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Density of eigenvalues of random band matrices

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Using methods of supersymmetry, we calculate the distribution of eigenvalues for random Hermitian band matrices. We show that, if the bandwidth b increases with the dimension of matrices N as $b \propto N^{\beta}$ with some $\beta > 0$, the resulting eigenvalue distribution is given by Wigner's semicircle law as in the case of full random matrices of the Gaussian unitary ensemble.

I. INTRODUCTION

Random matrix theory [1] has recently attracted a lot of interest due to its applications to chaotic systems [2]. Various studies [3,4] have shown that statistical properties of quantum spectra of classically chaotic systems are well described in terms of ensembles of random matrices. For autonomous systems one has to choose among three Gaussian ensembles: the orthogonal (GOE), the unitary (GUE), and the symplectic (GSE) ensemble, according to the symmetries present. In all three cases the elements of the matrices (real symmetric for GOE, Hermitian for GUE, and quaternion-real for GSE) have independent and identical normal distributions. Unfortunately, this approach gives satisfactory results for quantities like the level-spacing distribution or level correlation functions only if the corresponding classical system is fully chaotic in the whole phase space, for it is known [5] that quantum spectra of classically integrable (regular) systems tend to show Poissonian level-spacing distributions, just as a *diagonal* matrix with independently distributed elements. Since generic systems are neither fully chaotic nor integrable, one should expect for them some intermediate situation interpolating between the two cases. Various attempts aiming at a description of such mixed systems in terms of random matrices were undertaken [6–9]. In particular, it was proposed [6,9] that the intermediate situation could be well described by random matrices which have appreciably large elements only in a vicinity of the main diagonal. Such matrices, termed random band matrices, interpolate between diagonal and full ones.

An interesting complication sometimes arises: the so-called dynamical localization. This phenomenon, discovered for the first time in the kicked rotator [10], is similar to Anderson localization in disordered systems. There a random potential causes the energy eigenstates to be exponentially localized in the position representation. The degree of localization (measured with the help of the localization length of eigenvectors) depends on various parameters of the system. A transition between localized and delocalized states can also be described in terms of random band matrices [11–13]. A diagonal random matrix obviously has all states localized in the original basis. The same property holds for tridiagonal matrices [14]. On the other hand, a typical matrix from the Gaussian ensembles has delocalized eigenvectors.

The above remarks suggest investigations of random band matrices. In our paper we calculate the eigenvalue density for Hermitian random band matrices. We show that when the effective band width b (i.e., the number of appreciably large elements in a row or a column) of such a random matrix increases with the matrix dimension N as $b \propto N^{\beta}$ with some $\beta > 0$, the resulting distribution of eigenvalues is the same as for GUE and is given by Wigner's semicircle law [1]. Our result confirms recent numerical findings [15]. We employ Efetov's [16] supersymmetric analysis. These methods were first used in the theory of disordered metals and recently applied to various problems of nuclear physics [17,18]. In our calculations we closely follow the ideas and notations of Guhr [18]. A brief outline of this background material is given in Appendix A.

II. GRADED GENERATING FUNCTION

Let $P(H)d[H]$ denote a probability distribution in an ensemble of random matrices H and $\widetilde{H}(H)$ an arbitrary matrix function of H. For a particular matrix $\widetilde{H}$ the eigenvalue density $\rho(x)$ reads

$$\rho(x) = \sum_{j=1}^{N} \delta(x - x_j) = -\frac{1}{\pi} \text{Im Tr} \frac{1}{x^+ - \widetilde{H}}, \qquad (2.1)$$

where $x_1, \ldots, x_N$ denote the eigenvalues of $\widetilde{H}$ and $x^+ = x + i\epsilon$ with ϵ a positive infinitesimal constant. The ensemble averaged density $\bar{\rho}(x)$ can thus be obtained as the imaginary part of the Green's function

$$R(x) = -\frac{1}{\pi} \int d[H] P(H) \text{Tr} \frac{1}{x^+ - \widetilde{H}}. \qquad (2.2)$$

Using the formulas collected in Appendix A, one can represent $R(x)$ in the form

$$R(x) = -\frac{1}{2\pi}\frac{\partial}{\partial J}Z(J)\bigg|_{J=0}, \tag{2.3}$$

where the generating function $Z(J)$ is a Gaussian in a supersymmetric variable $\mathbf{\Psi} = (\mathbf{z}, \boldsymbol{\xi})^T$

$$Z(J) = \int d[H]P(H)\int d[\mathbf{\Psi}]\exp[2i\mathbf{\Psi}^\dagger(\mathbf{x}^+ - \widetilde{\mathbf{H}} + \mathbf{J})\mathbf{\Psi}]. \tag{2.4}$$

In the above formula, according to the notation described in Appendix A, $\mathbf{z}$ ($\boldsymbol{\xi}$) are N-component vectors of commuting (anticommuting) variables and $\mathbf{x} = \mathrm{diag}(xI, xI)$, $\widetilde{\mathbf{H}} = \mathrm{diag}(\widetilde{H}, \widetilde{H})$, $\mathbf{J} = \mathrm{diag}(-JI, JI)$ are $2N \times 2N$ supermatrices with I being the ordinary $N \times N$ unit matrix.

