(z)+\delta(y-y_0)A_i(z)+\int_{-\infty}^{\infty}\frac{(y-U)A_i(z)-A_r(z)}{1+(y-U)^2+A_r^2(z)+A_i^2(z)}\times D_0(U,z)dU\right] , \quad (48)$$

$$c_1\frac{dA_i(z)}{dz}=-\gamma_c\left[A_i(z)-\delta(y-y_0)A_r(z)-\int_{-\infty}^{\infty}\frac{(y-U)A_r(z)+A_i(z)}{1+(y-U)^2+A_r^2(z)+A_i^2(z)}\times D_0(U,z)dU\right] , \quad (49)$$

where $y_0=(\Omega-\omega_0)/\gamma$ is the normalized nondispersed frequency, $\delta=\gamma/\gamma_c$ is a dimensionless decay-rate ratio, and for notational convenience the lower limit on the frequency integrals has been extended to minus infinity.

Equations (48) and (49) represent a conventional model for amplification in a non-Doppler inhomogeneously broadened laser, and this model has been obtained using the standard approximations discussed above. If these approximations are not to be employed, it follows from Eqs. (26)–(29) that the field equations should be replaced by

$$c_1\frac{dB_r(z)}{dz}=-\gamma_c\left[2A_r(z)+\delta[(y+z_0)A_i(z)-(y_0+z_0)B_i(z)]+\frac{2(y+z_0)}{z_0}\int_{-\infty}^{\infty}\frac{(y-U)A_i(z)-A_r(z)}{1+(y-U)^2+A_r^2(z)+A_i^2(z)}D_0(U,z)dU\right] , \quad (50)$$

$$c_1\frac{dB_i(z)}{dz}=-\gamma_c\left[2A_i(z)-\delta[(y+z_0)A_r(z)-(y_0+z_0)B_r(z)]-\frac{2(y+z_0)}{z_0}\int_{-\infty}^{\infty}\frac{(y-U)A_r(z)+A_i(z)}{1+(y-U)^2+A_r^2(z)+A_i^2(z)}D_0(U,z)dU\right], \tag{51}$$

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\delta[(y+z_0)B_i(z)-(y_0+z_0)A_i(z)], \tag{52}$$

$$c_1\frac{dA_i(z)}{dz}=\gamma_c\delta[(y+z_0)B_r(z)-(y_0+z_0)A_r(z)], \tag{53}$$

where $B_r(z)$ and $B_i(z)$ are, respectively, the real and imaginary components of the magnetic field with the same normalization as used for the electric field, and $z_0=\omega_0/\gamma$ is the normalized center frequency of the transition.

The simplest special cases of Eqs. (48)–(53) occur for the limit of homogeneous broadening. With $U=0$ and an obvious redefinition of the population difference, these equations reduce to

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\left[A_r(z)+\delta(y-y_0)A_i(z)+\frac{yA_i(z)-A_r(z)}{1+y^2+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{54}$$

$$c_1\frac{dA_i(z)}{dz}=-\gamma_c\left[A_i(z)-\delta(y-y_0)A_r(z)-\frac{yA_r(z)+A_i(z)}{1+y^2+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{55}$$

$$c_1\frac{dB_r(z)}{dz}=-\gamma_c\left[2A_r(z)+\delta[(y+z_0)A_i(z)-(y_0+z_0)B_i(z)]+\frac{2(y+z_0)}{z_0}\frac{yA_i(z)-A_r(z)}{1+y^2+A_r^2(z)+A_i^2(z)}\times D_0(z)\right], \tag{56}$$

$$c_1\frac{dB_i(z)}{dz}=-\gamma_c\left[2A_i(z)-\delta[(y+z_0)A_r(z)-(y_0+z_0)B_r(z)]-\frac{2(y+z_0)}{z_0}\frac{yA_r(z)+A_i(z)}{1+y^2+A_r^2(z)+A_i^2(z)}\times D_0(z)\right], \tag{57}$$

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\delta[(y+z_0)B_i(z)-(y_0+z_0)A_i(z)], \tag{58}$$

$$c_1\frac{dA_i(z)}{dz}=\gamma_c\delta[(y+z_0)B_r(z)-(y_0+z_0)A_r(z)]. \tag{59}$$

As a further possible simplification, the frequency Ω can be set equal to the optical frequency ω of the atomic transition ($y_0=y$). Then Eqs. (54)–(59) become

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\left[A_r(z)+\frac{yA_i(z)-A_r(z)}{1+y^2+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{60}$$

$$c_1\frac{dA_i(z)}{dz}=-\gamma_c\left[A_i(z)-\frac{yA_r(z)+A_i(z)}{1+y^2+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{61}$$

$$c_1\frac{dB_r(z)}{dz}=-\gamma_c\left[2A_r(z)+\delta(y+z_0)[A_i(z)-B_i(z)]+\frac{2(y+z_0)}{z_0}\frac{yA_i(z)-A_r(z)}{1+y^2+A_r^2(z)+A_i^2(z)}\times D_0(z)\right], \tag{62}$$

$$c_1\frac{dB_i(z)}{dz}=-\gamma_c\left[2A_i(z)-\delta(y+z_0)[A_r(z)-B_r(z)]-\frac{2(y+z_0)}{z_0}\frac{yA_r(z)+A_i(z)}{1+y^2+A_r^2(z)+A_i^2(z)}\times D_0(z)\right], \tag{63}$$

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\delta(y+z_0)[B_i(z)-A_i(z)], \tag{64}$$

$$c_1\frac{dA_i(z)}{dz}=\gamma_c\delta(y+z_0)[B_r(z)-A_r(z)]. \tag{65}$$

This restriction on the frequency Ω could, of course, have been made arbitrarily at the outset of this calculation. However, maintaining frequency flexibility has important advantages for analytical studies and makes possible nontrivial steady-state solutions of the equation set [9]. Most of our numerical and analytical results are based on Eqs. (54)–(59) or a generalization of these equations discussed in Sec. IV.

