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Electric-field dynamics in plasmas: Theory

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The dynamics of electric fields at a neutral or charged point in a one-component plasma is considered. The equilibrium joint probability density for electric-field values at two different times is defined, and several formally exact limits are described in some detail. The asymptotic short-time behavior for both neutral and charged-point cases is shown to be Gaussian with respect to the field differences, but with a half-width depending on their sum. In the strong-coupling limit, the joint probability density is dominated by weak fields (charged-point case), leading to a Gaussian distribution with time dependence entirely determined from the electric-field time-correlation function. The limit of large fields is shown to be determined by the time-dependent autocorrelation function for the density of ions around the field point; for the special case of fields at a neutral point, this result implies that the joint distribution at large fields is determined entirely by the dynamic structure factor. Finally, the full distribution (all field values and times) is studied in the weak-coupling limit.

I. INTRODUCTION

The total electric field $\mathbf{E}(r)$ at a point $\mathbf{r}$, due to all charges in an equilibrium plasma, is a fundamental microscopic variable for the analysis of many problems [1,2]. It determines the force on each particle and is therefore relevant for calculation of transport properties; it also determines the coupling of the charged particles to internal degrees of freedom via multipole moments and thus provides the mechanism for atomic radiative processes. The statistical time-independent properties of the field are completely characterized by the distribution of field values $Q(\varepsilon)$. There are now quite accurate and practical methods for calculating this distribution, even for conditions of strong coupling [3]. Similarly, the dynamical properties of the fields at two different times are characterized by the joint distribution $Q(\varepsilon,t;\varepsilon',t')$. The objective here is to develop a theory for this joint distribution. As there are few previous discussions of this time-dependent distribution, the present paper is primarily concerned with establishing its behavior in several well-defined limits. Specific applications and approximations are deferred to a planned subsequent paper [4].

The system considered here is an equilibrium one-component plasma (OCP) of N positive ions with charge Ze, a single impurity particle of charge Z_0e, and a neutralizing uniform negative background charge. All interactions are assumed to be via Coulomb potentials, and the state conditions are assumed to be such that classical mechanics is applicable. The one-component plasma state conditions are completely specified by a single parameter $\Gamma \equiv (Ze)^2/(r_0 k_B T)$, where T is the temperature and r_0 is the ion sphere radius ($4\pi n r_0^3/3=1$). This plasma parameter measures the strength of the Coulomb coupling between ions relative to the average kinetic energy; the corresponding ion-impurity coupling is $(Z_0\Gamma/Z)$. Two cases are of interest, fields at a charged point ($Z_0 \neq 0$) and fields at a neutral point ($Z_0=0$). The distributions are quite different for these two cases, but much of the analysis can proceed for Z_0 finite, with the neutral point case recovered in the limit $Z_0 \to 0$.

In Sec. II, the electric-field distributions and their generating functions are defined, and some general properties associated with basic symmetries are noted. The exact short-time limit is studied in Sec. III, and an accurate method to calculate the associated coefficients is described. As an example (to motivate future computer simulations) the short-time behavior is studied for a neutral point at $\Gamma=1$. In Sec. IV, it is noted that a weak-field limit applies when $Z_0\Gamma/Z \gg 1$, for field distributions at a charged point. The result is a Gaussian distribution whose time dependence is determined from the electric-field autocorrelation function $\langle \mathbf{E}\cdot\mathbf{E}(t)\rangle$. The result is illustrated using computer-simulation data at $\Gamma=10$. Next, in Sec. V, the limit of one or both fields being very large is considered. It is shown that the joint distribution function then can be determined from the time-correlation function for the density of ions around the impurity. This result includes the nearest-neighbor approximation in the limit of both short times and large fields. The results of Secs. III and V agree in their common domain of validity (short times and large fields). Finally, the global distribution for all times and field values is analyzed in the weak-coupling (mean-field) limit in Sec. VI. The resulting generating function is expressed in terms of the ion density autocorrelation function, calculated from a weak-coupling kinetic equation (Vlasov equation, for the neutral-point case).

Some of our results have been noted already in the brief review of Ref. [5]. Also, Alastuey, Lebowitz, and Levesque [6] have proposed recently an approximate model for electric-field dynamics at a neutral point, and have performed some molecular-dynamics simulations for comparison at strong coupling. Smith, Stamm, and Cooper have considered the case of neutral-point electric-field dynamics, neglecting all interactions among the ions [7]. Prior work has been limited primarily to models based on an assumed stochastic process for the electric field [8]. The only computer-simulation results we are aware of for the joint distribution function are the limited data reported in Refs. [6] and [7]. We hope the present study will encourage further analysis by simulation.

II. DEFINITIONS AND GENERAL PROPERTIES

The microscopic state of the system of $N+1$ ions is specified by their positions and velocities, $\{\mathbf{q}_0,\mathbf{p}_0;\ldots;\mathbf{q}_N,\mathbf{p}_N\}$, where the subscript zero labels the impurity. The electric field at the impurity is given by

$$\mathbf{E}\equiv\sum_i^N \mathbf{e}(\mathbf{r}_i)+\mathbf{E}_b,\quad \mathbf{e}(i)\equiv Ze(\hat{\mathbf{r}}_i/r_i^2),\quad \mathbf{r}_i\equiv\mathbf{q}_i-\mathbf{q}_0\ , \tag{2.1}$$

where E_b is the field due to the uniform background, and $\mathbf{e}(i)$ is the electric field due to the ith ion. In the equilibrium ensemble there are equivalence classes of configurations corresponding to the same value for $\mathbf{E}$, and the distribution of values for $\mathbf{E}$ is determined from the relative weights of these classes. Therefore, the definitions of the probability density $Q(\varepsilon)$ and joint probability density $Q(\varepsilon,t;\varepsilon',t')$ are

$$Q(\varepsilon;t)\equiv\langle\delta(\varepsilon-\mathbf{E}(t))\rangle\ , \tag{2.2}$$

$$Q(\varepsilon,t;\varepsilon',t')\equiv\langle\delta(\varepsilon-\mathbf{E}(t))\delta(\varepsilon'-\mathbf{E}(t'))\rangle\ , \tag{2.3}$$

where the brackets denote an equilibrium ensemble average. There are some simplifications of the general form for these joint probability densities that arise from the invariance properties of the equilibrium state. Stationarity and time-reversal invariance lead to

$$\begin{aligned} &Q(\varepsilon,t)=Q(\varepsilon,0)\ ,\\ &Q(\varepsilon,t;\varepsilon',t')=Q(\varepsilon,|t-t'|;\varepsilon',0)\\ &\qquad=Q(\varepsilon',|t-t'|;\varepsilon,0)\ , \end{aligned} \tag{2.4}$$

