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# Relativistic multiphoton bremsstrahlung

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It was shown some time ago by Brown and Goble [Phys. Rev. **173**, 1505 (1968)] that the low-frequency approximation for single-photon bremsstrahlung derived originally by Low [Phys. Rev. **110**, 974 (1958)] could be extended to the two-photon case. A further extension to the case of $N$-photon bremsstrahlung is obtained here. For definiteness the scattering system is taken to be that of a Dirac electron in the presence of a static potential, but greater generality is possible, since, as in the earlier treatments, the argument is based on the generally valid gauge-invariance requirement. The first two terms in the low-frequency expansion may be constructed from a knowledge of the on-shell field-free scattering amplitude. If, in addition, the physical single-photon bremsstrahlung amplitude is known to first order in the frequency, the first *four* terms in the expansion of the $N$-photon amplitude, for $N>1$, may be determined. Some remarks are included concerning the possible utility of the theorem in external-field problems and in the derivation of infrared radiative corrections to scattering processes.

## I. INTRODUCTION

While single-photon spontaneous bremsstrahlung has long been subject to experimental and theoretical study, free-free transitions of higher order are very difficult to detect. As a result, the theory of such higher-order processes has received relatively little attention, perhaps less than it deserves. What has been done may be summarized very briefly. Calculations of two-photon emission in the nonrelativistic scattering by a pure Coulomb potential have been performed [1,2], motivated in part by the initiation of experimental studies [3]. An approximation for multiphoton bremsstrahlung, limited to low frequencies but applicable to scattering by an arbitrary short-range potential, was developed subsequently [4]. A simplified model was used in this latter work; it was restricted to nonrelativistic scattering energies, within the dipole approximation, and with spin effects ignored. To remove these restrictions we have reformulated the problem in the context of the scattering of a Dirac electron, as described below. To carry out this extension it was necessary to adopt a different approach, one making use of the gauge-invariance requirement in a more direct fashion, in a manner first employed by Brown and Goble [5] in their treatment of two-photon bremsstrahlung by a spin-zero particle.

The restriction to potential scattering is retained here for reasons of notational simplicity and definiteness. As emphasized by Low [6] in his original derivation of the low-frequency approximation for single-photon bremsstrahlung, the gauge-invariance argument is of general applicability so that the detailed structure of the target need not be considered. In fact, the underlying scattering process may itself involve the emission of soft photons, suggesting the possibility, realized here, of an $N$-photon generalization of Low's theorem. Aside from its methodological interest the result may be of some practical use in applications where the electric charge is not the relevant measure of the strength of the perturbation. In such cases higher-order (multiphoton) processes can play a more significant role. The infrared divergence problem provides one such example; here one must consider real and virtual processes involving an arbitrary number of photons. It is sufficient, for the purpose of removing the divergence, to retain only that contribution to the $N$-photon amplitude which has the strongest singularity in the zero-frequency limit. Finite corrections of higher order may be obtained by employing improved low-frequency approximations for the bremsstrahlung amplitudes. Self-energy corrections due to emission and reabsorption of virtual photons are ignored in the following derivation of the soft-photon approximation. This is not a significant limitation since such corrections may be generated from a given multiphoton amplitude in which they do not appear by including the appropriate photon propagation factors and performing an integral over photon momenta, in a manner described in detail previously [7]. An illustration of this procedure has been provided earlier in a derivation of second-order infrared radiative corrections for the scattering of a spin-zero particle [8].

The strength of the perturbation is enhanced in the *stimulated* bremsstrahlung process. There will be a range of external field intensities for which it will be necessary to include the effect of multiphoton contributions within the context of finite-order perturbation theory. The approximation developed here can be useful in such circumstances though it will, of course, break down for sufficiently intense fields. If the field is not of the plane-wave type, that is, if the different modes do not have the same propagation direction, the exact solutions describing a Dirac electron in the presence of such a field are not available. Perturbation theory will then serve as a useful guide, and the low-frequency approximation can be of particular value.