In the following we want to consider a particular class of random band matrices:

$$\widetilde{H}_{jk} = H_{jk}F_{jk} = H_{jk}F(j-k), \tag{2.5}$$

where the matrix H_{jk} belongs to the Gaussian unitary ensemble, i.e.,

$$P(H) = \mathcal{N}\exp(-\mathrm{Tr}H^2), \quad H^\dagger = H. \tag{2.6}$$

In (2.6) $\mathcal{N}$ is a normalization constant determined by the condition $\int P(H)d[H] = 1$, while $F(k)$ is a smooth, bell-shaped function. In the present paper we choose the Gaussian

$$F(k) = \frac{1}{(2\pi b^2)^{1/4}}\exp\left(-\frac{k^2}{4b^2}\right) = F(-k), \tag{2.7}$$

with b playing a role of a bandwidth. The results, however, to a great extent do not depend on the particular choice of $F(k)$. We define further $V(k) = F^2(k)$ and observe that $V(j-k)$ has the discrete Fourier expansion

$$V(j-k) = \sum_\mu g(\mu)\widetilde{\epsilon}_\mu(j)\widetilde{\epsilon}_\mu(k), \tag{2.8}$$

with

$$\widetilde{\epsilon}_\mu(j) = \begin{cases} \sqrt{\frac{2}{N}}\sin\omega_\mu j & \text{for } \mu \neq 0 \\ \sqrt{\frac{2}{N}}\cos\omega_\mu j & \text{for } \mu \neq 0\ , \end{cases}$$

$$\widetilde{\epsilon}_0(j) = \frac{1}{\sqrt{N}}, \tag{2.9}$$

and

$$\omega_\mu = \frac{2\pi\mu}{N}, \quad g(\mu) = \exp\left(-\frac{b^2\omega_\mu^2}{2}\right). \tag{2.10}$$

The latter expression for $g(\mu)$ holds, of course, only in the limit of large N.

In order to perform the integration over H in (2.4), let us observe that

$$\begin{aligned}\mathbf{\Psi}^\dagger\widetilde{\mathbf{H}}\mathbf{\Psi} &= \sum_{j,k}(z_j^*\widetilde{H}_{jk}z_k + \xi_j^*\widetilde{H}_{jk}\xi_k) \\ &= \sum_{j,k}H_{jk}F_{kj}(z_kz_j^* - \xi_k\xi_j^*) = \mathrm{Tr}H\widetilde{B},\end{aligned} \tag{2.11}$$

where

$$\widetilde{B}_{jk} = F_{jk}(z_jz_k^* - \xi_j\xi_k^*). \tag{2.12}$$

Hence, completing the square in the exponent we obtain

$$\int d[H]P(H)\exp(-2i\mathrm{Tr}H\widetilde{B}) = \exp(-\mathrm{Tr}\widetilde{B}^2) \tag{2.13}$$

and have thus achieved the average over the GUE in (2.4). Unfortunately, the remaining integral over the graded vector $\mathbf{\Psi}$ is no longer a Gaussian one since the matrix $\tilde{B}$ itself is bilinear in $\mathbf{\Psi}$ and $\mathbf{\Psi}^\dagger$. At the expense of introducing further auxiliary integrals, we can replace the quadratic form in $\tilde{B}$ by a linear one after the fashion of (A15). To that end we define $N \times N$ matrices $(D_\mu)_{jk} = \widetilde{\epsilon}_\mu(j)\delta_{jk}$ and 2×2 supermatrices,

$$\mathbf{B}_\mu = \begin{bmatrix} \mathbf{z}^\dagger D_\mu\mathbf{z} & \boldsymbol{\xi}^\dagger D_\mu\mathbf{z} \\ \mathbf{z}^\dagger D_\mu\boldsymbol{\xi} & \boldsymbol{\xi}^\dagger D_\mu\boldsymbol{\xi}\end{bmatrix}.$$

One proves easily with the help of (2.8) that $\mathrm{Tr}\widetilde{B}^2 = \sum_\mu g(\mu)\mathrm{Tr}_g\mathbf{B}_\mu^2$. We represent each of the exponentials containing $\mathrm{Tr}_g\mathbf{B}_\mu^2$ using (A15) and obtain

$$\begin{aligned}Z(J) = \int & d[\mathbf{\Psi}]\exp[2i\mathbf{\Psi}^\dagger(\mathbf{x}^+ + \mathbf{J})\mathbf{\Psi}] \\ &\times\int d[\sigma]\exp\left(-\sum_\mu\mathrm{Tr}_g\sigma_\mu^2\right) \\ &\times\exp\left(-2i\sum_\mu\sqrt{g(\mu)}\mathrm{Tr}_g\mathbf{B}_\mu\sigma_\mu^\dagger\right),\end{aligned} \tag{2.14}$$

where

$$d[\sigma] = \prod_\mu d[\sigma_\mu], \quad \sigma_\mu = \begin{bmatrix} v_\mu & \alpha_\mu^* \\ \alpha_\mu & iy_\mu\end{bmatrix}. \tag{2.15}$$

The integral over the graded vector $\mathbf{\Psi}$ has now assumed a Gaussian form. Its evaluation proceeds slightly more conveniently if we replace the $(2N)$-component vector $\mathbf{\Psi}$ by N two-component vectors $\mathbf{\Phi}_j = (z_j, \xi_j)^T$. To achieve such a transformation we write

$$\begin{aligned}\mathrm{Tr}_g\mathbf{B}_\mu\sigma_\mu^\dagger &= \mathbf{z}^\dagger D_\mu\mathbf{z}v_\mu + \boldsymbol{\xi}^\dagger D_\mu\mathbf{z}\alpha_\mu \\ &\quad -\mathbf{z}^\dagger D_\mu\boldsymbol{\xi}\alpha_\mu^* + i\boldsymbol{\xi}^\dagger D_\mu\boldsymbol{\xi}y_\mu \\ &= \mathbf{\Psi}^\dagger(\sigma_\mu\otimes D_\mu)\mathbf{\Psi},\end{aligned} \tag{2.16}$$

where $\otimes$ denotes the tensor product. Now we may express the right-hand side of (2.16) with the help of the two-component graded vectors $\mathbf{\Phi}_j$