i.e., $Q(\varepsilon,t)$ is time independent and $Q(\varepsilon,t;\varepsilon',t')$ is a function of $|t-t'|$ that is symmetric with respect to an interchange of ε and ε'. Furthermore, rotational invariance implies that $Q(\varepsilon;t)$ is a function only of the field magnitude, while $Q(\varepsilon,t;\varepsilon',t')$ depends on the fields only through their magnitudes and the angle between them. From the assumed mixing property of the dynamics, the asymptotic long- and short-time limits are

$$\begin{aligned} &Q(\varepsilon,t;\varepsilon',t)=\delta(\varepsilon-\varepsilon')Q(\varepsilon)\ ,\\ &Q(\varepsilon,\infty;\varepsilon'0)=Q(\varepsilon)Q(\varepsilon')\ , \end{aligned} \tag{2.5}$$

$$Q(\varepsilon)\equiv Q(\varepsilon,t)\ . \tag{2.6}$$

The probability density $Q(\varepsilon)$ determines all time-independent properties of the electric field. Similarly, the dynamical properties of the field at two different times are completely determined from the joint probability density $Q(\varepsilon,t;\varepsilon',t')$.

The theoretical analysis of $Q(\varepsilon)$ and $Q(\varepsilon,t;\varepsilon',t')$ is facilitated by considering instead their associated generating functions,

$$Q(\varepsilon)=(2\pi)^{-3}\int d\boldsymbol{\lambda}\, e^{-i\boldsymbol{\lambda}\cdot\varepsilon}e^{G(\boldsymbol{\lambda})}\ , \tag{2.7}$$

$$Q(\varepsilon,t;\varepsilon',0)=(2\pi)^{-6}\int d\boldsymbol{\lambda}\, d\boldsymbol{\lambda}' e^{-i\boldsymbol{\lambda}\cdot\varepsilon-i\boldsymbol{\lambda}'\cdot\varepsilon'}e^{G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)}\ . \tag{2.8}$$

These generating functions are given by

$$G(\boldsymbol{\lambda})=\ln(\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle)\ , \tag{2.9}$$

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=\ln(\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}(t)}e^{i\boldsymbol{\lambda}'\cdot\mathbf{E}}\rangle)\ . \tag{2.10}$$

The properties (2.4) and (2.5) imply, respectively,

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=G(\boldsymbol{\lambda},\boldsymbol{\lambda}';|t|)=G(\boldsymbol{\lambda}',\boldsymbol{\lambda};|t|)\ , \tag{2.11}$$

$$\begin{aligned} &G(\boldsymbol{\lambda},\boldsymbol{\lambda}';0)=G(|\boldsymbol{\lambda}+\boldsymbol{\lambda}'|),\\ &G(\boldsymbol{\lambda},\boldsymbol{\lambda}';\infty)=G(\boldsymbol{\lambda})+G(\boldsymbol{\lambda}')\ . \end{aligned} \tag{2.12}$$

Two further identifies are easily verified,

$$\left[\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}G(\boldsymbol{\lambda})\right]_{\boldsymbol{\lambda}=0}=-\langle E_iE_j\rangle\ , \tag{2.13}$$

$$\begin{aligned} \left[\frac{\partial^2}{\partial\lambda_i\partial\lambda_j'}G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)\right]_{\boldsymbol{\lambda}=\boldsymbol{\lambda}'=0}&=-\langle E_iE_j(t)\rangle\\ &=-\langle E_i(t)E_j\rangle\ . \end{aligned} \tag{2.14}$$

Equation (2.13) applies only for $Z_0\neq0$, as the right-hand side diverges otherwise.

The notation above implies that $G(\boldsymbol{\lambda})$ depends on $\boldsymbol{\lambda}$ only through its magnitude; similarly, $G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)$ depends on the magnitudes of λ, λ', and $y\equiv\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}}'$,

$$e^{G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)}\equiv F(\lambda,\lambda',y;t)\ . \tag{2.15}$$

The field distributions then have the representations

$$Q(\varepsilon)=(2\pi^2)^{-1}\int_0^\infty d\lambda\,\lambda^2e^{G(\lambda)}j_0(\lambda\varepsilon)\ , \tag{2.16}$$

$$Q(\varepsilon,t;\varepsilon',0)=\sum_{l=0}^\infty P_l(\hat{\varepsilon}\cdot\hat{\varepsilon}')Q_l(\varepsilon,\varepsilon';t)\ , \tag{2.17}$$

where $P_l(x)$ are Legendre polynomials and $Q_l(\varepsilon,\varepsilon';t)$ is

$$Q_l(\varepsilon,\varepsilon';t)\equiv2(2l+1)(2\pi)^{-4}(-1)^l\int_0^\infty d\lambda\,\lambda^2j_l(\lambda\varepsilon)\int_0^\infty d\lambda'\lambda'^2j_l(\lambda'\varepsilon')F_l(\lambda,\lambda';t)\ , \tag{2.18}$$

$$F_l(\lambda,\lambda';t)\equiv\int_{-1}^1 dy\, P_l(y)F(\lambda,\lambda',y;t)\ . \tag{2.19}$$

The generating functions can be reduced to quadratures for the special case of electric fields at a neutral point due to noninteracting particles. The result is given in Appendix A.

III. SHORT-TIME LIMIT

The initial distribution $Q(\varepsilon,0;\varepsilon',0)$ is proportional to a δ function at $\varepsilon=\varepsilon'$. Subsequently, this sharp distribution broadens and shifts towards the product of equilibrium distributions $Q(\varepsilon)Q(\varepsilon')$. To describe the early stages of the microfield dynamics it is useful to expand $G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)$ as a power series in t. Since $G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)$ is an even function of t, the leading time dependence is of order t^2. From Eq. (2.10) and stationarity, the leading terms are found to be

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=G(|\boldsymbol{\lambda}+\boldsymbol{\lambda}'|)+\tfrac{1}{2}\lambda_i\lambda_j' F_{ij}(|\boldsymbol{\lambda}+\boldsymbol{\lambda}'|)t^2+O(t^4)\,, \tag{3.1}$$

$$F_{ij}(\boldsymbol{\lambda})\equiv\langle \dot{E}_i\dot{E}_j e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle/\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle\,, \tag{3.2}$$

where $\dot{E}_i$ denotes the time derivative of the ith component of the electric field at $t=0$. The correlation function in (3.2) can be reduced further to (see Appendix B)

$$\begin{aligned}F_{ij}(\boldsymbol{\lambda})&=(e_0^2/\beta m_0 r_0^2)\delta_{ij}\\&\quad+(nk_BT/\mu)\int d\mathbf{r}\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r})}{\partial r_l}\right]\tilde{g}(\mathbf{r};\boldsymbol{\lambda})\\&\quad+(n^2k_BT/m_0)\int d\mathbf{r}\,d\mathbf{r}'\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r}')}{\partial r_l'}\right]\\&\quad\times\tilde{g}(\mathbf{r},\mathbf{r}';\boldsymbol{\lambda})\,,\end{aligned} \tag{3.3}$$