For the sake of clarity, the approximation will be developed in stages, with separate analyses given for the emission of one, two, and three soft photons. (The results are easily modified to include absorption processes as well.) In the simplest version of the approximation one expresses the $N$-photon amplitude in terms of the on-shell field-free scattering amplitude. The first two terms in an expansion in powers of the frequency are obtained correctly in this manner. The next two terms in the expansion may be included provided the single-photon bremsstrahlung amplitude, with initial and final electron momenta on the mass shell, is known to first order in the frequency. The momenta of the emitted photons are not assumed to be on the mass shell, thus allowing for applications involving virtual photons.

## II. SOFT-PHOTON APPROXIMATIONS

### A. One-photon amplitude

While Low's theorem for single-photon bremsstrahlung [6] is well known, it will now be rederived to establish notation and to indicate the approach to be followed in the multiphoton case. We first introduce the physical, field-free amplitude for the potential scattering of a Dirac electron in the form $t(p',p)=\bar{w}(p')f(p',p)w(p)$, where, following standard notation [9], the equations satisfied by the Dirac spinors are taken to be

$$(\not p-m)w(p)=0,\quad \bar{w}(p')(\not p'-m)=0\ , \tag{2.1}$$

with $\not p\equiv p\cdot\gamma\equiv p^{\mu}\gamma_{\mu}$ and $p^2-m^2=0$, $p'^2-m^2=0$. The amplitude $M^{(1)}(p',p;k\epsilon)$ for the emission of a photon of momentum $k$ and polarization $\epsilon$ during the scattering event is represented as the sum

$$M^{(1)}=M_1^{(1)}+M_2^{(1)}+M_3^{(1)}\ . \tag{2.2}$$

The first term is chosen so that it contains the two pole singularities at $2p\cdot k-k^2=0$ and $2p'\cdot k+k^2=0$ that appear in the exact amplitude in the zero-frequency limit; we define

$$M_1^{(1)}(p',p;k\epsilon)=\bar{w}(p')[\not\epsilon(\not p'+\not k-m)^{-1}f(p'+k,p)+f(p',p-k)(\not p-\not k-m)^{-1}\not\epsilon]w(p). \tag{2.3}$$

The gauge-invariance condition $M^{(1)}(p',p;kk)=0$ is not satisfied in this approximation; one finds, in particular, that

$$M_1^{(1)}(p',p;kk)=\bar{w}(p')[f(p'+k,p)-f(p',p-k)]w(p)\ . \tag{2.4}$$

In arriving at this result we have made use of the relations

$$(\not p-\not k-m)^{-1}\not k w(p)=-w(p)\ , \tag{2.5a}$$

$$\bar{w}(p')\not k(\not p'+\not k-m)^{-1}=\bar{w}(p')\ , \tag{2.5b}$$

each of which is verified immediately by multiplying through by the inverse of the propagator and making use of Eq. (2.1).

The amplitude $M_2^{(1)}$ is chosen to make the sum $M_1^{(1)}+M_2^{(1)}$ gauge invariant [5]. To emphasize that the scattering amplitude is on the mass shell we express it in terms of the scalar variables $\nu=n\cdot p$ and $\tau=(p'-p)^2$, with $n_{\mu}=\delta_{\mu 0}$. The variables $\xi=p^2-m^2$ and $\xi'=p'^2-m^2$ vanish on shell and are suppressed to simplify notation. Then, with $f(p',p)$ replaced by $f(\nu,\tau)$, we have

$$M_2^{(1)}(p',p;k\epsilon)=\frac{\epsilon_0}{\omega}\bar{w}(p')[f(\nu-\omega,\tau)-f(\nu,\tau)]w(p)\ , \tag{2.6}$$