$$\Psi^\dagger\left(\mathbf{x}^+ + \mathbf{J} - \sum_\mu \sqrt{g(\mu)}(\sigma_\mu \otimes D_\mu)\right)\Psi = \sum_j \Phi_j^\dagger\left(x^+\mathbf{I} + J\mathbf{E} - \sum_\mu \sqrt{g(\mu)}\widetilde{\epsilon}_\mu(j)\sigma_\mu\right)\Phi_j, \tag{2.17}$$

where

$$\Phi_j = \begin{pmatrix} z_j \\ \xi_j \end{pmatrix}, \quad \mathbf{E} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.18}$$

and $\mathbf{I}$ is the unit 2×2 graded matrix. The generating function thus takes the form

$$Z(J) = \int d[\sigma]\exp\left(-\sum_\mu \mathrm{Tr}_g\sigma_\mu^2\right)\int d[\Phi]\exp\left[2i\sum_j \Phi_j^\dagger\left(x^+\mathbf{I} + J\mathbf{E} - \sum_\mu \sqrt{g(\mu)}\widetilde{\epsilon}_\mu(j)\sigma_\mu\right)\Phi_j\right]. \tag{2.19}$$

The Gaussian integral over the Φ variables can be done according to (A9)

$$Z(J) = \int d[\sigma]\exp\left[-\sum_\mu \mathrm{Tr}_g\sigma_\mu^2 - \sum_j \ln\det{}_g\left(x^+\mathbf{I} + J\mathbf{E} - \sum_\mu \sqrt{g(\mu)}\widetilde{\epsilon}_\mu(j)\sigma_\mu\right)\right]. \tag{2.20}$$

III. CALCULATION OF THE DENSITY

In the integral (2.20) we shall treat differently the integration over σ_0 and over other σ_μ with $\mu \neq 0$, introducing a new integration variable $Y = \sigma_0 - \sqrt{N}(x^+\mathbf{I} - J\mathbf{E})$. The generating function then takes the form

$$Z(J) = \int d[Y]\prod_{\mu\neq 0} d[\sigma_\mu]\exp\{-\mathrm{Tr}_g[Y + \sqrt{N}(x^+\mathbf{I} + J\mathbf{E})]^2\} \times \exp\left[-\sum_{\mu\neq 0}\mathrm{Tr}_g\,\sigma_\mu^2 - \sum_j \ln\det{}_g\left(\frac{Y}{\sqrt{N}} - \sum_{\mu\neq 0}\sqrt{g(\mu)}\widetilde{\epsilon}_\mu(j)\sigma_\mu\right)\right]. \tag{3.1}$$

We further change the integration variables by performing a diagonalization of Y

$$Y = U^{-1}SU, \quad S = \begin{bmatrix} s_1 & 0 \\ 0 & is_2 \end{bmatrix}, \tag{3.2}$$

where U is a graded unitary matrix. Simultaneously we change the variables σ_μ into $U^{-1}\sigma_\mu U$ keeping for them the old name σ_μ. Note that such unitary transformation of all integration variables does not change any graded determinant entering the second exponential in (3.1). Denoting the Jacobian of the transformation (3.2) by $\mathcal{D}(S,U)$ we get

$$Z(J) = \int \mathcal{D}(S,U)d[S]d[U]\prod_{\mu\neq 0} d[\sigma_\mu]\exp\{-\mathrm{Tr}_g[S + \sqrt{N}U(x^+\mathbf{I} + J\mathbf{E})U^{-1}]^2\} \times \exp\left[-\sum_{\mu\neq 0}\mathrm{Tr}_g\,\sigma_\mu^2 - \sum_j \ln\det{}_g\left(S/\sqrt{N} - \sum_{\mu\neq 0}\sqrt{g(\mu)}\widetilde{\epsilon}_\mu(j)\sigma_\mu\right)\right], \tag{3.3}$$

where $d[S] = ds_1 ds_2$. We follow Guhr [18] and express the Jacobian as the squared Berezinian, $\mathcal{D}(S,U) = B^2(S)$,

$$B(S) = \frac{1}{s_1 - is_2}. \tag{3.4}$$

The integration over the graded U matrix can be performed using the Guhr's [18] generalization of the Itzykson-Zuber [19] and Mehta [20] formulas concerning integrations over the ordinary unitary groups. By rescaling the eigenvalues of Y as $s_1 \to \sqrt{N}s_1$ and $s_2 \to \sqrt{N}s_2$, we arrive at

$$Z(J) = N\int d[S]\frac{B(S)}{\pi B(x\mathbf{I} + J\mathbf{E})}\exp[-N\mathrm{Tr}_g(S + x^+\mathbf{I} + J\mathbf{E})^2]\mathcal{F}(S), \tag{3.5}$$

where we have introduced the shorthand notation

$$\mathcal{F}(S) = \int \prod_{\mu\neq 0} d[\sigma_\mu] \exp\left[-\sum_{\mu\neq 0} \mathrm{Tr}_g\, \sigma_\mu^2 - \sum_j \ln \det{}_g \left(S - \sum_{\mu\neq 0} \sqrt{g(\mu)}\tilde{\epsilon}_\mu(j)\sigma_\mu \right) \right] . \tag{3.6}$$

The Berezinian $B(x\mathbf{I} + J\mathbf{E})$ is to be understood in the sense of (3.4) with s_1 and is_2 as the eigenvalues of $x\mathbf{I} + J\mathbf{E}$.