As a final simplification which still retains most of the essence of the problem, it might sometimes be assumed that the optical frequency ω is at the center of the atomic transition ($y=0$). Then Eqs. (60)–(65) would reduce further to

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\left[A_r(z)-\frac{A_r(z)}{1+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{66}$$

$$c_1\frac{dA_i(z)}{dz}=-\gamma_c\left[A_i(z)-\frac{A_i(z)}{1+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{67}$$

$$c_1\frac{dB_r(z)}{dz}=-\gamma_c\left[2A_r(z)+\delta z_0[A_i(z)-B_i(z)]-2\frac{A_r(z)}{1+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{68}$$

$$c_1\frac{dB_i(z)}{dz}=-\gamma_c\left[2A_i(z)-\delta z_0[A_r(z)-B_r(z)]-2\frac{A_i(z)}{1+A_r^2(z)+A_i^2(z)}D_0(z)\right], \tag{69}$$

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\delta z_0[B_i(z)-A_i(z)], \tag{70}$$

$$c_1\frac{dA_i(z)}{dz}=\gamma_c\delta z_0[B_r(z)-A_r(z)]. \tag{71}$$

The approximate set consisting of Eqs. (66) and (67) actually simplifies a little further, since for an appropriate choice of phase $A_i(z)$ can be put equal to zero. The single remaining equation is equivalent to the amplifier model which was solved originally for the case that D_0 is a constant [1,2] and it is considered further in Ref. [9]. While the more accurate model consisting of Eqs. (68)–71) is larger, the parameters involved are basically the same, and the equations have the same structure as the approximate set. For laser amplifier studies it is sometimes convenient to express the laser behavior in terms of a threshold parameter r, which is the ratio of the constant pumping rate D_0 to its value when the unsaturated $[A_r^2(z)+A_i^2(z)=0]$ gain at line center ($y=0$) is just sufficient to make the field derivatives vanish. However, one readily finds that D_0 under these conditions has the value unity for all of the homogeneously broadened amplifier models considered here. Therefore the parameter D_0 may simply be replaced by the threshold parameter r in the constant-pump models.

The preceding analysis provides a formalism for calculating the electric and magnetic fields in a laser amplifier. It is also helpful to identify for display some other quantities of practical interest. In particular, it is convenient to introduce intensity and energy density functions, and suitable definitions of these quantities were discussed in Ref. [8]. Like the fields themselves, these energy-related quantities are a bit too complicated to provide a direct representation of the laser behavior. However, averaging over a time of one-half optical period eliminates certain complex exponentials and leads to the simple normalized formulas

$$I(z)=A_r(z)B_r(z)+A_i(z)B_i(z), \tag{72}$$

$$U_e(z)=A_r^2(z)+A_i^2(z), \tag{73}$$

$$U_m(z)=B_r^2(z)+B_i^2(z), \tag{74}$$

where $I(z)$ is the intensity at a location z in the amplifier, $U_e(z)$ is the local energy density of the electric field, and $U_m(z)$ is the local energy density of the magnetic field. For ordinary plane-wave applications in low-gain media, the normalized electric-field amplitudes $A_r(z)$ and $A_i(z)$ are almost equal to the magnetic-field amplitudes $B_r(z)$ and $B_i(z)$. In that limit the three energy measures given in Eqs. (72)–(74) are also almost equal. However, for the very-high-gain systems of interest here, it will be found that the electric and magnetic energies may differ substantially.

The functions given in Eqs. (72)–(74) focus, in effect, on the amplitude characteristics of the electromagnetic fields. The phase characteristics are also of interest. One convenient way to explore the phase properties of the rapidly varying laser fields is to introduce the concept of local wavelength. This wavelength will be defined in terms of the z derivative of the total phase of the field and is analogous to the instantaneous frequency employed in time dependent laser studies. For the electric field given in Eq. (5), the local propagation constant can be written

$$k_e(z)=k+\frac{d}{dz}\tan^{-1}\left[\frac{A_i(z)}{A_r(z)}\right]. \tag{75}$$

Thus the local wavelength $\lambda_e=2\pi/k_e$ can be related to the ordinary wavelength $\lambda=2\pi/k=2\pi c_1/\Omega$ by the equation

$$\frac{\lambda_e(z)}{\lambda}=\left[1+\frac{c_1}{\Omega}\frac{d}{dz}\tan^{-1}\left[\frac{A_i(z)}{A_r(z)}\right]\right]^{-1}. \tag{76}$$

In this study the electric and magnetic fields are partially decoupled, and it is possible for the local wavelength of the electric field to be different from the local wavelength of the magnetic field, which is given by

$$\frac{\lambda_m(z)}{\lambda}=\left[1+\frac{c_1}{\Omega}\frac{d}{dz}\tan^{-1}\left[\frac{B_i(z)}{B_r(z)}\right]\right]^{-1}. \tag{77}$$

The derivative terms in the above models from Eq. (48) and onward simplify if one introduces the normalized length coordinate $\zeta=\gamma_c z/c_1$. In terms of this coordinate and using other notation introduced previously, the difference from unity of the normalized local wavelengths $\lambda_e'(\zeta)=\lambda_e(\zeta)/\lambda$ and $\lambda_m'(\zeta)=\lambda_m(\zeta)/\lambda$ given in Eqs. (76) and (77) can be written

$$\lambda_e'(\zeta)-1=-\left\{1+\delta(y_0+z_0)\times\left[\frac{d}{d\zeta}\tan^{-1}\left[\frac{A_i(\zeta)}{A_r(\zeta)}\right]\right]^{-1}\right\}^{-1}, \quad (78)$$

$$\lambda_m'(\zeta)-1=-\left\{1+\delta(y_0+z_0)\times\left[\frac{d}{d\zeta}\tan^{-1}\left[\frac{B_i(\zeta)}{B_r(\zeta)}\right]\right]^{-1}\right\}^{-1}. \quad (79)$$

These formulas are evaluated in Sec. IV for a high-gain laser amplifier.

