Multifractality of self-affine fractals

Albert-László Barabási
Department of Atomic Physics, Eötvös University, P.O. Box 327, Budapest 1445, Hungary

Tamás Vicsek
Institute for Technical Physics, P.O. Box 76, Budapest 1325, Hungary
and Department of Atomic Physics, Eötvös University, P.O. Box 327, Budapest 1445, Hungary

(Received 5 December 1990)

The concept of multifractality is extended to self-affine fractals in order to provide a more complete description of fractal surfaces. We show that for a class of iteratively constructed self-affine functions there exists an infinite hierarchy of exponents H_q describing the scaling of the qth order height-height correlation function $c_q(x) \sim x^{qH_q}$. Possible applications to random walks and turbulent flows are discussed. It is demonstrated on the example of random walks along a chain that for stochastic lattice models leading to self-affine fractals H_q exhibits phase-transition-like behavior.

Many recent investigations have demonstrated that a wide class of processes lead to complex objects that can be described in terms of self-affine fractals. The examples range from plots of various kinds of random walks [1,2] to interfaces developing in marginally stable, far-from-equilibrium systems [3–6]. Growth processes resulting in self-affine interfaces have attracted particular interest during the last few years because of their relevance to a number of phenomena of practical importance, including thin-film growth by vapor deposition, two-phase viscous flow in porous media, formation of biological patterns, and sedimentation of granular materials (see, e.g., Refs. [3–6] and references cited therein).

The properties of a fractal can be best characterized by a set of exponents describing the scaling behavior of the quantities defined for the fractal. Recent studies have shown that in addition to the fractal dimension, there exists an infinite hierarchy of exponents that allows a much more complete representation of the so-called fractal measures [7–9]. Although this approach has been very successful, up to now the related ideas (multifractality, multiscaling) have only been applied to self-similar sets. Here we intend to treat the self-affine case and use an analogous analysis for functions which, in general, can be considered as complicated signals of arbitrary origin.

A single-valued standard self-affine function $h(x)$ satisfies the relation $h(x) \simeq \lambda^{-H} h(\lambda x)$, where λ is a parameter and H is the Hölder or roughness exponent. Alternatively, the height-height correlation function $c(h)$ defined for $h(x)$ scales as $c(x) = \langle [h(x') - h(x'+x)]^2 \rangle_{x'} \sim x^{2H}$. Characterizing $h(x)$ with many exponents instead of a single H is expected to result in a more complete description of fractal surfaces and advance our understanding of their relevant features.

In this paper we extend the concept of multifractality to self-affine fractals and demonstrate its applicability on selected examples. We show that for a class of iteratively constructed self-affine functions there exists an infinite hierarchy of exponents H_q describing the scaling of the qth-order correlation function $c_q(x) \sim x^{qH_q}$, which is defined through the qth moments of the distribution of height differences. Furthermore, it is demonstrated on the example of random walks on a lattice that any discrete model of self-affine fractals (including surfaces in growth models) results in a phase-transition-like behavior of H_q.

The multiscaling properties of the self-affine function $h(x)$ can be investigated by calculating the *qth-order height-height correlation function,* which we define as

$$c_q(x) = \frac{1}{N} \sum_{i=1}^{N} |h(x_i) - h(x_i + x)|^q , \quad (1)$$

where $N \gg 1$ is the number of points over which the average is taken, and only terms with $|h(x_i) - h(x_i + x)| > 0$ are considered. This correlation function exhibits a nontrivial multiscaling behavior if

$$c_q(x) \sim x^{qH_q} , \quad (2)$$

with H_q changing continuously with q at least for some region of the q values. For self-affine functions defined on a finite interval the above scaling is expected to hold in the $x \ll 1$ limit, while in the discrete case (where there exists a lower cutoff length Δx as, e.g., in the lattice growth models) Eq. (2) is satisfied in the $x \to \infty$ limit. It can be shown that a continuous spectrum of H_q values is not consistent with the expression $h(x) \simeq \lambda^{-H} h(\lambda x)$, which is valid for standard self-affine functions with a single exponent H. For self-affine functions with multifractal properties this expression is modified, since the local scaling of the height differences depends on x in analogy with the local scaling of the measure in a box for fractal measures. A detailed analysis of this type of scaling behavior and its implications will be discussed in Ref. [10].