In order to simplify the notation we define a graded 2×2 matrix $\sigma(j)$ with matrix elements

$$\alpha(j) = \sum_{\mu\neq 0} \sqrt{g(\mu)}\tilde{\epsilon}_\mu(j)\alpha_\mu, \quad \alpha^*(j) = \sum_{\mu\neq 0} \sqrt{g(\mu)}\tilde{\epsilon}_\mu(j)\alpha_\mu^*,$$
$$v(j) = \sum_{\mu\neq 0} \sqrt{g(\mu)}\tilde{\epsilon}_\mu(j)v_\mu, \quad y(j) = \sum_{\mu\neq 0} \sqrt{g(\mu)}\tilde{\epsilon}_\mu(j)y_\mu. \tag{3.7}$$

By recalling $\alpha(j)^2 = [\alpha^*(j)]^2 = 0$ we can rewrite the logarithm of the graded determinant in (3.6) as

$$\ln \det{}_g[S - \sigma(j)] = \ln[s_1 - v(j)] - \ln[is_2 - iy(j)] - \frac{\alpha^*(j)\alpha(j)}{[s_1 - v(j)][is_2 - iy(j)]}. \tag{3.8}$$

Note also that the integrand in the (3.5) can be understood as an exponential of a quadratic form in the anticommuting variables. We can therefore integrate out these variables as it is done in Ref. [21]. The result reads

$$\mathcal{F}(S) = \left(\frac{N}{2\pi}\right)^{N-1} \int d[v]d[y] \exp\left(-\frac{N}{2} \sum_{\mu\neq 0} (v_\mu^2 + y_\mu^2) \right)$$
$$\times \exp\left\{ -\sum_j \left[\ln\left(s_1 - \sqrt{\frac{N}{2}} v(j) \right) - \ln\left(is_2 - i\sqrt{\frac{N}{2}} y(j) \right) \right] + \ln \det \hat{M} \right\}, \tag{3.9}$$

where the $(N-1) \times (N-1)$ matrix $\hat{M}$ is given as

$$\hat{M}_{\mu\mu'} = \delta_{\mu\mu'} - \frac{1}{N} \sum_j \frac{\sqrt{g(\mu)}\epsilon_\mu(j)\sqrt{g(\mu')}\epsilon_{\mu'}(j)}{\left(s_1 - \sum_{\mu''\neq 0} \sqrt{g(\mu'')}\epsilon_{\mu''}(j)v_{\mu''} \right)\left(is_2 - \sum_{\mu''\neq 0} \sqrt{g(\mu'')}\epsilon_{\mu''}(j)iy_{\mu''} \right)}. \tag{3.10}$$

In the above equations we have rescaled the integration variables $v_\mu \to \sqrt{N/2}v_\mu$ and $y_\mu \to \sqrt{N/2}y_\mu$, anticipating a saddle-point integral. We have also introduced the notation

$$\epsilon_\mu(j) = \sqrt{\frac{N}{2}}\tilde{\epsilon}_\mu(j) \tag{3.11}$$

for $\mu \neq 0$.

The integration over the commuting variables in (3.9) is a very difficult task in general. In this paper we consider the case when the bandwidth b behaves as N^β, with some $0 < \beta \le 1$. In such a case the coefficients $g(\mu)$ are exponentially small for $\mu = O(N^{\beta^*})$ for $N \to \infty$, provided $\beta^* > 1 - \beta$. We expect that out of the integration variables u_μ and y_μ, only the first $N^{1-\beta} = N/b$ give a nontrivial (i.e., $s_{1,2}$-dependent) contribution to the integral in (3.9). Therefore, if we performed the integration over the variables u_μ and y_μ using the saddle-point method, we should obtain the asymptotic expansion of the form

$$\ln \mathcal{F}(S) = O(N) + O(N \ln N/b) + \cdots . \tag{3.12}$$

The first term in this expansion is the standard contribution from the saddle point, the second term and/or the higher-order terms come from the corrections to the saddle point. Contrary to the usual situation when the corrections are of the order of $\ln N$, here they may in the worst case contain an additional factor equal to the effective number of integration variables. Still the corrections are negligible in the limit of large N.

In Appendix B we show that the stable saddle point corresponds to $v_\mu = y_\mu = 0$. This is a direct consequence of the orthogonality of the eigenvectors, especially

$$\sum_j \epsilon_\mu(j) = 0 \tag{3.13}$$

for $\mu \neq 0$. The leading contribution to the asymptotic equation is thus

$$\ln \mathcal{F}(S) = N \ln s_1 - N \ln is_2. \tag{3.14}$$

The fluctuations around the saddle point are described by the Gaussian integral, which is obtained by expanding the exponent in the integrand of Eq. (3.9) up to the quadratic terms in the variables u_μ and y_μ. The resulting expression for $\mathcal{F}$ reads

$$\mathcal{F}(S) = \exp(N \ln s_1 - N \ln is_2 + \sum_\mu \ln\left(1 - \frac{1}{2}\frac{g(\mu)}{s_1 is_2}\right) - \tfrac{1}{2}\sum_\mu \ln\det[\mathbf{I} + g(\mu)\mathbf{a}(\mu,S)], \quad (3.15)$$

where the explicit form of the matrix $\mathbf{a}(\mu,S)$ is given in Appendix B. For now it is important to note that this matrix has elements of the order of 1 at most. For increasing μ the corrections to the saddle-point value [Eq. (3.15)] behave as $\sum_\mu g(\mu)a(\mu,s)$ with $a(\mu,S) = O(1)$. Obviously, in the limit $N \to \infty$ the overall effect of those corrections is indeed of the order of relevant terms in the sums over μ, i.e., N/b. This shows the self-consistency of our approach and proves the correctness of the saddle-point method applied to the integration over the variables v_μ and y_μ.

The remaining integral over s_1 and s_2 can be again evaluated in the limit of large N using the saddle-point method. This integral can be written in the form

$$Z(J) = -\int d[S]\frac{2J}{\pi(s_1 - is_2 + i\epsilon)}\exp\left(-N(s_1 + x - J)^2 + N(is_2 + x + J)^2 - N\ln s_1 + N\ln is_2 + \frac{N}{b}\mathcal{L}(s_1, is_2)\right). \quad (3.16)$$

We have included now the small imaginary term in the Berezinian to ensure the proper integration contour. In the evaluation of the saddle-point values we keep the correction terms of the order of $1/b$ to the saddle-point values. It is worth stressing, however, that the results can be generalized to arbitrary order in $1/b$.