where $\tau=(p'+k-p)^2$ and $\omega\equiv k_0$. Since this correction term is nonsingular in the limit of zero frequency, it follows that the remainder $M_3^{(1)}=\epsilon^{\mu}M_{3\mu}^{(1)}$ is both nonsingular and gauge invariant, so that $k^{\mu}M_{3\mu}^{(1)}=0$. One concludes, after differentiation of this last equation with respect to the photon momentum, that $M_3^{(1)}$ vanishes in the zero-frequency limit. This is the content of Low's theorem. It will be convenient to write

$$M_3^{(1)}(p',p;k\epsilon)=\bar{w}(p')R(p',p;k\epsilon)w(p)\ , \tag{2.7}$$

with the matrix $R$ understood to be of first order in $k$. The leading term $M_1^{(1)}$ is represented diagrammatically in Fig. 1(a); the nonsingular remainder $M_2^{(1)}+M_3^{(1)}$ is pictured in Fig. 1(b).

### B. Two-photon amplitude

The two-photon emission amplitude is represented as

$$M^{(2)}(p',p;k_1\epsilon_1k_2\epsilon_2)=\bar{w}(p')F^{(2)}(k_1\epsilon_1k_2\epsilon_2)w(p)\ , \tag{2.8}$$

where, to simplify notation, the dependence of $F^{(2)}$ on the electron momenta is suppressed. Following the pattern established above, we look for $F^{(2)}$ in the form

$$F^{(2)}=F_1^{(2)}+F_2^{(2)}+F_3^{(2)}\ , \tag{2.9}$$

where $F_1^{(2)}$ contains the double-pole singularity that ap-

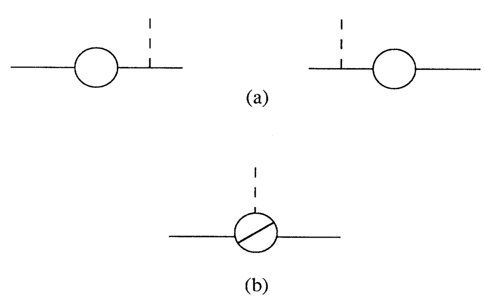


FIG. 1. Diagrams representing (a) the dominant contributions to the amplitude for the emission of a single low-frequency photon either before or after the collision, whose amplitude is represented by the open circle, and (b) the correction to the dominant term; the diagonal line through the circle is meant to indicate that the singular part of the emission amplitude has been removed.

pears in the zero-frequency limit, $F_2^{(2)}$ is chosen to correct for the violation of gauge invariance in the leading term, and $F_3^{(2)}$ accounts for terms of higher order in a gauge-invariant manner. There are three types of double-pole terms, corresponding to the emission of both photons in the initial state, both in the final state, and one each in the initial and final states. Accordingly, we write

$$F_1^{(2)}=F_{1A}^{(2)}+F_{1B}^{(2)}+F_{1C}^{(2)} . \tag{2.10}$$

In providing the explicit forms of these terms we set $\omega_i=k_{i0}$, $i=1,2$, and $\nu=p_0$, and we suppress the dependence of the on-shell scattering amplitude $f(\nu,\tau)$ on the momentum-transfer variable, which is fixed at the value $\tau=(p'-p+k_1+k_2)^2$. We have

$$F_{1A}^{(2)}=f(\nu-\omega_1-\omega_2)(\not p-\not k_1-\not k_2-m)^{-1}[\not\epsilon_1(\not p-\not k_2-m)^{-1}\not\epsilon_2+\not\epsilon_2(\not p-\not k_1-m)^{-1}\not\epsilon_1] , \tag{2.11a}$$

$$F_{1B}^{(2)}=[\not\epsilon_2(\not p'+\not k_2-m)^{-1}\not\epsilon_1+\not\epsilon_1(\not p'+\not k_1-m)^{-1}\not\epsilon_2](\not p'+\not k_2+\not k_1-m)^{-1}f(\nu) , \tag{2.11b}$$

$$F_{1C}^{(2)}=\not\epsilon_1(\not p'+\not k_1-m)^{-1}f(\nu-\omega_2)(\not p-\not k_2-m)^{-1}\not\epsilon_2+\not\epsilon_2(\not p'+\not k_2-m)^{-1}f(\nu-\omega_1)(\not p-\not k_1-m)^{-1}\not\epsilon_1 . \tag{2.11c}$$