where r_0 is the ion sphere radius, $e_0\equiv Ze/r_0^2$ is the field magnitude of a particle at this distance, and μ is the reduced mass for an ion-impurity pair. The correlation functions $\tilde{g}(\mathbf{r},\boldsymbol{\lambda})$ and $\tilde{g}(\mathbf{r},\mathbf{r}';\boldsymbol{\lambda})$ are defined by

$$n\tilde{g}(\mathbf{r};\boldsymbol{\lambda})=N\langle\delta(\mathbf{r}-\mathbf{r}_1)e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle/\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle\,, \tag{3.4}$$

$$n^2\tilde{g}(\mathbf{r},\mathbf{r}';\boldsymbol{\lambda})=N\langle\delta(\mathbf{r}-\mathbf{r}_1)\delta(\mathbf{r}'-\mathbf{r}_2)e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle/\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle\,. \tag{3.5}$$

It is noted below that these functions are the Fourier transforms of the pair and triplet correlation functions in the presence of a specified field at the impurity. Further discussion of these expressions is given in Appendix B.

Equation (2.8) now determines the joint distribution function for the fields. It is useful to introduce the new integration and field coordinates,

$$\boldsymbol{\eta}\equiv(\boldsymbol{\lambda}-\boldsymbol{\lambda}')/2,\ \ \boldsymbol{\sigma}\equiv\boldsymbol{\lambda}+\boldsymbol{\lambda}',\ \ \Delta\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}',\ \ \mathbf{a}\equiv(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}')/2\,,$$

$$\begin{aligned}Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)&=(2\pi)^{-6}\int d\boldsymbol{\eta}\,d\boldsymbol{\sigma}\,e^{-i\boldsymbol{\eta}\cdot\Delta\boldsymbol{\varepsilon}-i\boldsymbol{\sigma}\cdot\mathbf{a}}\exp\left[G(\sigma)-\frac{t^2}{2}(\eta_i+\sigma_i/2)(\eta_j-\sigma_j/2)F_{ij}(\boldsymbol{\sigma})\right]\\&\equiv(2\pi)^{-3}\int d\boldsymbol{\eta}\,e^{-i\boldsymbol{\eta}\cdot\Delta\boldsymbol{\varepsilon}}Q(a)\exp[-\mathcal{F}(\mathbf{a},\boldsymbol{\eta};t)]\,,\end{aligned} \tag{3.6}$$

$$\mathcal{F}(\mathbf{a},\boldsymbol{\eta};t)=\frac{t^2}{2Q(a)}(2\pi)^{-3}\int d\boldsymbol{\sigma}\,e^{-i\boldsymbol{\sigma}\cdot\mathbf{a}}e^{G(\sigma)}(\eta_i+\sigma_i/2)(\eta_j-\sigma_j/2)F_{ij}(\boldsymbol{\sigma})+O(t^4)\,. \tag{3.7}$$

Use of (3.7) in (3.6) gives the desired short-time limit [9],

$$Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)=Q(a)\left[\frac{e^{-(\Delta\varepsilon)_\perp^2/2A_1(a)t^2}}{2\pi t^2A_1(a)}\right]\left[\frac{e^{-(\Delta\varepsilon)_\parallel^2/2A_2(a)t^2}}{[2\pi t^2A_2(a)]^{1/2}}\right][1+O(t^2)]\,. \tag{3.8}$$

Here $(\Delta\boldsymbol{\varepsilon})_\perp$ and $(\Delta\boldsymbol{\varepsilon})_\parallel$ are the components of $\Delta\boldsymbol{\varepsilon}$ perpendicular and parallel to $\mathbf{a}$, respectively. The scalar functions $A_1(a)$ and $A_2(a)$ characterize a second-order tensor $A_{ij}(\mathbf{a})$,

$$\begin{aligned}A_1(a)&=\tfrac{1}{2}[A_{ii}(\mathbf{a})-\hat{\mathbf{a}}_i\hat{\mathbf{a}}_jA_{ij}(\mathbf{a})]\,,\\A_2(a)&=\hat{\mathbf{a}}_i\hat{\mathbf{a}}_jA_{ij}(\mathbf{a})\end{aligned} \tag{3.9}$$

and $A_{ij}(\mathbf{a})$ is related to $F_{ij}(\boldsymbol{\lambda})$ by a Fourier transformation,

$$Q(a)A_{ij}(\mathbf{a})\equiv(2\pi)^{-3}\int d\boldsymbol{\lambda}\,e^{-i\boldsymbol{\lambda}\cdot\mathbf{a}}e^{G(\lambda)}F_{ij}(\boldsymbol{\lambda})\,. \tag{3.10}$$

From the definition of $F_{ij}(\boldsymbol{\lambda})$ it is straightforward to show that $A_{ij}(\mathbf{a})$ is the conditional average,

$$A_{ij}(\boldsymbol{\varepsilon})=\langle\dot{E}_i\dot{E}_j\rangle_{\boldsymbol{\varepsilon}}\equiv\langle\dot{E}_i\dot{E}_j\delta(\boldsymbol{\varepsilon}-\mathbf{E})\rangle/Q(\varepsilon)\,. \tag{3.11}$$

The result (3.8) is a Gaussian distribution with respect to the variable $\Delta\boldsymbol{\varepsilon}$, but whose amplitude and half-width depend on the variable $\mathbf{a}$. For very short times the distribution function is nonzero only for $\boldsymbol{\varepsilon}\simeq\boldsymbol{\varepsilon}'$, in which case $\mathbf{a}\sim\boldsymbol{\varepsilon}'$. The distribution function is then a symmetric Gaussian about $\boldsymbol{\varepsilon}'$ with a half-width equal to $A_2(\varepsilon')t^2$. It is found below that $A_2(\varepsilon')$ is a monotonically increasing function of ε', so the initial decay for large fields is more rapid than that for small fields. This is expected, since large fields are due to rather unlikely configurations of the ions. For slightly larger times, values of $\mathbf{a}\neq\boldsymbol{\varepsilon}'$ become relevant and the Gaussian is no longer symmetric about $\boldsymbol{\varepsilon}'$; furthermore, the maximum shifts monotonically towards smaller fields as the distribution broadens. For long times the distribution becomes uniform, but (3.8) is no longer valid on this time scale.