The function $\mathcal{L}$ in the lowest order of the expansion in $1/b$ may be written as

$$\mathcal{L}(s_1, is_2) = \frac{b}{2N}\sum_{\mu\neq 0}\ln\left(\frac{\left(1 - \frac{g(\mu)}{2s_1 is_2}\right)^2}{\left(1 - \frac{g(\mu)}{2s_1^2}\right)\left(1 - \frac{g(\mu)}{2(is_2)^2}\right)}\right) + O\left(\frac{1}{b}\right) + \cdots. \quad (3.17)$$

It is worth noticing that $\mathcal{L}$ is symmetric with respect to the exchange of s_1 and is_2. Moreover, $\mathcal{L}(s,s) = 0$ for arbitrary s.

The saddle-point equations take the form

$$-2(s_1 + x - J) - \frac{1}{s_1} + \frac{1}{b}\frac{\partial\mathcal{L}(s_1, is_2)}{\partial s_1} = 0 \quad (3.18)$$

and

$$+2(is_2 + x - J) + \frac{1}{is_2} - \frac{1}{b}\frac{\partial\mathcal{L}(s_1, is_2)}{\partial is_2} = 0. \quad (3.19)$$

We can now easily compute the expansion of the saddle-point values

$$s_1 = s_{10} + \frac{1}{b}s_{11} + \cdots, \quad (3.20)$$

$$is_2 = is_{20} + \frac{1}{b}is_{21} + \cdots. \quad (3.21)$$

At the same time, in order to calculate the density (2.3), it is sufficient to expand each of the terms in Eqs. (3.20) and (3.21) up to linear terms in J. After an elementary calculation we obtain

$$s_{10} = s(x) + \frac{2J}{2 - 1/s^2(x)}, \quad (3.22)$$

$$is_{20} = s(x) - \frac{2J}{2 - 1/s^2(x)}, \quad (3.23)$$

and

$$s_{11} = is_{21} = \frac{2bJ}{N}\frac{1}{s^4(x)[2 - 1/s^2(x)]}\sum_{\mu\neq 0}\frac{g(\mu)}{\left(1 - \frac{g(\mu)}{2s^2(x)}\right)^2}, \quad (3.24)$$

where

$$s(x) = -\frac{1}{2}x \pm \frac{i}{2}\sqrt{2 - x^2}. \quad (3.25)$$

Note that

$$s(x) + x = -s^*(x), \quad (3.26)$$

and $|s(x)|^2 = \frac{1}{2}$. Note also that s_{11} and is_{21} are already linear in J. The difference $s_1 - is_2$ is not at all affected by the $1/b$ contributions and is proportional to J. The latter property remains intact when we extend our calculation to the order of $1/b^2$. Out of the two possible choices of $s(x)$ in (3.25), the one that corresponds to the negative imaginary part, contributes to the saddle-point integral [18]. In the limit $J \to \infty$ the prefactor $2J$ in the expression (3.16) cancels with the analogous one that enters the Berezinian, i.e., $2J/(s_1 - is_2) = [1 - 1/2s^2(x)]$. The final result is obtained by calculating the saddle-point value of the exponent in (3.16) up to the linear terms in J, and evaluating the lowest-order corrections in $1/N$ that come from Gaussian fluctuations around the saddle point. Note that due to the symmetry of $\mathcal{L}$, the contribution of $\mathcal{L}(s_1, is_2)$ to the saddle-point value of the exponent is null.

Using the explicit expressions for s_1, and is_2, we obtain

$$Z(J) = N\left(2 - \frac{1}{s^2(x)}\right)[D(x)]^{-1/2}\exp[-4NJs^*(x)], \quad (3.27)$$

where $D(x)$ is the determinant of the 2×2 matrix of the second derivatives of the exponent. This term results from the Gaussian integration over the fluctuations around the saddle point. In order to calculate $D(x)$ up to the first-order terms in $1/b$, we observe first that the off-diagonal matrix elements of the fluctuations matrix are proportional to the mixed derivative $\partial^2/\partial s_1 \partial(is_2)$ of the exponent in (3.16). For this reason they contribute to $D(x)$ in the order $1/b^2$, and may be neglected. On the other hand, using the symmetry properties of $\mathcal{L}$ it is easy to show that the diagonal terms of the fluctuations matrix are given by

$$D_{11}(x) = N\left(2 - \frac{1}{s^2(x)} - \frac{1}{b}\frac{\partial^2 \mathcal{L}(s(x), s(x))}{\partial s_1^2}\right),$$
$$D_{22}(x) = N\left(2 - \frac{1}{s^2(x)} + \frac{1}{b}\frac{\partial^2 \mathcal{L}(s(x), s(x))}{\partial s_1^2}\right), \tag{3.28}$$

so that

$$D(x)^{1/2} = N\left[\left(2 - \frac{1}{s^2(x)}\right) + O\left(\frac{1}{b^2}\right)\right]. \tag{3.29}$$

Using the above expression we immediately obtain the final result

$$\frac{1}{N}R(x) = \frac{2}{\pi}s^*(x) + O\left(\frac{1}{b^2}\right) + \cdots \tag{3.30}$$

and

$$\frac{1}{N}\bar{\rho}(x) = \frac{2}{\pi}\sqrt{2 - x^2} + O\left(\frac{1}{b^2}\right) + \cdots. \tag{3.31}$$

The above expression is the main result of the present paper. It states that the density of eigenvalues of band matrices with $b \propto N^\beta$ obeys Wigner's law. Moreover, the corrections to the semicircular law of the order $O(1/b)$ vanish.