IV. RESULTS

Several numerical studies have been performed to examine the implications of the derivative approximation for amplification in a high-gain laser. A typical set of intensity wave forms for a homogeneously broadened amplifier is plotted in Fig. 1. The computations in this case are based on the model given in Eqs. (56)–(59), and the intensity is defined in Eq. (72). The solutions are obtained using a second-order Runge-Kutta method. The laser parameters used in these examples include the decay rate ratio $\delta=\gamma/\gamma_c=0.3$, threshold parameter $r=20$, initial intensity $I(0)=10^{-4}$, and line center tuning $y=y_0=0$. The intensity is plotted against the normalized distance $\zeta=\gamma_c z/c_1$ for several values of the normalized center frequency $z_0=\omega_0/\gamma$.

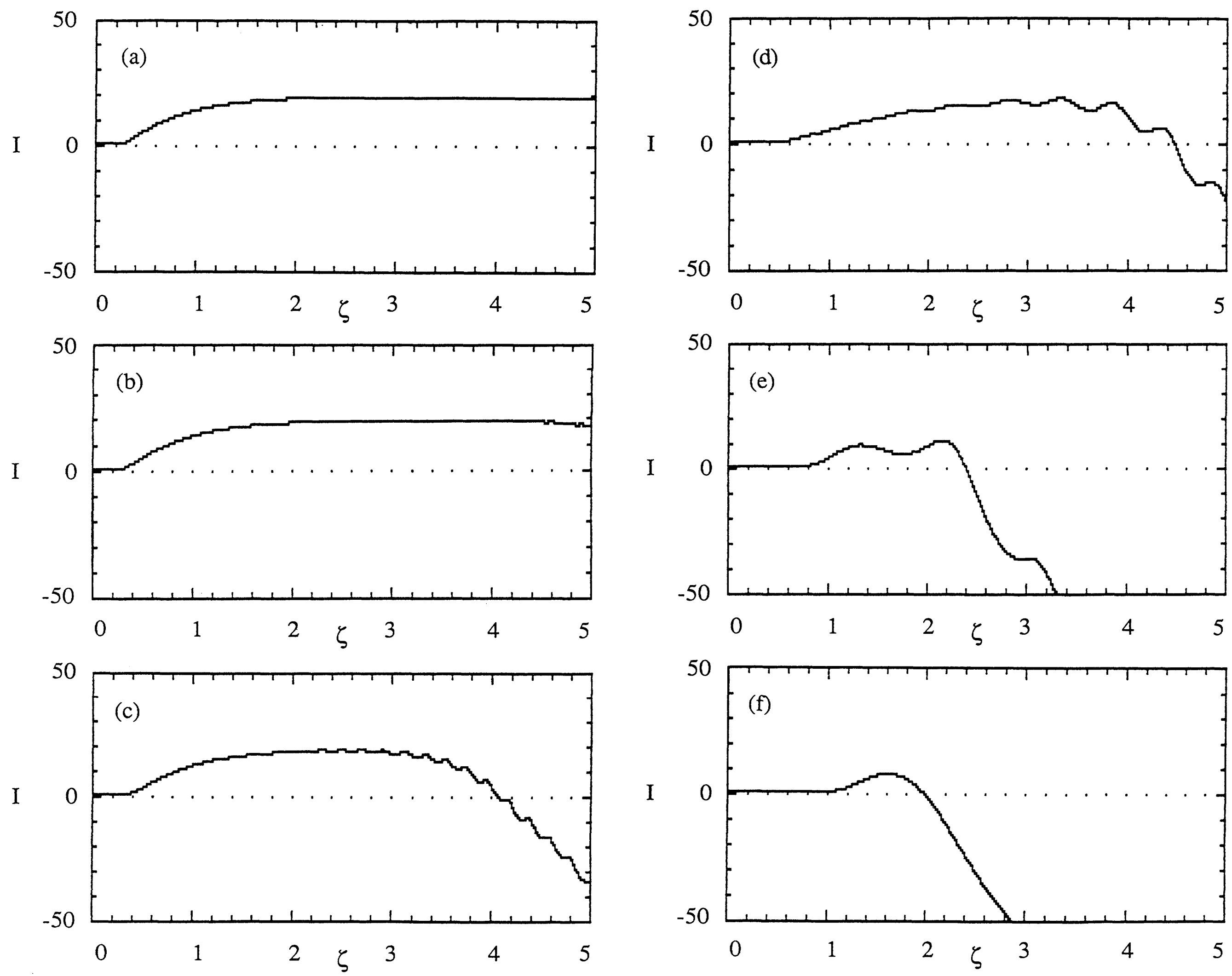

FIG. 1. Normalized intensity as a function of the normalized distance $\zeta=\gamma_c z/c_1$ for a laser amplifier having the decay rate ratio $\delta=\gamma/\gamma_c=0.3$ and the threshold parameter $r=20$. The normalized center frequencies $z_0=\omega_0/\gamma$ for these plots are (a) ∞, (b) 100, (c) 50, (d) 20, (e) 10, and (f) 5. It is clear from these plots that decreasing frequency (or increasing homogeneous linewidth) for this range of operation causes oscillatory behavior and eventually negative or left-traveling intensity.

The curve in Fig. 1(a) is obtained for the limit $z_0=\infty$, and in this limit the field equations given in Eqs. (56)–(59) reduce to Eqs. (54) and (55). This reduction may be understood by considering first Eqs. (58) and (59). In order for the magnetic-field derivatives in these equations to remain finite for large values of z_0, the electric and magnetic fields must approach equality. Thus the terms in Eqs. (56) and (57) which are products of the small differences between the electric and magnetic fields and the large frequency z_o may be replaced by the corresponding electric-field derivatives. Then Eqs. (56) and (57) reduce easily to Eqs. (54) and (55), which are the basis of the more familiar amplifier models.