To be more quantitative, it is necessary to evaluate the coefficients in this short-time expansion. First, substitute (3.3) into (3.10) to get

$$A_{ij}(\mathbf{a})=(e_0^2/\beta m_0 r_0^2)\delta_{ij} + (nk_B T/\mu)\int d\mathbf{r}\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r})}{\partial r_l}\right]g(\mathbf{r};\mathbf{a}) + (n^2 k_B T/m_0)\int d\mathbf{r}\,d\mathbf{r}'\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r}')}{\partial r_l'}\right] \times g(\mathbf{r},\mathbf{r}';\mathbf{a}) , \quad (3.12)$$

where $g(\mathbf{r};\mathbf{a})$ and $g(\mathbf{r},\mathbf{r}';\mathbf{a})$ are the pair and triplet correlation functions in the presence of a field $\mathbf{a}$ at the impurity,

$$ng(\mathbf{r};\mathbf{a})=N\langle\delta(\mathbf{r}-\mathbf{r}_1)\rangle_a , \quad (3.13)$$

$$n^2 g(\mathbf{r},\mathbf{r}';\mathbf{a})=N(N-1)\langle\delta(\mathbf{r}-\mathbf{r}_1)\delta(\mathbf{r}'-\mathbf{r}_2)\rangle_a , \quad (3.14)$$

where the conditional average $\langle\ \rangle_a$ is the same as that defined in Eq. (3.11). The functions $\tilde{g}$ of (3.4) and (3.5) are related to those of (3.13) and (3.14) by Fourier transformation, as indicated by (3.10). It has been shown [10] that $g(\mathbf{r};\mathbf{a})$ can be determined directly from the theory for $Q(a)$. For the most accurate independent-particle models [3], $g(\mathbf{r};\mathbf{a})$ is given by

$$g(\mathbf{r};\mathbf{a})=g(r)Q(|\mathbf{a}-\mathbf{e}^*(\mathbf{r})|)/Q(a) , \quad (3.15)$$

where $g(r)$ is the usual equilibrium radial distribution function for ions at a distance f from the impurity, and $\mathbf{e}^*(r)$ is an effective field of the independent-particle model chosen to satisfy condition (2.13) (see also Appendix B). A similar expression relating $g(\mathbf{r},\mathbf{r}';\mathbf{a})$ to the three-particle correlation function $g(\mathbf{r},\mathbf{r}')$ and the microfield distribution Q can also be obtained. The detailed forms for the coefficients $A_1(a)$ and $A_2(a)$ are obtained in Appendix B. The corresponding dimensionless quantities $A^*\equiv A/(\omega_p e_0)^2$, where $\omega_p=(4\pi n Z^2 e^2/m)^{1/2}$ is the plasma frequency for the ions, are given in Table I for the neutral case at $\Gamma=1$. It is seen that both coefficients are monotonically increasing functions of a; their asymptotic behavior is found to be (neutral case)

$$A_1^*(a)\to\tfrac{1}{3}a^3, \quad A_2^*(a)\to\tfrac{4}{3}4a^3 . \quad (3.16)$$

To illustrate the main result of this section, Eq. (3.8), it is useful to introduce the characteristic times

$$t_\perp^2\equiv\Delta\varepsilon_\perp^2/2A_1(a) , \quad t_\parallel^2=\Delta\varepsilon_\parallel^2/2A_2(a) . \quad (3.17)$$

Then the joint probability distribution can be written as

TABLE I. Short-time data at $\Gamma=1$.

ε	$A_1(\varepsilon)$	$A_2(\varepsilon)$	$Q(\varepsilon)$ (units of 10^3)
0.0	0.857	0.857	147.0
0.25	0.899	0.916	132.0
0.5	1.03	1.11	98.9
0.75	1.27	1.48	64.4
1.0	1.64	2.09	38.6
1.25	2.16	3.04	22.4
1.5	2.88	4.45	13.0
1.75	3.83	6.44	7.70
2.0	5.03	9.15	4.69
2.25	6.53	12.7	2.95
2.5	8.34	17.3	1.91
2.75	10.5	23.0	1.28
3.0	13.1	30.1	0.876
3.5	19.5	48.5	0.444
4.0	27.8	73.8	0.244
4.5	38.3	107.0	0.144
5.0	51.2	149.0	0.089
6.0	85.2	264.0	0.039
7.0	132.0	428.0	0.019
8.0	193.0	648.0	0.010

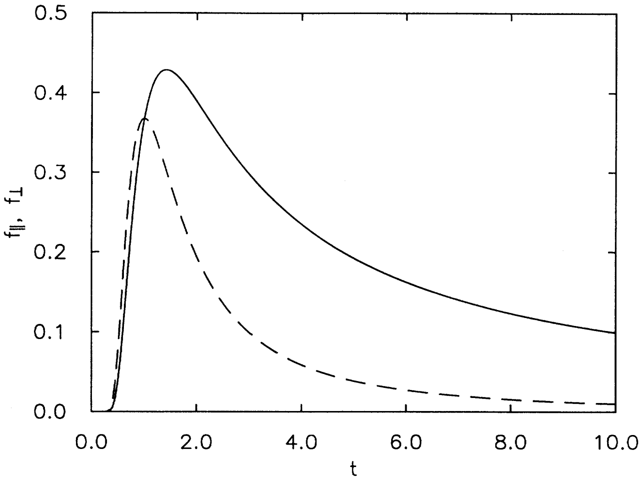

FIG. 1. Short-time functions $f_\perp(t)$ (——), and $f_\parallel(t)$ (— — —), Eq. (3.19).

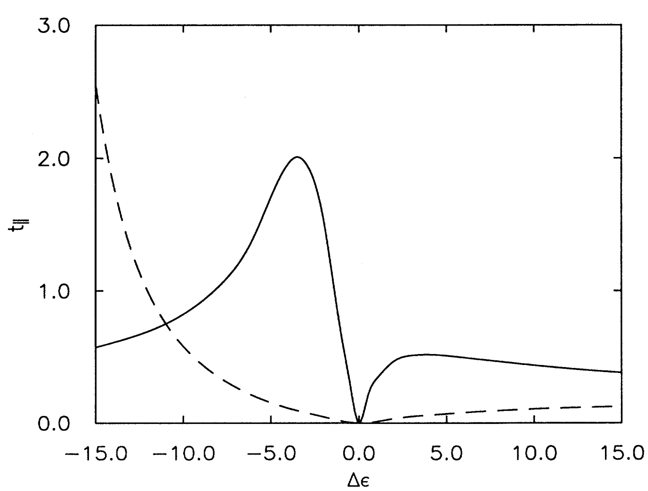

FIG. 2. Characteristic time $t_\parallel$, Eq. (3.20), as a function of $\Delta\varepsilon$ for $\varepsilon'/e_0=1$ (——) and $\varepsilon'/e_0=10$ (— — —); both calculated at $\Gamma=1$.