Gauge invariance for the sum $F_1^{(2)}+F_2^{(2)}$ is achieved with the choice

$$F_2^{(2)}(k_1\epsilon_1k_2\epsilon_2)=-F_1^{(2)}(k_1\epsilon_1k_2k_2)\frac{\epsilon_{20}}{\omega_2}-F_1^{(2)}(k_1k_1k_2\epsilon_2)\frac{\epsilon_{10}}{\omega_1}+F_1^{(2)}(k_1k_1k_2k_2)\frac{\epsilon_{10}}{\omega_1}\frac{\epsilon_{20}}{\omega_2} . \tag{2.12}$$

To put this expression in more explicit form we make use of Eq. (2.5a) to write

$$F_{1A}^{(2)}(k_1\epsilon_1k_2k_2)=f(\nu-\omega_1-\omega_2)(\not p-\not k_1-\not k_2-m)^{-1}\times[-\not\epsilon_1+\not k_2(\not p-\not k_1-m)^{-1}\not\epsilon_1] . \tag{2.13}$$

(The eventual inclusion of initial- and final-state spinors is implied here and in the remainder of this calculation.) Further simplification is achieved by making use of the identity

$$(\not p-\not k_1-\not k_2-m)^{-1}\not k_2(\not p-\not k_1-m)^{-1}=(\not p-\not k_1-\not k_2-m)^{-1}-(\not p-\not k_1-m)^{-1} , \tag{2.14}$$

which leads to the relation

$$F_{1A}^{(2)}(k_1\epsilon_1k_2k_2)=-f(\nu-\omega_1-\omega_2)(\not p-\not k_1-m)^{-1}\not\epsilon_1 . \tag{2.15}$$

Proceeding in this fashion for the remaining terms in Eq. (2.12), we arrive at the result

$$\begin{aligned}F_2^{(2)}(k_1\epsilon_1k_2\epsilon_2)=&\{[f(\nu-\omega_1-\omega_2)-f(\nu-\omega_1)](\not p-\not k_1-m)^{-1}\not\epsilon_1\\&+\not\epsilon_1(\not p'+\not k_1-m)^{-1}[f(\nu-\omega_2)-f(\nu)]\}\frac{\epsilon_{20}}{\omega_2}+(1\leftrightarrow2)\\&+[f(\nu-\omega_1-\omega_2)+f(\nu)-f(\nu-\omega_1)-f(\nu-\omega_2)]\frac{\epsilon_{10}}{\omega_1}\frac{\epsilon_{20}}{\omega_2} ,\end{aligned} \tag{2.16}$$

where $(1\leftrightarrow2)$ denotes the preceding terms contained within braces with indices 1 and 2 interchanged.

The correction term $F_3^{(2)}$ is decomposed as

$$F_3^{(2)}=F_{31}^{(2)}+F_{32}^{(2)}+F_{33}^{(2)} . \tag{2.17}$$

$F_{31}^{(2)}$ is chosen so that the amplitude takes on the correct value at the position of the single poles that arise in the zero-frequency limit when one of the photons is emitted in either the initial or final state with the other photon emitted in an intermediate state. We have

$$\begin{aligned}F_{31}^{(2)}(k_1\epsilon_1k_2\epsilon_2)=&[R(p',p-k_1;k_2\epsilon_2)(\not p-\not k_1-m)^{-1}\not\epsilon_1\\&+\not\epsilon_1(\not p'+\not k_1-m)^{-1}R(p'+k_1,p;k_2\epsilon_2)]\\&+(1\leftrightarrow2) .\end{aligned} \tag{2.18}$$

This form fails to satisfy gauge invariance; setting $R(p',p;kk)=0$, to be justified below, we find, for example, that

$$F_{31}^{(2)}(k_1k_1k_2\epsilon_2)=-R(p',p-k_1;k_2\epsilon_2)+R(p'+k_1,p;k_2\epsilon_2) . \tag{2.19}$$