Let us examine the validity of the assumption (2) on a recursively constructed fractal which is a straightforward generalization of Mandelbrot's deterministic self-affine function [11], imitating the main features of a Brownian plot with $H=\frac{1}{2}$. The iteration procedure is demonstrated in Fig. 1. In each step of the recursion the intervals obtained in the previous step are replaced with the properly rescaled version of the generator, which has the form of an asymmetric letter z made of four intervals. During this procedure every interval is regarded as a diagonal of a rectangle becoming more elongated as the number of iterations k increases. The basis of the rectangle is divided into four parts and the generator replaces the intervals in such a way that its turnovers are always at analogous positions (at the first quarter and the middle of the basis). The function becomes self-affine in the $k\to\infty$ limit. Depending on the parameter b_1 very different structures can be generated.

After the kth step, the number of intervals (boxes) along the x axis is 4^k and their length is equal to 4^{-k}. Denoting with $N(\Delta h)$ the number of boxes in which $|h(x)-h(x+\Delta x)|=\Delta h$, we have $N(b_1^n b_2^{k-n})=2^k C_k^n$,

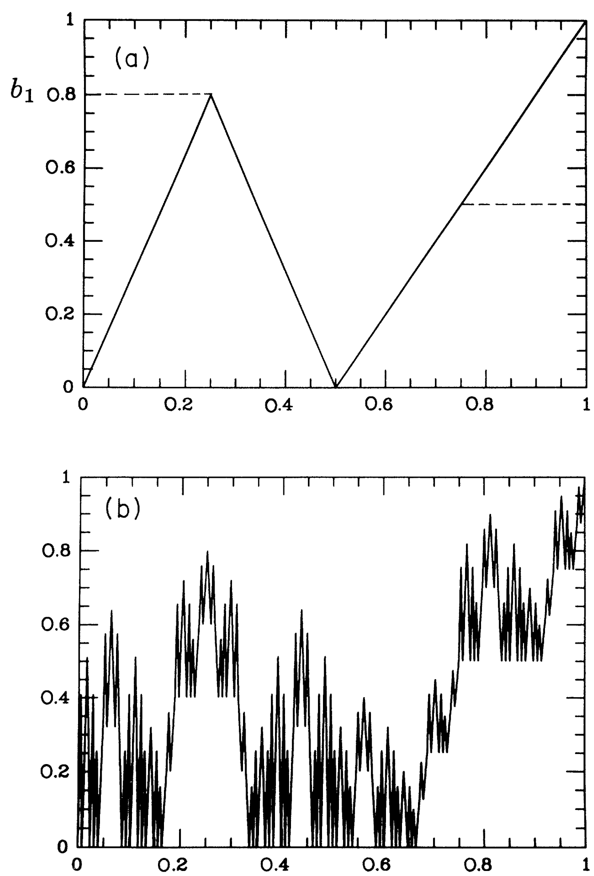

FIG. 1. Construction of a self-affine curve with a height-height correlation function that has multiscaling properties. In each step of the recursion the intervals obtained in the previous step are replaced with the properly rescaled version of the initial configuration shown in (a). The function becomes self-affine in the $k\to\infty$ limit [here the fourth step is shown in (b)]. Depending on the parameters b_1 different structures with arbitrary roughness exponent $H>0$ can be generated.

where $n=0,\ldots,k$. Thus

$$c_q(\Delta x)=\sum_{n=0}^{k} 2^{-k} C_k^n b_1^{nq} b_2^{(k-n)q} , \tag{3}$$

with $\Delta x=4^{-k}$. Note that here $c_q(x)$ is given only for discrete values of its argument; however, because of scaling, this is not expected to influence the results obtained for H_q. Since Eq. (3) can be written as $c_q(\Delta x)=[(b_1^q+b_2^q)/2]^k$ we have

$$H_q=\frac{\ln[(b_1^q+b_2^q)/2]}{q\ln(\frac{1}{4})} . \tag{4}$$

In the present approach the roughness exponent introduced earlier is $H=H_1$. Figure 2 shows H_q calculated from (4) for $b_1=0.8$ and $b_2=0.5$ and its comparison with the numerically determined data [using Eqs. (1) and (2)] for a prefractal obtained after $k=9$ steps of the construction. There is a good agreement between the results; however, it is possible to carry out the numerical calculations only for $q>0$ because of the large uncertainties.