$$Q(\varepsilon,t;\varepsilon',0)=Q(a)(\pi^{3/2}\Delta\varepsilon_\perp^2\Delta\varepsilon_\parallel)^{-1}f_\perp(t/t_\perp)f_\parallel(t/t_\parallel)\ , \tag{3.18}$$

where the time dependence is determined from the two functions

$$f_\perp(t)\equiv t^{-2}e^{-t^{-2}},\quad f_\parallel(t)=t^{-1}e^{-t^{-2}}\ . \tag{3.19}$$

These are shown in Fig. 1. For given ε' both functions show that the probability of finding a different field ε rises from zero to a maximum and then decreases again to zero. The times at which the maxima occur for $f_\perp$ and $f_\parallel$ are determined by the time scales $t_\perp$ and $t_\parallel$, which are functions of ε and ε'. Consider the special case of colinear fields, i.e., $\varepsilon'=\varepsilon'\hat{\varepsilon}$. Then $\Delta\varepsilon_\perp=0$ and the time dependence is governed by $f_\parallel(t/t_\parallel)$ [note that $(\Delta\varepsilon_\perp^2)^{-1}f_\perp(t/t_\perp)$ is finite in this limit.] The time scale $t_\parallel$ can be written as

$$t_\parallel^2=\Delta\varepsilon^2/2A_2(|\varepsilon'+\tfrac{1}{2}\Delta\varepsilon|)\ . \tag{3.20}$$

For given ε', $t_\parallel$ vanishes for small $\Delta\varepsilon$, rises to a maximum, and approaches zero as $\Delta\varepsilon^{-1/2}$ for large $\Delta\varepsilon$ according to (3.16). Similarly, for fixed $\Delta\varepsilon$, $t_\parallel$ decreases monotonically for $\varepsilon'>\Delta\varepsilon$ as $(\varepsilon')^{-3/2}$. Figure 2 shows $\omega_p t_\parallel$ as a function of $\Delta\varepsilon/e_0$, for $\varepsilon'/e_0=1$ and $\varepsilon'/e_0=10$. Since (3.17) is only valid for short times, $\omega_p t\ll 1$, the relevant portion of the graph in Fig. 1 depends strongly on both ε and ε'.

Figure 2 shows that the short-time limit is not uniformly valid with respect to t for all field values. A related qualification appears in the Fourier representation: consider the use of (3.1) in (2.14) to get the short-time expansion of the field correlation function,

$$\begin{aligned}\langle E_iE_j(t)\rangle&=\langle E_iE_j\rangle-\tfrac{1}{2}F_{ij}(0)t^2+O(t^4)\\&=\langle E_iE_j\rangle-\tfrac{1}{2}\langle\dot{E}_i\dot{E}_j\rangle t^2+O(t^4)\ .\end{aligned} \tag{3.21}$$

This is the correct result for $Z_0\neq 0$, but not for the neutral-point case where the coefficients diverge. For example, in the limit of small plasma parameter and for $Z_0=0$, the right-hand side of (3.1) is easily evaluated to give

$$\begin{aligned}G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=&-c_1|\boldsymbol{\lambda}+\boldsymbol{\lambda}'|^{3/2}\\&-c_2(\lambda\lambda')^2(1-x^2)|\boldsymbol{\lambda}+\boldsymbol{\lambda}'|^{-7/2}(\omega_p t)^2+O(t^4)\ ,\end{aligned} \tag{3.22}$$

where $x=\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}}'$, $c_1=e_0^{3/2}2(2\pi)^{1/2}/5$, and $c_2=e_0^{1/2}9(2\pi)^{1/2}/8$. The Taylor series in time has coefficients with a nonanalytic dependence on $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$. Similarly, it can be shown that a Taylor series in $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ has coefficients that are nonanalytic in t.

IV. GAUSSIAN LIMIT

The distributions $Q(\varepsilon)$ and $Q(\varepsilon,t;\varepsilon',0)$ become simple Gaussians if their generating functions can be expanded to second order in $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$. The definitions (2.9) and (2.10) show that this is possible only if the dominant configurations correspond to small values of E. This is never the case for field distributions at a neutral point, as there is nothing to prohibit one or more ions from being close to the impurity. However, for fields at a charged point there is a repulsion that excludes particles from a sphere of radius $r_c/r_0\simeq\Gamma(Z_0/Z)$. The associated fields are then of the order $E\simeq e_0(Z/Z_0\Gamma)$. Thus, for conditions of strong coupling and/or large Z_0/Z the Gaussian limit is applicable. The calculation is straightforward, with the results

$$Q_g(\varepsilon)=\left[\frac{3}{2\pi\langle E^2\rangle}\right]^{3/2}\exp\left[-\frac{3\varepsilon^2}{2\langle E^2\rangle}\right]\ , \tag{4.1}$$

$$Q_g(\varepsilon,t;\varepsilon',0)=[1-\alpha^2(t)]^{-3/2}Q_g\left[\frac{\varepsilon-\alpha(t)\varepsilon'}{[1-\alpha^2(t)]^{1/2}}\right]Q_g(\varepsilon')\ . \tag{4.2}$$

Here, $\alpha(t)$ is the normalized electric-field autocorrelation function

$$\alpha(t)\equiv\langle\mathbf{E}(t)\cdot\mathbf{E}\rangle/\langle E^2\rangle\ . \tag{4.3}$$

The Gaussian limit of $Q(\varepsilon)$ is entirely specified by $\langle E^2\rangle$. It has been shown elsewhere [10,11] that the $\langle E^2\rangle$ is given exactly by

$$\langle E^2\rangle=e_0^2 3(Z/Z_0\Gamma)\ . \tag{4.4}$$

The distribution of field values $P(\varepsilon)\equiv 4\pi\varepsilon^2Q(\varepsilon)$ in dimensionless form is then found to be

$$P^*(\beta)=e_0P(\varepsilon)=(4/x\sqrt{\pi})(\beta/x)^2e^{-(\beta/x)^2}\ , \tag{4.5}$$

with $\beta\equiv\varepsilon/e_0$ and $x\equiv(2Z/Z_0\Gamma)^{1/2}$. The peak occurs at $\beta_{\max}=x$ and $\beta_{\max}P^*(\beta_{\max})=0.83$. This Gaussian-limit form for the location and value of the maximum is in qualitative agreement with Monte Carlo simulation results, even for relatively small Γ. For example, Fig. 3 shows a comparison of the Gaussian limit to simulation results for $Z=Z_0=1$ and $\Gamma=10$. The agreement for

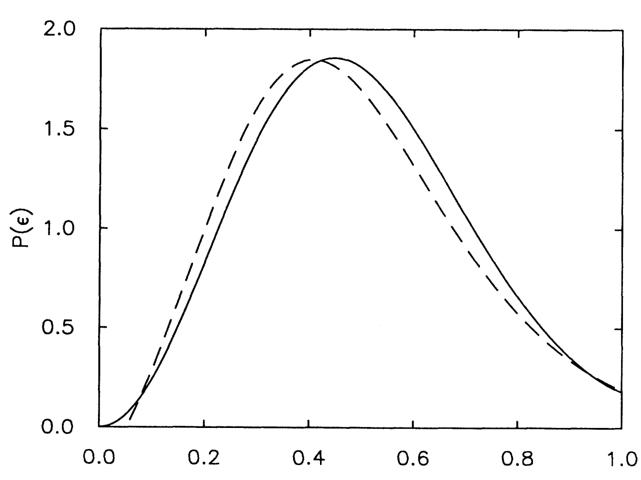

FIG. 3. Distribution of field magnitudes $P(\varepsilon)\equiv 4\pi\varepsilon^2Q(\varepsilon)$, for $Z=Z_0=1$ and $\Gamma=10$ in units of e_0; Gaussian limit, Eq. (4.5) (——), and computer simulation (— — —).