The intensity function shown in Fig. 1(a) represents a small signal entering the amplifier at $\zeta=0$ and growing rapidly with ζ until approaching its steady-state loss-limited value, which in this case is $r-1=19$ [9]. This behavior is consistent with previous studies of laser amplifiers; and, as discussed in the following paper, various analytic solutions are possible for this limit. However, it is found that this behavior may change substantially for noninfinite values of the normalized center frequency z_0, or equivalently for values of the homogeneous linewidth ($\Delta\nu_h=\gamma/\pi$) not so much smaller than the optical frequency ω_0. The intensity function shown in Fig. 1(b) is obtained at the value $z_0=100$. In this case the intensity is almost the same as in Fig. 1(a) except that there is a hint of an intensity change toward the right-hand side of the figure. In Fig. 1(c) the frequency is $z_0=50$, and in this case there is a conspicuous oscillatory instability. In the normalized distance of $\zeta=5$ the intensity has gone negative contrary to predictions of conventional models. In this formalism the meaning of a negative intensity is that the total field solution is dominated by a traveling wave moving in the negative z direction. As the frequency z_0 is further reduced, there is a steady trend toward larger and slower spatial oscillations; and, for the value $z_0=5$ represented in Fig. 1(f), the intensity variations bear little resemblance to normally expected amplifier behavior. The main conclusion to be drawn from Fig. 1 is that, at least for this general range of parameters, a more accurate laser amplifier model, which avoids the derivative approximation based on the assumption of slowly varying amplitudes, exhibits a striking instability, large intensity magnitudes, and an apparent reflection of the propagating wave. A more detailed interpretation of this behavior will be given in Sec. V.

The relationship between the electric and magnetic fields can be explored by examining the local energy densities and wavelengths of these fields. Figure 2(a) shows the local energy density $U_m(\zeta)$ and the local wavelength shift $\Delta\lambda'_m(\zeta)=\lambda'_m(\zeta)-1$ associated with the magnetic field for a high-gain laser amplifier under the same conditions as discussed previously and with the normalized frequency $z_0=5$. A comparison of the energy density curve of Fig. 2(a) with the intensity curve of Fig. 1(f) shows that the magnetic energy density is very different from the intensity; and, in particular, the energy density is always positive. Also, from Fig. 2(a) one finds that the local wavelength shift for the magnetic field is about -0.5 within the region of unsaturated growth ($\lambda'_m\approx0.5$), becoming positive with saturation ($\lambda'_m>1$), and negative again when the intensity becomes negative ($\lambda'_m\approx-0.8$). Thus, for the somewhat extreme parameter values adopted here, the local wavelength $\lambda'_m(\zeta)$ in the amplifying medium may be dramatically different from the background value λ.

Qualitatively similar results are obtained for the electric energy density U_e and local wavelength $\lambda'_e(\zeta)$, as shown in Fig. 2(b). The parameters in the model for Fig. 2(b) are the same as used in Figs. 1(f) and 2(a). Quantitatively, however, the electric energy density is quite

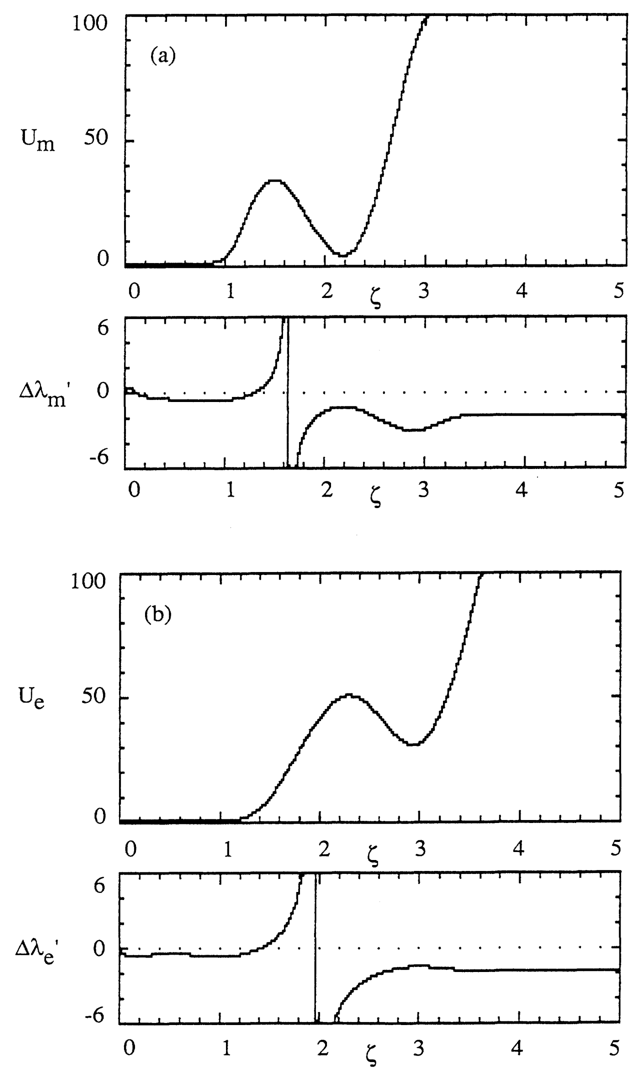

FIG. 2. Energy density and local wavelength shift for the same laser as Fig. 1(f). (a) shows the energy and wavelength shift of the magnetic field, and (b) shows the same quantities for the electric field. These energy densities are very different from each other and may differ in sign from the intensity. Both fields have a downward shift in wavelength, going upward with saturation and reversing sign with the intensity.

different from both the intensity and magnetic energy density. The detailed variations of the electric local wavelength are also quite different from the behavior of the magnetic local wavelength. These differences become less significant for larger values of the lifetime ratio δ and the normalized frequency z_0.