Next we discuss the multifractal aspects of Brownian plots. These self-affine functions are obtained by plotting the displacement $h(x)$ of a particle randomly walking in one dimension as a function of time (denoted here by x). In this case the probability that $h(x)$ changes by Δh on an interval of length Δx is $P(\Delta h,\Delta x)=(1/\sqrt{2\pi\Delta x})\exp(-\Delta h^2/2\Delta x)$. Changing the summation in (1) for integration and using the variable $u=\Delta h/\sqrt{2\Delta x}$ we have

$$c_q(\Delta x)=\int_0^\infty P(\Delta h,\Delta x)\Delta h^q d\Delta h$$

$$=\frac{(2\Delta x)^{q/2}}{\sqrt{\pi}}\int_0^\infty u^q e^{-u^2}du \sim I\Delta x^{q/2} . \tag{5}$$

The integral for u is equal to $I=\Gamma[(q+1)/2]/2$ if $q>-1$, while it is divergent for $q<-1$. This means that

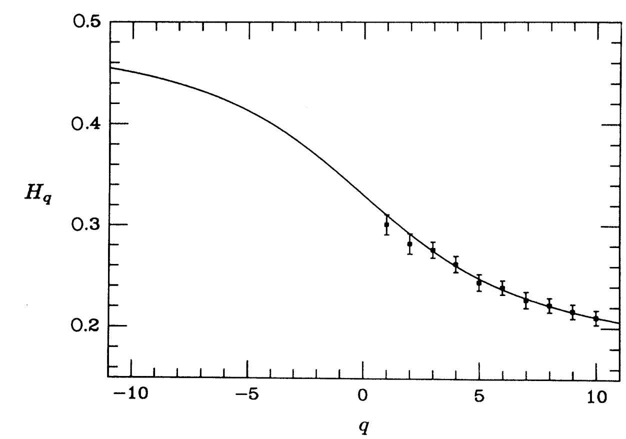

FIG. 2. This figure shows the H_q spectrum calculated from Eq. (4) for the self-affine fractal of Fig. 1 with $b_1=0.8$ and $b_2=0.5$ (solid line). The theoretical result is compared with the numerically determined data obtained using Eqs. (1) and (2) for a prefractal generated in the ninth step of the construction. The numerical calculations can be carried out only for $q>0$ because of the large uncertainties.

$H_q=\frac{1}{2}$ for all values of q larger than -1, below which H_q is undefined. The same calculation can be repeated for arbitrary fractional Brownian motion, leading to the same result with H instead of $\frac{1}{2}$.

The situation is rather different in the *discrete* case, where we can assume without the loss of generality that the particle makes a jump of unit length at every time step of unit duration. We studied this version by analytical and simulation techniques and found that the results correspond to the following behavior:

$$c_q(x)\simeq\exp\left[\frac{\operatorname{sgn}(q)}{|q|}\right]x^{qH_q}, \tag{6}$$

with

$$H_q=\begin{cases}H_0 & \text{for } q>-1\\ -H_0/q & \text{for } q<-1,\end{cases} \tag{7}$$

with $H_0=\frac{1}{2}$. Thus, in this case H_q is defined for all q and it tends to zero as $q\to-\infty$.

It follows from (7) that H_q manifests *phase-transition-like behavior* in the vicinity of $q=-1$, at which point it is not differentiable. Analogous phase transitions can be observed in some dynamical systems as well. For example, a singular point at $q=-1$ followed by a constant behavior for $q>-1$ was found in the D_q spectrum of some simple dynamical systems described by a family of one-dimensional, piecewise parabolic maps [12]. A similar dependence of the multifractal spectrum was observed in Ref. [13]. The above H_q spectrum is obviously not monotonously decreasing, which is due to the fact that the distribution of the differences in $h(x)$ is not a normalized measure. Figure 3 shows the comparison of simulation results with the expression (7).

Equation (7) for $q>-1$ is just the standard scaling of the distance of a random walker from the origin. For the $q<-1$ case the following argument can be used: The terms in the sum for the height correlation function are essentially the probabilities of a given sequence of length $x=n=k+l$ of "up" (k) and "down" (l) jumps multiplied by the absolute value of the resulting height difference $n-k$ in power of q. Thus, $c_q(n)=(1/2)^n\sum_{k=0}^{n/2}2n!/[k!(n-k)!]|n-k|^q$. It is easy to show (by taking the derivative of the logarithm of the kth term of the sum) that for $q<-1$ the dominant contribution to c_q comes from the term with $k=n/2-1$ (for $k=n/2$, $\Delta h=0$, and these sequences are not considered by definition). From this we get $H_q=\lim_{n\to\infty}1/(q\ln n)\ln(2^q n!/\{2^n[(n/2)!]^2\})$. Using Stirling's formula $n!=(n/e)^n(2\pi n)^{1/2}$ leads to the required result $H_q=-1/2q$. Thus, the origin of the phase transition in our case is entirely due to the discrete nature of the $h(x)$ function for walks on a chain, and it is not analogous to the phase-transition-like behavior observed for growth models [14].