It is obvious that our result may be generalized in many ways. In particular it can be obtained for arbitrary smooth, bell-shaped functions $F(i-j)$ [see Eq. (2.7)]. In principle, the method presented here can be used for the generation of systematic expansion in $1/b$ for large but finite b. This could allow one to study the transition from the semicircle distribution to the Gaussian distribution of diagonal matrix elements. The method presented here may be applied also to the calculation of the density-density correlation function. We hope that this application will explain the scaling of the level-spacing distribution for $b \propto N^{1/2}$ which was observed in the recent numerical studies of Casati, Molinari, and Izrailev [15]. The asymptotic analysis in that case, however, is much more complicated [22], and one necessarily has to go beyond the first-order terms in the expansion in $1/b$.

After completion of this paper we have learned that a similar result in the case of $b \propto N$, i.e., $\beta = 1$, has been obtained recently by Casati and Girko [23]. Our theory seems more general since it applies to arbitrary β.

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APPENDIX A

In this appendix we summarize the most important formulas concerning supersymmetric analysis used in our calculations. In our presentation we closely follow the expositions contained in Refs. [16–18].

We use lower-case greek letters to denote anticommuting (Grassmann) variables and lower-case roman letters for commuting ones, reserving the boldface characters for the corresponding vectors, e.g., $\mathbf{z} = (z_1, \ldots, z_N)^T, \boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)^T$. Upper-case boldface letters are used for supersymmetric (graded) vectors with commuting and anticommuting components, e.g., $\boldsymbol{\Psi} = (\mathbf{z}, \boldsymbol{\xi})^T$. Linear transformations of graded vectors are defined with the use of graded $2N \times 2N$ matrices of the block form

$$\mathbf{A} = \begin{bmatrix} a & \sigma \\ \eta & b \end{bmatrix},$$

where a, b, σ, and η are $N \times N$ matrices with commuting (a and b) and anticommuting (σ and η) elements.

If ξ is a Grassmann variable, then ξ^* is an independent one with the property $(\xi^*)^* = -\xi$. Consequently, one defines the Hermitian conjugate of a graded matrix A as

$$\mathbf{A}^\dagger = \begin{bmatrix} a^\dagger & \eta^\dagger \\ -\sigma^\dagger & b^\dagger \end{bmatrix}.$$

The Hermitian conjugates of $N \times N$ matrices are constructed according to the standard rules via transposition and conjugation. For graded matrices one introduces supersymmetric counterparts of trace and determinant

$$\mathrm{Tr}_g \mathbf{A} = \mathrm{Tr} a - \mathrm{Tr} b, \tag{A1}$$

$$\det_g \mathbf{A} = \det(a - \sigma b^{-1}\eta)(\det b)^{-1}. \tag{A2}$$

Simple calculations show that Trg and detg have the properties of their ordinary counterparts when applied to products of graded matrices, i.e.,

$$\mathrm{Tr}_g(\mathbf{A} \cdot \mathbf{B}) = \mathrm{Tr}_g(\mathbf{B} \cdot \mathbf{A}),$$
$$\det_g(\mathbf{A} \cdot \mathbf{B}) = \det_g \mathbf{A} \cdot \det_g \mathbf{B}. \tag{A3}$$

As for ordinary matrices one has

$$\det_g \mathbf{A} = \exp[\mathrm{Tr}_g \ln(\mathbf{A})]. \tag{A4}$$

Integrals over anticommuting variables are defined by

$$\int d\xi = 0 = \int d\xi^*, \quad \int \xi d\xi = (2\pi)^{-1/2} = \int \xi^* d\xi^*. \tag{A5}$$

The differentials of anticommuting variables are Grassmann variables as well, hence

$$\int \xi_1\xi_2 d\xi_1 d\xi_2 = -\left(\int \xi_1 d\xi_1\right)\left(\int \xi_2 d\xi_2\right) = (2\pi)^{-1}. \tag{A6}$$

If A is an $N\times N$ ordinary matrix, then, by diagonalization of a quadratic form and series expansion, one proves

$$\int \exp(i\boldsymbol{\xi}^\dagger A\boldsymbol{\xi})d[\xi] = \det\left(\frac{A}{2i\pi}\right), \tag{A7}$$

where $d[\xi] = d\xi_1^* d\xi_1, \ldots, d\xi_N^* d\xi_N$. This can be compared with its ordinary counterpart

$$\int \exp(i\mathbf{z}^\dagger A\mathbf{z})d[z] = \det\left(\frac{A}{2i\pi}\right)^{-1}, \tag{A8}$$

with

$$d[z] = \prod_{n=1}^{N} dz_n^* dz_n = 2^N \prod_{n=1}^{N} d\,\mathrm{Re}z_n\, d\,\mathrm{Im}z_n.$$

An analogous formula for the graded vectors and matrices reads

$$\int \exp(i\boldsymbol{\Psi}^\dagger \mathbf{A}\boldsymbol{\Psi})d[\boldsymbol{\Psi}] = \det{}_{\mathbf{g}}\mathbf{A}, \tag{A9}$$

where $\boldsymbol{\Psi} = (\mathbf{z}, \boldsymbol{\xi})^T$ and $d[\boldsymbol{\Psi}] = d[z]d[\xi]$.

Let $\mathbf{A} = \mathrm{diag}(A, A)$ be a block-diagonal graded matrix with A being an ordinary $N\times N$ matrix, $\mathbf{E} = \mathrm{diag}(-I, I)$, where I is the unit $N \times N$ matrix, and J is a scalar parameter. Then

$$\begin{aligned} Z(J) &\equiv \exp[i\boldsymbol{\Psi}^\dagger(\mathbf{A} + J\mathbf{E})\boldsymbol{\Psi}]d[\boldsymbol{\Psi}] = \frac{\det(A+J)}{\det(A-J)} \\ &= \exp \mathrm{Tr}[\ln(I + JA^{-1}) - \ln(I - JA^{-1})] \end{aligned} \tag{A10}$$

and

$$\left.\frac{\partial}{\partial J}Z(J)\right|_{J=0} = 2\,\mathrm{Tr}A^{-1}, \tag{A11}$$

which gives a convenient representation of the trace of an inverse matrix in terms of a Gaussian integral.