As noted in our recent study of laser dynamics [8], one of the consequences of using a more accurate model of the field equations is that one must also be more specific in describing the cavity losses. It is conventional to simply generalize the conductivity losses to include all of the other losses that might occur due to scattering and absorption, and that idea was employed in the above derivations. However, as the corrections implied by the more exact treatment become larger, it may be important to distinguish between the loss rates for the electric and magnetic fields. For the case of both electric and magnetic losses, Eqs. (56)–(59) may be generalized to

$$c_1\frac{dB_r(z)}{dz}=-\gamma_c\left[2(1-f)A_r(z)+\delta[(y+z_0)A_i(z)-(y_0+z_0)B_i(z)]+\frac{2(y+z_0)}{z_0}\frac{yA_i(z)-A_r(z)}{1+y^2+A_r^2(z)+A_i^2(z)}D_0(z)\right], \quad (80)$$

$$c_1\frac{dB_i(z)}{dz}=-\gamma_c\left[2(1-f)A_i(z)-\delta[(y+z_0)A_r(z)-(y_0+z_0)B_r(z)]-\frac{2(y+z_0)}{z_0}\frac{yA_r(z)+A_i(z)}{1+y^2+A_r^2(z)+A_i^2(z)}D_0(z)\right], \quad (81)$$

$$c_1\frac{dA_r(z)}{dz}=-\gamma_c\{2fB_r(z)+\delta[(y+z_0)B_i(z)-(y_0+z_0)A_i(z)]\}, \quad (82)$$

$$c_1\frac{dA_i(z)}{dz}=-\gamma_c\{2fB_i(z)-\delta[(y+z_0)B_r(z)-(y_0+z_0)A_r(z)]\}, \quad (83)$$

where the factor f identifies the fraction of the losses that are associated with the magnetic rather than the electric field. One readily finds that in the limit of large z_0 the set of equations given as Eqs. (80)–(83) reduces to Eqs. (54) and (55). Thus conventional analyses are independent of f; but, as in our laser dynamics study, one can show that for smaller values of z_0 this factor can have a significant effect on the amplifier behavior.

V. PLUS AND MINUS WAVE COMPONENTS

The solutions of the high-gain laser model given in Eqs. (56)–(59) or more generally in Eqs. (80)–(83) show very clearly that for an injected signal propagating in the positive direction there may develop one or more regions of negative intensity. As indicated above, the interpretation of this negative intensity is that the wave energy is moving primarily in the negative direction. This interesting behavior cannot occur spontaneously in a more conventional amplifier model as represented by Eqs. (54) and (55), and thus it merits special attention.

The fundamental reason that the apparently positive propagating wave assumed in Eqs. (5), (6), and (23) permits negatively traveling solutions is that the assumed wave allows an arbitrarily rapid spatial dependence of the complex field and polarization amplitudes. It is clear, for example, that an amplitude variation $\exp(-2ikz)$, when multiplied by the usual traveling-wave exponential $\exp(ikz-i\omega t)$, yields in effect a wave propagating in the negative direction. In spite of this directional ambiguity of rapidly varying waves in the rigorous field equations, it is still possible to construct a useful separation of the plus and minus wave components. For this purpose one must keep track of the amplitudes and phases of both the electric and magnetic fields.

As a first step in the field separation, it will be assumed that the total electric field can be written as the sum of separate plus and minus traveling-wave field components. Then with Eq. (5) one has

$$\begin{aligned}E(z,t)&=\tfrac{1}{2}E'(z)\exp(ikz-i\omega t)+\text{c.c.}\\&=\tfrac{1}{2}[E'^{+}(z)\exp(ikz-i\omega t)+E'^{-}(z)\exp(-ikz-i\omega t)]+\text{c.c.}\\&=\tfrac{1}{2}[E'^{+}(z)+E'^{-}(z)\exp(-2ikz)]\exp(ikz-i\omega t)+\text{c.c.}\end{aligned} \quad (84)$$

It follows from this equation that the new plus and minus amplitudes $E'^{+}(z)$ and $E'^{-}(z)$ are related to the amplitude function $E'(z)$ of Eq. (5) by

$$E'(z)=E'^{+}(z)+E'^{-}(z)\exp(-2ikz). \quad (85)$$

In normalized units this is

$$A(z)=A^{+}(z)+A^{-}(z)\exp(-2ikz). \quad (86)$$

Similarly for the magnetic field one obtains

$$B(z)=B^{+}(z)+B^{-}(z)\exp(-2ikz). \quad (87)$$

The functions $A(z)$ and $B(z)$ may be considered to be known, since, as shown in Sec. IV, they may be obtained from straightforward numerical solutions of the differential equations. Our purpose now is to find expressions for the new amplitude functions of the plus and minus wave components in terms of the original field amplitudes.

Equations (86) and (87) can be interpreted as two complex equations for the four complex unknowns $A^+(z)$, $A^-(z)$, $B^+(z)$, and $B^-(z)$. Other relations between the electric- and magnetic-field amplitudes may be sought in Maxwell's equations. While the plus and minus waves cannot separately be exact solutions of the wave equations in the amplifying medium, simple relationships exist when the gain and loss terms vanish. Thus we will imagine the amplifying medium to contain thin lossless and gainless regions in which the plus and minus waves are well defined. In these regions the simplified forms of Eqs. (1) and (2) lead to the two additional constraints

$$E'^+(z)=H'^+(z) , \tag{88}$$

$$E'^-(z)=-H'^-(z) . \tag{89}$$

In the normalized units introduced previously, these constraints are

$$A^+(z)=B^+(z) , \tag{90}$$

$$A^-(z)=-B^-(z) . \tag{91}$$

Equations (86), (87), (90), and (91) are a complete set that one can use to obtain the amplitudes of the positive and negative wave components. First, Eqs. (90) and (91) may be used to eliminate the magnetic amplitudes, and Eq. (87) becomes