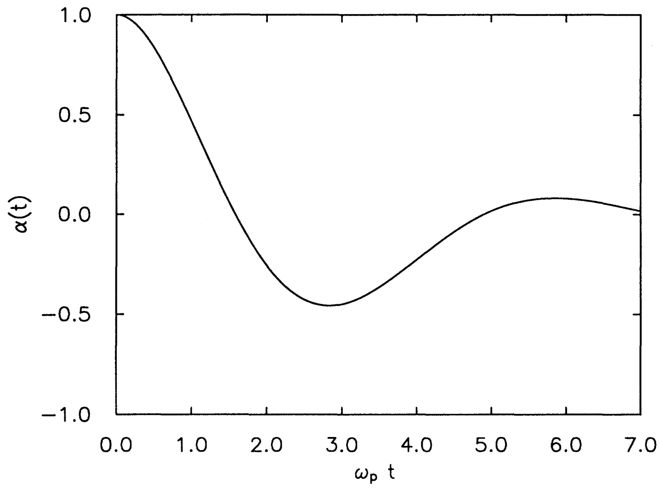

FIG. 4. Electric-field autocorrelation function $\alpha(t)$ for the same conditions as in Fig. 3.

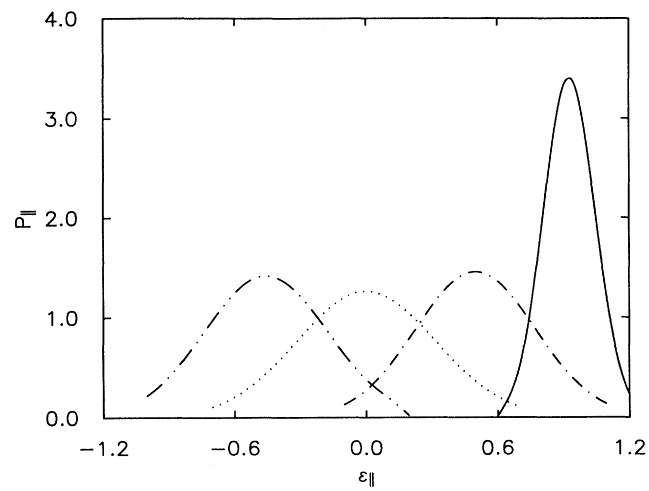

FIG. 5. Conditional probability density $P_{\parallel}(\varepsilon,t;\varepsilon',0)$, Eq. (4.8), for $\Gamma=10$, $Z=Z_0=1$, as a function of ε with $\varepsilon'=1$ (in units of e_0), at $\omega_p t=0.32$ (——), 0.96 (—·—), 2.88 (—··—), and $\omega_p t=\infty$ (···).

most field values is reasonable, considering that these conditions are only marginal for the validity of the Gaussian limit.

The time-dependent distribution requires, in addition, specification of $\alpha(t)$. An accurate model for this function has been proposed [1] based on the first few time derivatives and the relationship of $\alpha(t)$ to the self-diffusion coefficient, giving good agreement with results from computer simulation [12] even at strong coupling. Figure 4 shows the results from this model for $Z=Z_0=1$ and $\Gamma=10$. The time integral of $\alpha(t)$ vanishes as a consequence of the relationship of the electric field to the impurity momentum, and consequently it must change sign. In Figure 4 this occurs at $\omega_p t\simeq 1.6$. Consequently, the Gaussian probability density given by (4.2) approaches its asymptotic limit as $\omega_p t\to 1.6$ continues towards fields in the opposite direction of the initial field and finally approaches again the asymptotic limit. To illustrate this, the expression (4.2) can be factored into distributions for the components of the field ε, parallel and perpendicular to the initial field ε',

$$P(\varepsilon,t;\varepsilon',0)\equiv Q_g(\varepsilon,t;\varepsilon',0)/Q_g(\varepsilon') = P_{\perp}(\varepsilon_{\perp},t)P_{\parallel}(\varepsilon_{\parallel},t;\varepsilon',0)\ , \tag{4.6}$$

$$P_{\perp}(\varepsilon_{\perp},t)=\{3/2\pi\langle E^2\rangle[1-\alpha^2(t)]\} \times\exp\{-3\varepsilon_{\perp}^2/2\langle E^2\rangle[1-\alpha^2(t)]\}\ , \tag{4.7}$$

$$P_{\parallel}(\varepsilon_{\parallel},t;\varepsilon',0)=\{3/2\pi\langle E^2\rangle[1-\alpha^2(t)]\}^{1/2} \times\exp\{-3(\varepsilon_{\parallel}-\varepsilon')^2/2\langle E^2\rangle[1-\alpha^2(t)]\}\ . \tag{4.8}$$

Figure 5 illustrates $P_{\parallel}(\varepsilon_{\parallel},t;\varepsilon',0)$ for the conditions above and $\varepsilon'=1$. The initial δ function has broadened somewhat at $\omega_p t=0.32$, broadens and shifts towards smaller fields at $\omega_p t=0.96$, passes the asymptotic distribution to favor negative fields at $\omega_p t=2.88$, and finally approaches the stationary distribution for very long times.

In the more general case, the distribution $P_{\perp}(\varepsilon_{\perp},t)$ is independent of the initial field, with a δ-function distribution at $\varepsilon_{\perp}=0$ for $t=0$. This distribution simply broadens (not monotonically) to the stationary distribution, but shows no shift. The Gaussian evolution described by (4.2) is clearly not well represented by a Markovian stochastic process.