$F_{32}^{(2)}$ is to be chosen to restore gauge invariance, up to terms of first order in the photon momenta. For this purpose it will be convenient to expand the right-hand side of Eq. (2.19) in powers of $k_1$, subject to the mass-shell conditions $2p\cdot k_1-k_1^2=0$ and $2p'\cdot k_1+k_1^2=0$. This may be accomplished by expressing the function $R$ in terms of the scalar variables

$$\nu=n\cdot p,\quad \sigma=\epsilon_2\cdot p\ ,\quad \sigma'=\epsilon_2\cdot p' . \tag{2.20}$$

This is not a complete set of variables but it is sufficient for first-order accuracy; the contributions to the expansion associated with the momentum-transfer variable cancel to first order and terms proportional to $k_2$ contribute only in second order. With Eq. (2.19) rewritten as

$$F_{31}^{(2)}(k_1k_1k_2\epsilon_2)=k_1\cdot\left[n\frac{\partial R}{\partial \nu}+\epsilon_2\left[\frac{\partial R}{\partial \sigma}+\frac{\partial R}{\partial \sigma'}\right]\right]\,, \tag{2.19')}$$

correct to first order, we are led to the choice

$$F_{32}^{(2)}(k_1\epsilon_1k_2\epsilon_2)=-\epsilon_1\cdot\left[n\frac{\partial R}{\partial \nu}+\epsilon_2\left[\frac{\partial R}{\partial \sigma}+\frac{\partial R}{\partial \sigma'}\right]\right]+(1\leftrightarrow 2)\;. \tag{2.21}$$

It may be verified that, to first order,

$$\overline{w}(p')(F_{31}^{(2)}+F_{32}^{(2)})w(p)=\left[M_3^{(1)}\left[p',p-k_1+\epsilon_1\frac{p\cdot k_1}{p\cdot\epsilon_1};k_2\epsilon_2\right]\frac{2p\cdot\epsilon_1}{-2p\cdot k_1+k_1^2}+\frac{2p'\cdot\epsilon_1}{2p'\cdot k_1+k_1^2}M_3^{(1)}\left[p'+k_1-\epsilon_1\frac{p'\cdot k_1}{p'\cdot\epsilon_1},p;k_2\epsilon_2\right]\right]+(1\leftrightarrow 2)\;.$$

This amplitude is gauge invariant to first order. (Recall that the on-shell amplitude $M_3^{(1)}$ is gauge invariant.) It follows that the nonsingular remainder $\overline{w}(p')F_{33}^{(2)}w(p)$ must vanish for either one of the photon momenta going to zero; that is, it is of second order in these momenta. The low-frequency approximation is obtained by ignoring this second-order term.

By regrouping terms in the low-frequency approximation one arrives at a more easily vizualized structure, as pictured in Fig. 2. The diagrams in Figs. 2(a) and 2(b) represent the result of attaching a second photon to either the initial or final electron line in the diagrams of Fig. 1. (A doubling of terms arising from the interchange of photon indices is not indicated explicitly in these graphs.) The violation of gauge invariance thereby introduced is corrected through the addition of the term pictured in Fig. 2(c); this represents the sum of the last term shown in Eq. (2.16) and the amplitude given by Eq. (2.21). An iterative procedure of this type may be used to construct the low-frequency approximation for the $N$-photon bremsstrahlung amplitude, as described below.