Let

$$\sigma = \begin{bmatrix} v & \alpha^* \\ \alpha & iy \end{bmatrix} \tag{A12}$$

be a 2×2 graded matrix with v and y real. Then

$$\mathrm{Tr}_g\sigma^2 = v^2 + y^2 + 2\alpha^*\alpha = \mathrm{Tr}_g\sigma^{\dagger 2}. \tag{A13}$$

Denoting $d[\sigma] = d[\alpha]dvdy$, we have, according to (A7),

$$\begin{aligned} \int \exp(-\mathrm{Tr}_g\sigma^{\dagger 2})d[\sigma] &= \int \exp(-\mathrm{Tr}_g\sigma^2)d[\sigma] \\ &= \int e^{-(v^2+y^2)}dvdy \int e^{-2\alpha^*\alpha}d[\alpha] = \frac{\pi}{\pi} = 1. \end{aligned} \tag{A14}$$

Hence, completing the square in the exponential and shifting the variables we obtain, from the linearity and the cyclic property of the Trg operation,

$$\begin{aligned} \int \exp[-\mathrm{Tr}_g\sigma^2 - 2i\sqrt{g}\mathrm{Tr}_g(\rho\sigma^\dagger)]d[\sigma] &= \int \exp[-\mathrm{Tr}_g\sigma^{\dagger 2} - 2i\sqrt{g}\mathrm{Tr}_g(\rho\sigma^\dagger)]d[\sigma] \\ &= e^{-g\mathrm{Tr}_g\rho^2}\int \exp[-\mathrm{Tr}_g(\sigma^\dagger - i\sqrt{g}\rho)^2]d[\sigma] \\ &= e^{-g\mathrm{Tr}_g\rho^2} \end{aligned} \tag{A15}$$

for an arbitrary 2×2 graded matrix ρ and scalar real g. It is easy to show that the above formula remains valid for the case of $2N \times 2N$ graded matrices σ and ρ where v and y are ordinary $N \times N$ Hermitian matrices if we define

$$\begin{aligned} d[v] &= \prod_{j=1}^{N} dv_{jj} \prod_{j>k}^{N} d\,\mathrm{Re}\,v_{jk} d\,\mathrm{Im}\,v_{jk}, \\ d[y] &= \prod_{j=1}^{N} dy_{jj} \prod_{j>k}^{N} d\,\mathrm{Re}\,y_{jk} d\,\mathrm{Im}\,y_{jk}, \\ d[\alpha] &= \prod_{j,k}^{N} d\alpha_{jk}^* d\alpha_{jk}, \end{aligned}$$

and

$$d[\sigma] = 2^{N(N-1)}d[v]d[y]d[\alpha]. \tag{A16}$$

APPENDIX B

In this appendix we prove that the saddle point of the integral (3.9) is given by $v_\mu = y_\mu = 0$. We also perform the Gaussian integral over the fluctuations around this saddle point.

To this end we first expand the logarithms entering the exponent in the expression (3.9). We obtain

$$\sum_j \ln\left(s_1 - \sqrt{\frac{N}{2}}v(j)\right) = N\,\ln s_1 - \sqrt{\frac{N}{2}}\sum_j \frac{v(j)}{s_1} - \frac{N}{4}\sum_j\left(\frac{v(j)}{s_1}\right)^2 - \cdots. \tag{B1}$$

Using the property (3.13) of the Fourier eigenvectors, we easily see that $\sum_j v(j) = 0$, i.e., the first-order term in Eq. (B1) indeed vanishes. The second-order term may be easily summed up, using

$$\sum_j \tilde{\epsilon}_\mu(j)\tilde{\epsilon}_{\mu'}(j) = \delta_{\mu\mu'}. \tag{B2}$$

The result is

$$\sum_j \ln\left(s_1 - \sqrt{\frac{N}{2}}v(j)\right) = N \ln s_1 - N\sum_\mu \frac{g(\mu)v_\mu^2}{4s_1^2}. \tag{B3}$$

An analogous term arises from the second logarithm in the exponent of (3.9).

It is a little more difficult to expand $\ln \det \hat{M}$. First, we expand the full matrix up to the second-order terms in v_μ and y_μ,

$$\hat{M} = \hat{M}_0 + \hat{M}_1 + \hat{M}_2 + \cdots, \tag{B4}$$

where

$$\hat{M}_{0,\mu\mu'} = \left(1 - \frac{g(\mu)}{2s_1 is_2}\right)\delta_{\mu\mu'} = f(\mu,S)\delta_{\mu\mu'}, \tag{B5}$$

$$\hat{M}_{1,\mu\mu'} = -\frac{1}{\sqrt{2N}}\sum_j \frac{\sqrt{g(\mu)}\epsilon_\mu(j)\sqrt{g(\mu')}\epsilon_{\mu'}(j)}{s_1 is_2} \times\left(\frac{v(j)}{s_1} + \frac{iy(j)}{is_2}\right), \tag{B6}$$

$$\hat{M}_{2,\mu\mu'} = -\tfrac{1}{2}\sum_j \frac{\sqrt{g(\mu)}\epsilon_\mu(j)\sqrt{g(\mu')}\epsilon_{\mu'}(j)}{s_1 is_2} \times\left[\left(\frac{v(j)}{s_1}\right)^2 + \left(\frac{iy(j)}{is_2}\right)^2 + \frac{v(j)iy(j)}{s_1 is_2}\right]. \tag{B7}$$