$$B(z)=A^+(z)-A^-(z)\exp(-2ikz) . \tag{92}$$

Now Eqs. (86) and (92) may be solved, and the results for the electric-field amplitudes are

$$A^+(z)=\tfrac{1}{2}[A(z)+B(z)] , \tag{93}$$

$$A^-(z)=\tfrac{1}{2}[A(z)-B(z)]\exp(2ikz) . \tag{94}$$

With Eqs. (90) and (91) the magnetic-field amplitudes are

$$B^+(z)=\tfrac{1}{2}[B(z)+A(z)] , \tag{95}$$

$$B^-(z)=\tfrac{1}{2}[B(z)-A(z)]\exp(2ikz) . \tag{96}$$

Thus the plus and minus field amplitudes are now expressed explicitly in terms of the functions obtained from the numerical models.

Quantities of particular interest here include the intensities of the plus and minus wave components. With Eqs. (72) and (93)–(96) one obtains

$$\begin{aligned} I^+(z)&=A_r^+(z)B_r^+(z)+A_i^+(z)B_i^+(z)\\ &=\tfrac{1}{4}[A_r(z)+B_r(z)]^2+\tfrac{1}{4}[A_i(z)+B_i(z)]^2 , \end{aligned} \tag{97}$$

$$\begin{aligned} I^-(z)&=-[A_r^-(z)B_r^-(z)+A_i^-(z)B_i^-(z)]\\ &=\tfrac{1}{4}[A_r(z)-B_r(z)]^2+\tfrac{1}{4}[A_i(z)-B_i(z)]^2 , \end{aligned} \tag{98}$$

where as usual the subscripts r and i refer to the real and imaginary parts of the corresponding functions. With these definitions the intensity component $I^+(z)$ propagating in the plus direction and the component $I^-(z)$ propagating in the minus direction are both always positive. From Eqs. (72), (97), and (98), one can readily verify the simple intensity conservation theorem

$$I(z)=I^+(z)-I^-(z) . \tag{99}$$

Other analytic relationships between the various intensities and energy densities are discussed in detail in the following paper [9].

It is now of interest to plot the component intensities $I^+(z)$ and $I^-(z)$ to see how they compare to the previous results. Figure 3 contains plots of $I^+(z)$ and $I^-(z)$ for exactly the same conditions as in the plots of Fig. 1 for the total intensity $I(z)$. Since these component intensities can not change sign the zero of the vertical scale is moved to the bottom of the plot. The plot of $I^+(z)$ in Fig. 3(a) (the upper curve) for the frequency limit $z_0=\infty$ is identical to the plot of the total intensity $I(z)$ in Fig. 1(a). The function $I^-(z)$ remains zero for this value of z_0 and is represented by the thickened base line of the figure.

The intensity functions shown in Fig. 3(b) are obtained for $z_0=100$. In this case there is evidence of small rapid oscillation of the plus intensity and growth of the negative intensity near $\zeta=5$. For the diminishing z_0 values represented by Figs. 3(c) and 3(d), the oscillations become slower, and the growth of the negative wave is more substantial. The magnitude of the negative wave may become much larger than the positive wave. For the still smaller values of z_0 represented by Figs. 3(e) and 3(f) the normally expected amplifier behavior is not discernible at all.

In some ways the results that have just been described appear quite strange. Of particular concern might be the fact that for the small values of z_0 and a small input signal both the plus and minus intensities and their sum $I(z)$ approach values which are far larger than required to saturate the gain of the laser to a negligible value. The interpretation of this behavior is that our apparently logical choice of initial conditions would be difficult to achieve in an experiment. Those conditions require, in effect, that at the input the negative wave has zero intensity. Because the amplifier tends to be reflective, the only way to achieve this initial condition is by injecting a precisely chosen and possibly very large negative wave at the output of the amplifier. The necessary magnitude of this injected wave is, of course, revealed by numerical solutions such as those in Fig. 3.

While the numerical results in Fig. 3 are of immediate academic interest, it is also worthwhile to obtain results that would correspond more closely to experiments. From the above interpretation it is evident that one should allow a nonzero reflected wave to emerge from the laser input. Instead of focusing only on the input, more attention must be applied to the amplifier output. For simplicity it will be assumed here that both ends of the laser amplifier are antireflection coated, or alternatively that the amplifier is immersed in a medium having the same refracting properties μ_1 and ϵ_1 as itself. In this case one of the fundamental boundary conditions for most practical applications would be that the negative wave is zero at the output. Thus we have a condition on the negative intensity at the output $[I^-(\zeta_{out})=0]$ and the positive intensity at the input $[I^+(\zeta=0)=I_{in}]$.

Integration of the differential equations of the laser model with these new boundary conditions requires a different procedure, since the actual field values are not known initially at either end of the laser. One possibility would be to obtain the fields at the input end by combining the given input intensity with a guessed value for the reflected wave. The output fields could then be computed and the guess varied until there is no negative wave at the output. One deficiency of this procedure, however, is that the intermediate results in which the negative wave has nonzero intensity at the output have no practical interest. We have found it more useful to simply integrate the equations backward from the output to the input using zero for the initial intensity of the negative wave at the output and a guess for the positive wave at the output. One can then iterate this procedure with various guesses until the positive wave at the input has the desired intensity. At that point both the net gain and reflection will have been determined. An advantage of this latter procedure is that each of the intermediate calculations corresponds to an experimental situation which, at least in principle, might be realized in practice.