### C. $N$-photon amplitude

The generalization of the low-frequency approximation to the $N$-photon case is straightforward, requiring little more than the introduction of the appropriate notation. To begin dealing with this matter we introduce a function $P^{(N)}$ describing the emission of $N$ photons by an electron of momentum $p$, prior to its interaction with the target. This takes the form of a sum, over all permutations of photon indices, of a product of propagators and emissions vertices. We define

$$P^{(N)}(p;k_1\epsilon_1k_2\epsilon_2\cdots k_N\epsilon_N)=\left[\not p-\sum_{j=1}^{N}\not k_j-m\right]^{-1}\not\epsilon_N\left[\not p-\sum_{j=1}^{N-1}\not k_j-m\right]^{-1}\not\epsilon_{N-1}\times\cdots(\not p-\not k_1-m)^{-1}\not\epsilon_1+\cdots\;. \tag{2.22}$$

The ellipsis is meant to indicate that all permutations are to be included. The corresponding emission function for the final state is

$$\widetilde{P}^{(N)}(p';k_1\epsilon_1k_2\epsilon_2\cdots k_N\epsilon_N)=\not\epsilon_1(\not p'+\not k_1-m)^{-1}\not\epsilon_2(\not p'+\not k_1+\not k_2-m)^{-1}\times\cdots\not\epsilon_N\left[\not p'+\sum_{j=1}^{N}\not k_j-m\right]^{-1}+\cdots \tag{2.23}$$

where the ellipsis again indicates permutation. These functions undergo simple transformations when any one of the polarization vectors is replaced by the corresponding momentum. Since the functions are symmetric in photon indices it suffices to state the result for $\epsilon_1\to k_1$, which is

$$P^{(N)}(p;k_1k_1k_2\epsilon_2\cdots k_N\epsilon_N)w(p)=-P^{(N-1)}(p;k_2\epsilon_2\cdots k_N\epsilon_N)w(p)\;, \tag{2.24a}$$

$$\overline{w}(p')\widetilde{P}^{(N)}(p';k_1k_1k_2\epsilon_2\cdots k_N\epsilon_N)=\overline{w}(p')\widetilde{P}^{(N-1)}(p';k_2\epsilon_2\cdots k_N\epsilon_N)\;. \tag{2.24b}$$

These relations are derived by induction with the aid of Eqs. (2.5) and (2.14).

To keep notational complexity at a minimum we focus in the following on the case $N=3$; the more general

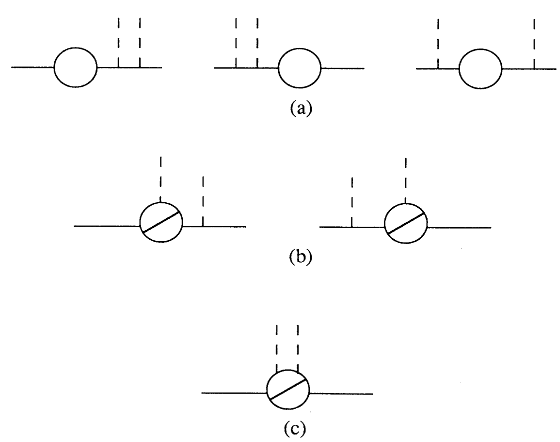


FIG. 2. Diagrams representing (a) the dominant, double-pole contribution to the two-photon bremsstrahlung amplitude, (b) the single-pole contribution, and (c) the nonsingular correction term that must be included to maintain gauge invariance.

treatment will then be clear. We represent the three-photon amplitude as

$$M^{(3)}(p',p;k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)=\bar{w}(p')(F_1^{(3)}+F_2^{(3)}+F_3^{(3)})w(p)\ . \quad (2.25)$$

In specifying the form of the function $F_1^{(3)}$, which contains the triple-pole singularity, we use the abbreviation $[g(123)+\text{c.p.}]\equiv g(123)+g(231)+g(312)$ to indicate a cyclic permutation of indices. We then have

$$F_1^{(3)}(k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)=f(\nu-\omega_1-\omega_2-\omega_3)P^{(3)}(p;k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)+\tilde{P}^{(3)}(p';k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)f(\nu)$$
$$+[\tilde{P}^{(2)}(p';k_1\epsilon_1k_2\epsilon_2)f(\nu-\omega_3)P^{(1)}(p;k_3\epsilon_3)+\text{c.p.}]$$
$$+[\tilde{P}^{(1)}(p';k_1\epsilon_1)f(\nu-\omega_3-\omega_2)P^{(2)}(p;k_2\epsilon_2k_3\epsilon_3)+\text{c.p.}]\ . \quad (2.26)$$