The logarithm of the determinant of $\hat{M}$ can be written as

$$\ln\det\hat{M} = \ln\det\hat{M}_0 + \mathrm{Tr}\frac{1}{\hat{M}_0}(\hat{M}_1 + \hat{M}_2) - \tfrac{1}{2}\mathrm{Tr}\left(\frac{1}{\hat{M}_0}\hat{M}_1\right)^2 + \cdots. \tag{B8}$$

The zeroth-order term is, due to (B5), equal to

$$\ln\det\hat{M}_0 = \sum_{\mu\neq 0}\ln\left(1 - \frac{g(\mu)}{2s_1 is_2}\right). \tag{B9}$$

The first order term reads

$$\mathrm{Tr}\frac{1}{\hat{M}_0}\hat{M}_1 = -\frac{1}{\sqrt{2N}}\sum_j\sum_{\mu\neq 0}\frac{g(\mu)\epsilon_\mu(j)\epsilon_\mu(j)}{s_1 is_2 f(\mu,S)} \times\left(\frac{v(j)}{s_1} + \frac{iy(j)}{is_2}\right). \tag{B10}$$

The sum over μ may be performed, using

$$\frac{1}{N}\sum_{\mu\neq 0}\frac{g(\mu)\epsilon_\mu(j)\epsilon_\mu(j)}{f(\mu,S)} = \tfrac{1}{2}\left(G(0,S) - \frac{g(0)}{Nf(0,S)}\right), \tag{B11}$$

where

$$G(k,S) = \frac{1}{N}\sum_\mu \frac{g(\mu)}{f(\mu,S)}e^{-i\omega_\mu k}, \tag{B12}$$

Elementary estimations show that the function $G(0,S)$ is of the order of $1/b$. It is essential that the result (B11) is j independent. The trace (B10) then becomes proportional to the $\sum_j v(j) = 0$ or $\sum_j iy(j) = 0$, and thus vanishes. This proves that all the first-order terms in the exponent of (3.9) are null and there exists indeed a saddle point at $v_\mu = iy_\mu = 0$.

Same algebra allows to calculate the $\mathrm{Tr}\hat{M}_0^{-1}\hat{M}_2$. The result is

$$\mathrm{Tr}\frac{1}{\hat{M}_0}\hat{M}_2 = -\frac{N}{4s_1 is_2}\left(G(0,S) - \frac{g(0)}{Nf(0,S)}\right) \times\sum_{\mu\neq 0}g(\mu)\left[\left(\frac{v_\mu}{s_1}\right)^2 + \left(\frac{iy_\mu}{is_2}\right)^2 + \frac{v_\mu iy_\mu}{s_1 is_2}\right]. \tag{B13}$$

Finally, the last term can be written as

$$\mathrm{Tr}\left(\frac{1}{\hat{M}_0}\hat{M}_1\right)^2 = \frac{N}{8(s_1 is_2)^2} \times\sum_{j,k}\left(G(j-k,S) - \frac{g(0)}{Nf(0,S)}\right)^2 \times\left(\frac{v(j)}{s_1} + \frac{iy(j)}{is_2}\right)\left(\frac{v(k)}{s_1} + \frac{iy(k)}{is_2}\right). \tag{B14}$$

Note that both $v(j)$ and $iy(j)$ are defined in terms of the Fourier expansion. The sum over j in (B14) can be treated as an action of the matrix whose matrix elements depend only on $j-k$, on the vectors $v(j)$ and $iy(j)$. Using the Fourier expansion of $v(j)$ and $iy(j)$ we easily perform such a task. The sum over k then becomes equivalent to the evaluation of scalar product of the vectors $v(k)$ or $iy(k)$ with the vector that results from the action of the matrix. Introducing

$$h(\mu,S)=\sum_k\left(G(k,S)-\frac{g(0)}{Nf(0,S)}\right)^2 e^{+i\omega_\mu k}, \tag{B15}$$

we obtain

$$\mathrm{Tr}\left(\frac{1}{\hat{M}_0}\hat{M}_1\right)^2=\frac{N}{8(s_1 is_2)^2}\sum_{\mu\neq 0}g(\mu)h(\mu,S)\left(\frac{v_\mu}{s_1}+\frac{iy_\mu}{is_2}\right)^2. \tag{B16}$$

The expressions (B3), (B13), and (B16) completely describe the fluctuation matrix. It is elementary to check that at the saddle point this matrix is positive definite, so that the saddle point is stable. The result of the Gaussian integration over v_μ and iy_μ can now be easily written in the form (3.15), i.e., in the form of the sum over $\mu\neq 0$ of logarithms of determinants of 2×2 matrices. The matrix $\mathbf{I}+\mathbf{a}(\mu,S)$ that enters the expression (3.15) is given by

$$1+g(\mu)a_{11}(\mu,S)=1-\frac{g(\mu)}{2s_1^2}\left(1-\frac{G(0,S)-\frac{g(0)}{Nf(0,S)}}{s_1 is_2}-\frac{h(\mu,S)}{4(s_1 is_2)^2}\right),$$

$$1+g(\mu)a_{22}(\mu,S)=1-\frac{g(\mu)}{2(is_2)^2}\left(1+\frac{G(0,S)-\frac{g(0)}{Nf(0,S)}}{s_1 is_2}+\frac{h(\mu,S)}{4(s_1 is_2)^2}\right),$$

$$g(\mu)a_{12}(\mu,S)=i\frac{g(\mu)}{4s_1 is_2}\left(\frac{G(0,S)-\frac{g(0)}{Nf(0,S)}}{s_1 is_2}+\frac{h(\mu,S)}{2(s_1 is_2)^2}\right),$$

$$g(\mu)a_{21}(\mu,S)=g(\mu)a_{12}(\mu,S).$$

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