A series of plots of the plus and minus intensities is given in Fig. 4 as a function of distance from the laser output at $\zeta=1.0$ to the input at $\zeta=0$. As before, the

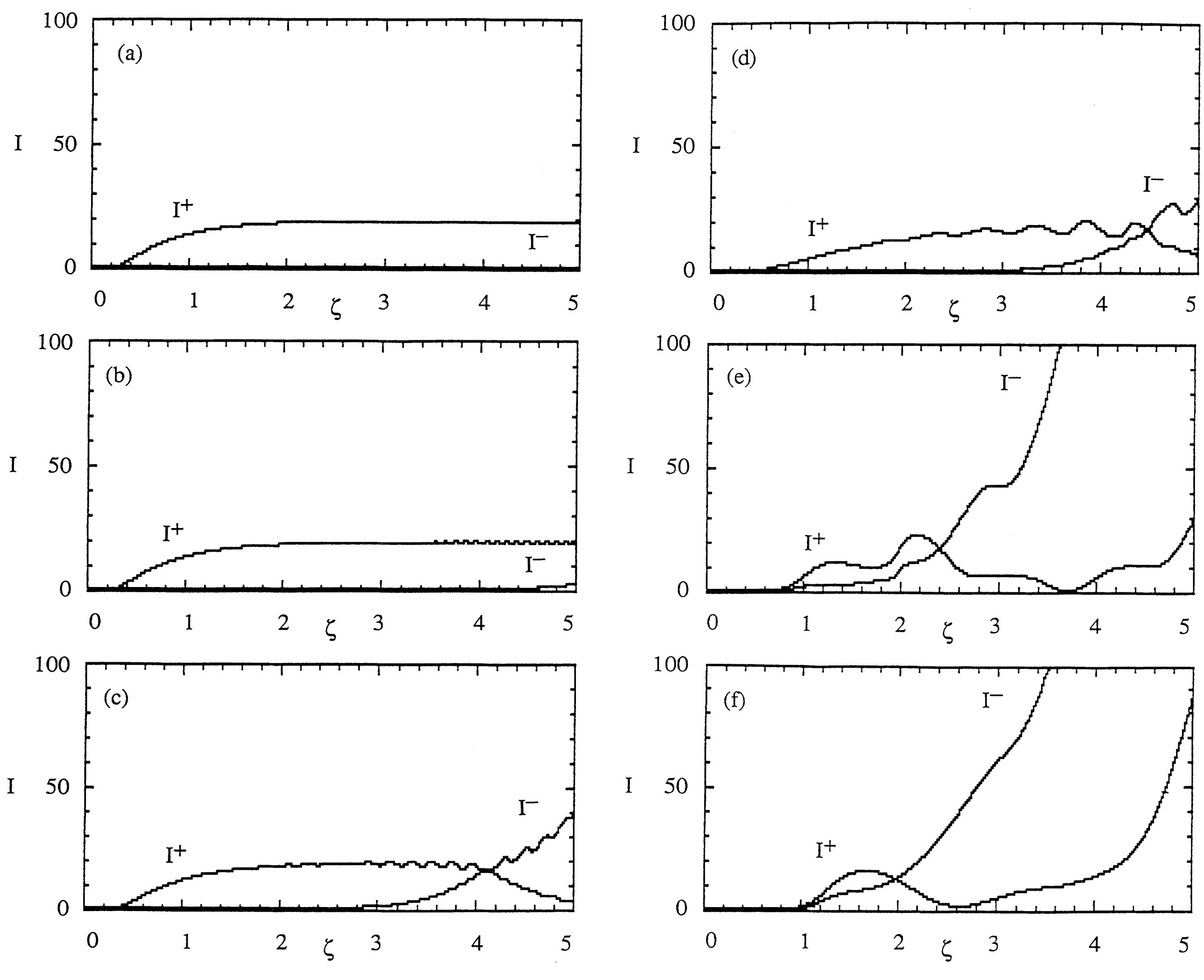

FIG. 3. Normalized intensity components of the plus and minus waves for the same conditions as Fig. 1. The normalized center frequencies for these plots are (a) ∞, (b) 100, (c) 50, (d) 20, (e) 10, and (f) 5. The curve that starts higher to the left side of the plots is always $I^+(z)$, while the curve that starts lower is always $I^-(z)$. The instability and oscillation of the plus and minus waves are evident.

basic amplifier parameters include the decay rate ratio $\delta=0.3$, pumping rate $D_0=20$, and line center tuning $y=y_0=0$; and here the normalized center frequency is $z_0=100$ for all of the parts of the figure. In Fig. 4(a) the assumed intensity of the plus wave at the output end is $I^+(1.0)=16$, which is a bit below the usual loss-limited value of $r-1=19$. The upper curve shows the ζ dependence of the intensity of the plus wave, and the thickened base line indicates that the intensity of the minus wave does not get far above zero. From the computation we find that the plus intensity at the input is $I^+(0)=1.37$ while the minus intensity at the input is $I^-(0)=0.022$. Thus the net gain of the amplifier in this case is $G=I^+(1.0)/I^+(0)\approx 11.7$, and the effective intensity reflectivity is $R=I^-(0)/I^+(0)\approx 0.016$.

In Fig. 4(b) the assumed output intensity of the plus wave is $I^+(1.0)=14$. The computed intensities at the input are $I^+(0)=7.24\times 10^{-3}$ and $I^-(0)=0.372$. Thus the gain and reflectivity in this case are $G\approx 1.934\times 10^3$ and $R\approx 5.14$. It is evident that the reflectivity can be much larger than unity. It is also clear that this field configuration could be reproduced with no external input at all if a mirror with a reflectivity of less than 2% were suitably positioned at the amplifier input end. Thus the more rigorous laser-amplifier models presented here permit the operation of laser oscillators having only one mirror or in some cases no mirrors at all.