We expect that the formalism presented in this paper will be useful in the analysis of various experimentally observed physical quantities. As an important example let us mention measurements of the distribution of (i) a passive scalar $\rho(\mathbf{x})$ and (ii) the velocity in turbulent flows. (i) It has recently been shown that the dissipation field in turbulent jets can be described in terms of multifractal spectra [15]. Since the local dissipation rate is proportional to the squared gradient of the passive scalar, on the basis of Eqs. (1) and (2) [in which $|h(x_i)-h(x_i+x)|$ is related to the local coarse-grained derivative of the function] we conclude that the distribution $\rho(\mathbf{x})$ is a self-affine function with multifractal properties. (ii) The definition of the so-called velocity structure functions in turbulent flows is analogous to Eq. (1). There is experimental and theoretical evidence for the q-dependent scaling of these functions [16]. This means that the velocity field itself has to be a multiscaling self-affine fractal and its properties can be investigated by the corresponding methods developed for such geometrical objects.

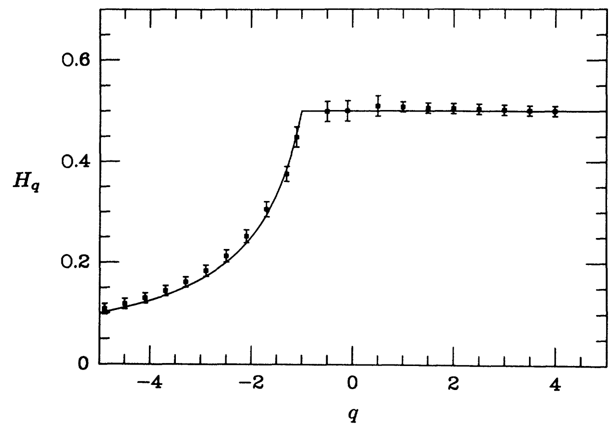

FIG. 3. The H_q spectrum for the plot of displacements vs time for a random walk on a one-dimensional lattice. The solid line corresponds to expression (7) of the text. The simulation results are also indicated, where the error bars were determined from the goodness of the fit to the numerical data.

The examples discussed in this paper demonstrate that the multifractal analysis described above provides relevant additional information about the scaling properties of self-affine functions. Furthermore, our approach has potentially interesting implications concerning the relationship of self-affine fractals and fractal measures. The elaboration of a complete multifractal-type formalism for the self-affine case is in progress. Finally, it is straightforward to extend the present multiscaling approach to higher dimensions.

ACKNOWLEDGMENTS

We thank F. Family, J. Kertész, P. Szépfalusy, and T. Tél for useful discussions. The present research was supported by the Hungarian Scientific Research Foundation Grant No. 693.

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[2] J. Feder, *Fractals* (Plenum, New York, 1988); R. F. Voss, in *The Science of Fractal Images,* edited by H. O. Peitgen and D. Saupe (Springer, Berlin, 1988), Ch. 1.
[3] *Random Fluctuations and Pattern Growth,* edited by H. E. Stanley and N. Ostrowsky (Kluwer, Dordrecht, 1988).
[4] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).
[5] P. Meakin, CRC Rev. Solid State Phys. **13**, 143 (1987).
[6] For recent reviews, see F. Family, Physica A **168**, (1990); J. Krug and H. Spohn, in *Solids Far From Equilibrium: Growth, Morphology and Defects,* edited by C. Godrèche (Cambridge University Press, Cambridge, England, 1990).
[7] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[8] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics,* edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985).
[9] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[10] A-L. Barabási, P. Szépfalusy, and T. Vicsek (unpublished).
[11] B. B. Mandelbrot, Phys. Scripta **32**, 257 (1985).
[12] A. Csordás and P. Szépfalusy, Phys. Rev. A **39**, 4767 (1989).
[13] K. J. Måløy, in *Time Dependent Effects in Disordered Materials,* edited by R. Pynn and T. Riste (Plenum, New York, 1987), p. 111.
[14] J. Lee and H. E. Stanley, Phys. Rev. Lett. **61**, 2945 (1988); R. Blumenfeld and A. Aharony, *ibid.* **62**, 2977 (1989).
[15] P. R. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988).
[16] See, e.g., M. Nelkin, J. Stat. Phys. **54**, 1 (1989), and references cited therein.