V. LARGE-FIELD LIMIT

The most probable configuration for creating a large field is expected to be such that most of the field is due to a single particle at a distance $r\simeq\sqrt{Ze/\varepsilon}$. Alternatively, it could be produced by several particles at larger distances, but they would have to be approximately collinear. However, these constraints on the superposition of fields due to more than one particle become increasingly restrictive as the field increases. To further justify and quantify these notions, consider first the distribution of fields $Q(\varepsilon)$. From the defining equations (2.7) and (2.9),

$$Q(\varepsilon)=(2\pi)^{-3}\int d\boldsymbol{\lambda}\, e^{-i\boldsymbol{\lambda}\cdot\boldsymbol{\varepsilon}}\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle =(2\pi)^{-3}\int d\boldsymbol{\lambda}\, e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\varepsilon}}\left\langle\prod_i^N[1+\phi(\boldsymbol{\lambda},\mathbf{r}_i)]\right\rangle \tag{5.1}$$

with $\phi(\boldsymbol{\lambda},\mathbf{r}_i)$ given by

$$\phi(\boldsymbol{\lambda},\mathbf{r}_i)\equiv -1+\exp[i\boldsymbol{\lambda}\cdot\mathbf{e}(\mathbf{r}_i)]\ . \tag{5.2}$$

Next, make the change of variables $\boldsymbol{\lambda}\varepsilon=\hat{\boldsymbol{\lambda}}l\equiv\mathbf{l}$ and $\mathbf{r}_i=\mathbf{z}/\sqrt{\varepsilon}$; then Eq. (5.1) becomes, for large ε,

$$Q(\varepsilon)=(2\pi\varepsilon)^{-3}\int d\mathbf{l}\, e^{-i\mathbf{l}\cdot\hat{\boldsymbol{\varepsilon}}}(1+\varepsilon^{-3/2}\int d\mathbf{z}\,\phi(\mathbf{l},\mathbf{z})\langle n(\mathbf{z}/\sqrt{\varepsilon})\rangle+O(\varepsilon^{-3})]\to\int d\mathbf{r}\langle n(\mathbf{r})\rangle\delta(\boldsymbol{\varepsilon}-\mathbf{e}(\mathbf{r}))\ , \tag{5.3}$$

where $n(\mathbf{r})$ is the number density of ions at $\mathbf{r}$ relative to the impurity,

$$n(\mathbf{r})=\sum_{i=1}^{N}\delta(\mathbf{r}-(\mathbf{q}_i-\mathbf{q}_0)) \ . \tag{5.4}$$

This is the expected result: the configuration space sampled is always such that the field ε is determined from a single particle. The configuration integrals can be performed in (5.3), leading to

$$\begin{aligned} Q(\varepsilon) &\to \tfrac{1}{2}(Ze)^{3/2}\varepsilon^{-9/2}\langle n(\hat{\varepsilon}(Ze/\varepsilon)^{1/2})\rangle \\ &= \tfrac{1}{2}(Ze)^{3/2}\varepsilon^{-9/2} n g(\hat{\varepsilon}(Ze/\varepsilon)^{1/2}) \ , \end{aligned} \tag{5.5}$$

where $g(r)$ is the equilibrium pair correlation function. For the neutral-point case $g(r)=1$ and (5.5) becomes the asymptotic form of the Holtsmark distribution. The field distribution at a charged point depends on the radial distribution function only at short distances, for large fields, and (5.5) approaches the nearest-neighbor distribution [12] asymptotically.

To discuss the joint distribution for large fields at two times, two cases are distinguished: (i) large ε but arbitrary ε', and (ii) both ε and ε' large.

(i) *Large* ε, *arbitrary* ε'. The defining equations are (2.8) and (2.10),

$$Q(\varepsilon,t;\varepsilon',0)=(2\pi)^{-6}\int d\boldsymbol{\lambda}\, d\boldsymbol{\lambda}' e^{-i\boldsymbol{\lambda}\cdot\varepsilon-i\boldsymbol{\lambda}'\cdot\varepsilon'}\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}(t)}e^{i\boldsymbol{\lambda}'\cdot\mathbf{E}}\rangle=(2\pi)^{-3}\int d\boldsymbol{\lambda}\, e^{-i\boldsymbol{\lambda}\cdot\varepsilon}\left\langle\prod_i^N[1+\phi(\boldsymbol{\lambda},\mathbf{r}_i)]\right\rangle_{\varepsilon'} Q(\varepsilon') \ . \tag{5.6}$$

The conditional average $\langle\ \rangle_{\varepsilon'}$ is defined by Eq. (3.11). The same change of variables as described below (5.2) leads to

$$\begin{aligned} Q(\varepsilon,t;\varepsilon',0) &\to (2\pi\varepsilon)^{-3}\int d\mathbf{l}\, e^{-i\mathbf{l}\cdot\hat{\varepsilon}}Q(\varepsilon')\left[1+\varepsilon^{-3/2}\int d\mathbf{z}\,\phi(\mathbf{l},\mathbf{z})\langle n(\mathbf{z}/\sqrt{\varepsilon})\rangle_{\varepsilon'}+O(\varepsilon^{-3})\right] \\ &\to \int d\mathbf{r}\,\delta(\varepsilon-\mathbf{e}(\mathbf{r}))\langle n(\mathbf{r},t)\rangle_{\varepsilon'}Q(\varepsilon') \ , \\ Q(\varepsilon,t;\varepsilon',0)/Q(\varepsilon') &\to N\langle\delta(\varepsilon-\mathbf{e}(\mathbf{r}_1,t))\rangle_{\varepsilon'} \ . \end{aligned} \tag{5.7}$$

This expresses most explicitly the fact that the large field ε is due to a single particle, as expected. The configuration integral in (5.7) can be performed with the result

$$Q(\varepsilon,t;\varepsilon',0)/Q(\varepsilon')\to\tfrac{1}{2}(Ze)^{3/2}\varepsilon^{-9/2}\langle n(\mathbf{r},t)\rangle_{\varepsilon'}\big|_{\mathbf{r}=\hat{\varepsilon}(Ze/\varepsilon)^{1/2}} \ . \tag{5.8}$$

(ii) *Large* ε *and large* ε'. In this case, Eq. (5.6) can be expanded further in the symmetric form

$$Q(\varepsilon,t;\varepsilon',0)=(2\pi)^{-6}\int d\boldsymbol{\lambda}\, d\boldsymbol{\lambda}' e^{-i\boldsymbol{\lambda}\cdot\varepsilon-i\boldsymbol{\lambda}'\cdot\varepsilon'}\left\langle\prod_i^N[1+\phi(\boldsymbol{\lambda},\mathbf{r}_i(t))]\prod_j^N[1+\phi(\boldsymbol{\lambda}',\mathbf{r}_j)]\right\rangle \ . \tag{5.9}$$