Further plots of intensity functions are given in Figs. 4(c)–4(f) with diminishing values of the assumed output intensity. In Fig. 4(c) the output intensity is $I^+(1.0)=12$, and the reflected intensity $I^-(0)=0.97$ is clearly visible at the laser input. In Figs. 4(d)–4(f) the assumed output decreases further and the reflected wave eventually becomes larger than the output. There is a near symmetry between the input and output for the results shown in Fig. 4(e).

VI. CONCLUSION

One of the oldest and most basic areas of study relating to lasers concerns the operation of simple cw one-

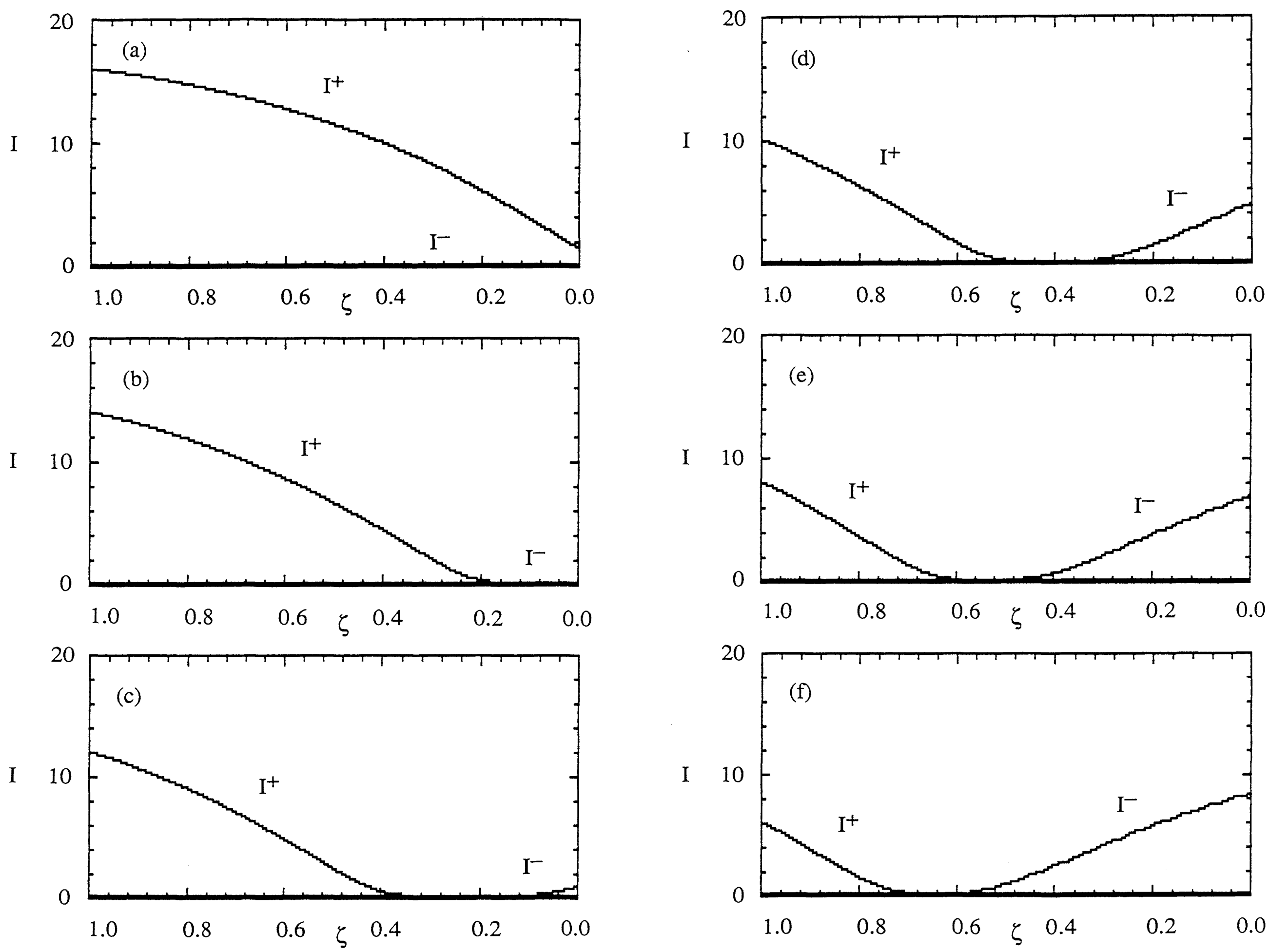

FIG. 4. Reversed plots of the intensity components of the plus and minus waves for an amplifier which has no minus wave at the output. The amplifier parameters are the same as in Fig. 1, and the center frequency is $z_0=100$. The output intensities $I^+(1.0)$ for these plots are (a) 16, (b) 14, (c) 12, (d) 10, (e) 8, and (f) 6. The growth from zero of the reflected wave is evident in (c)–(f).

dimensional laser amplifiers. In amplifier studies it is usual to employ a derivative approximation based on the assumption that the amplitude of the electric field varies slowly compared to a wavelength. The purpose of this research has been to develop laser amplifier models that do not make this approximation. It is found that in systems having high gain and large bandwidth the derivative approximation may cause significant errors and may obscure entirely some important aspects of amplifier behavior. The concept of local wavelength has been introduced, and it has been shown that for a given optical frequency the wavelength may vary with distance along a laser amplifier. It has also been shown by means of numerical solutions that unidirectional propagation may be unstable in laser amplifiers, and a formalism has been developed for treating the plus and minus intensity components for the general amplifier field configuration. The internally reflected waves may become much larger than the normally expected unidirectional signal, and laser oscillation with a single cavity mirror should be possible. The reflections may relate to the tendency of ring lasers to run in both directions, and they may also influence the behavior of high-gain mirrorless lasers that are understood to involve amplified spontaneous emission or superradiance.

The quantitative implications of the more rigorous model for any specific practical laser system cannot be fully assessed from the numerical examples given above. In the following paper analytic solutions and stability criteria are considered, which make these ideas more applicable to specific lasers [9].

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