The violation of gauge invariance is removed by adding

$$F_2^{(3)}(k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)=\left[-F_1^{(3)}(k_1k_1k_2\epsilon_2k_3\epsilon_3)\frac{\epsilon_{10}}{\omega_1}+\text{c.p.}\right]$$
$$+\left[F_1^{(3)}(k_1k_1k_2k_2k_3\epsilon_3)\frac{\epsilon_{10}}{\omega_1}\frac{\epsilon_{20}}{\omega_2}+\text{c.p.}\right]-F_1^{(3)}(k_1k_1k_2k_2k_3k_3)\frac{\epsilon_{10}}{\omega_1}\frac{\epsilon_{20}}{\omega_2}\frac{\epsilon_{30}}{\omega_3}\ . \quad (2.27)$$

More explicit forms for these functions are easily obtained with the aid of Eq. (2.24). We find that

$$F_1^{(3)}(k_1k_1k_2\epsilon_2k_3\epsilon_3)=[f(\nu-\omega_2-\omega_3)-f(\nu-\omega_1-\omega_2-\omega_3)]P^{(2)}(p;k_2\epsilon_2k_3\epsilon_3)+\tilde{P}^{(2)}(p';k_2\epsilon_2k_3\epsilon_3)[f(\nu)-f(\nu-\omega_1)]$$
$$+\{\tilde{P}^{(1)}(p';k_2\epsilon_2)[f(\nu-\omega_3)-f(\nu-\omega_1-\omega_3)]P^{(1)}(p;k_3\epsilon_3)+(2\leftrightarrow3)\}\ . \quad (2.28)$$

From this expression we immediately obtain

$$F_1^{(3)}(k_1k_1k_2k_2k_3\epsilon_3)=[f(\nu-\omega_1-\omega_2-\omega_3)-f(\nu-\omega_2-\omega_3)-f(\nu-\omega_1-\omega_3)+f(\nu-\omega_3)]P^{(1)}(p;k_3\epsilon_3)$$
$$+\tilde{P}^{(1)}(p';k_3\epsilon_3)[f(\nu-\omega_1-\omega_2)-f(\nu-\omega_2)-f(\nu-\omega_1)+f(\nu)]\ , \quad (2.29)$$

and this in turn leads us to the relation

$$F_1^{(3)}(k_1k_1k_2k_2k_3k_3)=f(\nu)-f(\nu-\omega_1)-f(\nu-\omega_2)-f(\nu-\omega_3)$$
$$+f(\nu-\omega_1-\omega_2)+f(\nu-\omega_1-\omega_3)+f(\nu-\omega_2-\omega_3)-f(\nu-\omega_1-\omega_2-\omega_3)\ . \quad (2.30)$$

The construction of the component $F_2^{(3)}$ according to Eq. (2.27) is determined by these relations.

To complete the specification of the low-frequency approximation we choose

$$F_3^{(3)}=F_{31}^{(3)}+F_{32}^{(3)}\ , \quad (2.31)$$

with

$$F_{31}^{(3)}(k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)=[R(p',p-k_2-k_3;k_1\epsilon_1)P^{(2)}(p;k_2\epsilon_2k_3\epsilon_3)+\text{c.p.}]$$
$$+[\tilde{P}^{(2)}(p';k_2\epsilon_2k_3\epsilon_3)R(p'+k_2+k_3,p;k_1\epsilon_1)+\text{c.p.}]$$
$$+[\tilde{P}^{(1)}(p';k_2\epsilon_2)R(p'+k_2,p-k_3;k_1\epsilon_1)P^{(1)}(p;k_3\epsilon_3)+\text{c.p.}]\ . \quad (2.32)$$

The violation of gauge invariance introduced by this amplitude is canceled by the appropriate choice of $F_{32}^{(3)}$. Following a procedure analogous to that described above in connection with the two-photon amplitude we find that gauge invariance is restored, up to terms which are nonvanishing in the zero-frequency limit, by the choice

$$F_{32}^{(3)}(k_1\epsilon_1k_2\epsilon_2k_3\epsilon_3)=[F_{32}^{(2)}(k_1\epsilon_1k_2\epsilon_2)P^{(1)}(p;k_3\epsilon_3)+\tilde{P}^{(1)}(p';k_3\epsilon_3)F_{32}^{(2)}(k_1\epsilon_1k_2\epsilon_2)+\text{c.p.}]\ , \quad (2.33)$$

where $F_{32}^{(2)}(k_1\epsilon_1k_2\epsilon_2)$ is given by Eq. (2.21), with electron momenta appropriately shifted to account for the presence of the third photon.

The structure of the low-frequency approximation for the three-photon amplitude is clarified by the observation that the diagrams representing this amplitude may be obtained from those shown in Fig. 2 for the two-photon amplitude by attaching an additional photon to either the incoming or outgoing electron line. This is to be done in all possible distinct ways (without overcounting). The construction is completed by adding the term

$$-F_1^{(3)}(k_1k_1k_2k_2k_3k_3)\frac{\epsilon_{10}}{\omega_1}\frac{\epsilon_{20}}{\omega_2}\frac{\epsilon_{30}}{\omega_3},$$

with $F_1^{(3)}(k_1k_1k_2k_2k_3k_3)$ given by Eq. (2.30). The violation of gauge invariance is then of first order in the photon momenta; this is acceptable since terms of this order are not accounted for in the low-frequency approximation for the three-photon amplitude. In a similar way the four-photon amplitude in the soft-photon approximation is obtained by allowing for an additional photon to be radiated in either the initial or final state, with the soft-photon approximation for the three-photon amplitude representing the underlying scattering process. No additional terms need be added since the violation of gauge invariance appears in the terms of order unity and the approximation accounts correctly for the first four terms in the expansion, of inverse fourth, third, second, and first order. This iterative procedure provides us with a recipe for determining the $N$-photon amplitude in the soft-photon approximation. For $N>3$ the terms of order $k^{-N}$, $k^{-N+1}$, $k^{-N+2}$, and $k^{-N+3}$, where $k$ is a representative photon momentum, are given correctly by attaching photons to external lines in the diagrams representing the low-frequency approximation for the emission of $(N-1)$ photons, with no need for additional gauge terms to be included.

## III. SUMMARY

The soft-photon approximation for single-photon bremsstrahlung derived some time ago by Low [6] provides a remarkable illustration of the power and generality of the gauge-invariance requirement. The first two terms in the low-frequency expansion of the bremsstrahlung amplitude are given correctly in Low's approximation in terms of the physical amplitude for scattering without the radiation of the low-frequency photon. Since this underlying scattering process is arbitrary, it may itself involve the emission of other soft photons, suggesting the possibility of an $N$-photon generalization of Low's theorem. The theorem derived here is of just this type. The prescription for constructing the $N$-photon amplitude is quite simple in principle; starting with the amplitude for single-photon emission, the radiation of an additional photon is assumed to take place either in the initial or final state, a process which is explicitly calculable. The first two terms in the low-frequency expansion are given correctly by this construction. Additional accuracy can be achieved (allowing for the determination of the first four terms in the expansion) if the starting point is not Low's approximation for the one-photon amplitude but rather one including a first order correction to it, and if appropriate terms are added to maintain gauge invariance to the required accuracy. To simplify the presentation the formulation was given here in terms of the potential scattering of a Dirac electron; greater generality is possible. As pointed out earlier [5], extensions of Low's theorem can be useful in obtaining more accurate infrared radiative corrections to scattering processes. They can also play a role in the development of low-frequency approximations for stimulated bremsstrahlung, as has been demonstrated previously [10] in the context of the nonrelativistic version of the problem of scattering in the presence of an intense laser field.

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