Changing variables again, as indicated following (5.2), gives

$$\begin{aligned} Q(\varepsilon,t;\varepsilon',0) &\to (2\pi\varepsilon)^{-6}\int d\mathbf{l}\, e^{-i\mathbf{l}\cdot\hat{\varepsilon}}\int d\mathbf{l}' e^{-i\mathbf{l}'\cdot\varepsilon'/\varepsilon}\Big[\langle e^{i\mathbf{l}'\cdot\mathbf{E}(t)/\varepsilon}\rangle+\langle e^{i\mathbf{l}\cdot\mathbf{E}/\varepsilon}\rangle \\ &\qquad +\varepsilon^{-3}\int d\mathbf{z}\, d\mathbf{z}'\phi(\mathbf{l},\mathbf{z})\phi(\mathbf{l}',\mathbf{z}')\langle n(\mathbf{z}/\sqrt{\varepsilon},t)n(\mathbf{z}'/\sqrt{\varepsilon})\rangle+O(\varepsilon^{-6})\Big] \ , \\ Q(\varepsilon,t;\varepsilon',0) &\to \int d\mathbf{r}\, d\mathbf{r}'\delta(\varepsilon-\mathbf{e}(\mathbf{r}))\delta(\varepsilon'-\mathbf{e}(\mathbf{r}'))\langle n(\mathbf{r},t)n(\mathbf{r}')\rangle \ . \end{aligned} \tag{5.10}$$

The time-dependent density fluctuations of ions near the impurity therefore determine the joint distribution function at large fields. Using the explicit form (5.4) for this density gives the alternative expression

$$Q(\varepsilon,t;\varepsilon',0)\to\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\delta(\varepsilon-\mathbf{e}(\mathbf{r}_i,t))\delta(\varepsilon'-\mathbf{e}(\mathbf{r}_j))\rangle \ . \tag{5.11}$$

The configuration space sampled in (5.11) is always such that the fields ε *and* ε' are determined from a single particle (not necessarily the same). It is possible to show that (5.11) has the proper long- and short-time limits, in terms of the asymptotic form for $Q(\varepsilon)$. In the neutral-point case (5.11) is simply related to the dynamic structure factor by

$$Q(\varepsilon,t;\varepsilon',0)\to Q(\varepsilon)+Q(\varepsilon')+\tfrac{1}{4}(Ze)^3\varepsilon^{-9/2}\varepsilon'^{-9/2}(2\pi)^{-3}\int d\mathbf{k}\, S(\mathbf{k},t)\exp\{-i\mathbf{k}\cdot[\hat{\varepsilon}(Ze/\varepsilon)^{1/2}-\hat{\varepsilon}'(Ze/\varepsilon')^{1/2}]\} \ , \tag{5.12}$$

$$S(\mathbf{k},t)\equiv\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}[\langle n(\mathbf{r},t)n(0)\rangle-n^2] \ . \tag{5.13}$$

For short times $\varepsilon\simeq\varepsilon'$, so both fields are due to the same particle and the dominant contribution to (5.12) becomes

$$\lim_{t\to 0}Q(\varepsilon,t;\varepsilon',0)\to N\langle[\delta(\varepsilon-\mathbf{e}(\mathbf{r}_1(t)))][\delta(\varepsilon'-\mathbf{e}(\mathbf{r}_i))]\rangle \ , \tag{5.14}$$

with $\mathbf{r}_1(t)\to\mathbf{r}_1+\mathbf{v}_1 t$, and $\mathbf{v}_1$ is the relative velocity. Evaluation of (5.14) is straightforward with the result [2]

$$\lim_{t\to 0}Q(\varepsilon,t;\varepsilon',0)\to\tfrac{1}{4}n\exp[-\beta ZZ_0e^2(\varepsilon/Ze)^{1/2}]\times(Ze)^3(\pi u^2t^2)^{-3/2}(\varepsilon\varepsilon')^{-9/2}\exp\left[-(Ze/u^2t^2)\left[\frac{\hat{\varepsilon}}{\sqrt{\varepsilon}}-\frac{\hat{\varepsilon}'}{\sqrt{\varepsilon'}}\right]^2\right] \ . \tag{5.15}$$

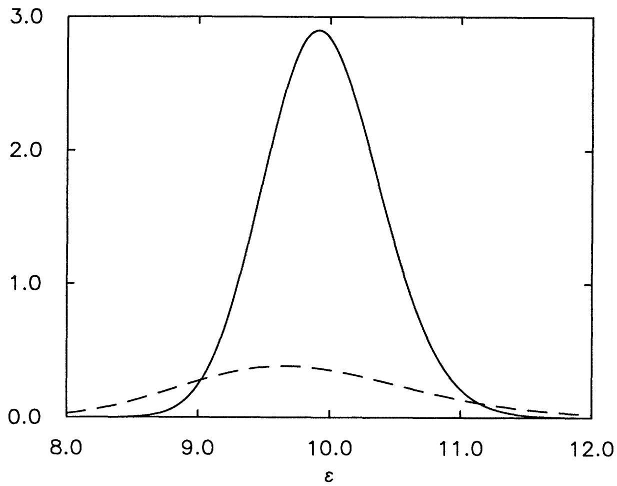

FIG. 6. Conditional probablity $P(\varepsilon,t;\varepsilon',0)\equiv Q(\varepsilon,t;\varepsilon',0)/Q(\varepsilon')$ in the high-field limit, Eq. (5.15), for $Z=Z_0$, as a function of ε with $\varepsilon'/e_0=10$ and $\hat{\varepsilon}\cdot\hat{\varepsilon}'=1$. Two times are shown, $ut/r_0=0.01$ (——) and 0.02 (— — —).

Here, $u=(2k_BT/m)^{1/2}$ is the thermal velocity. This result is in agreement with the short-time expansion of Sec. III, Eq. (3.8), as may be verified using the asymptotic forms (3.16) for the short-time coefficients. Equation (5.15) is illustrated in Fig. 6 for $Z_0=0$, $Z=1$, $\hat{\varepsilon}\cdot\hat{\varepsilon}'=1$, and $\varepsilon'=10$.

VI. WEAK-COUPLING LIMIT

For small plasma parameter, the correlations among particles are weak in spite of the long-range interactions. In this limit, a mean-field theory applies in which each particle moves independently in an average field due to all other particles. Consider first the generating function for the distribution of field values,

$$e^{G(\lambda)}=\langle e^{i\lambda\cdot\mathbf{E}}\rangle=\left\langle\prod_i^N[1+\phi(\lambda,\mathbf{r}_i)]\right\rangle$$
$$=1+\sum_{p=1}^N\frac{n^p}{p!}\int d\mathbf{r}_1\cdots d\mathbf{r}_p\left[\prod_{i=1}^p\phi(\lambda,\mathbf{r}_i)\right]\times g^{(p)}(\mathbf{r}_1,\ldots,\mathbf{r}_p)\ , \qquad (6.1)$$

where the equilibrium configurational correlation functions have been introduced,

$$n^pg(\mathbf{r}_1,\ldots,\mathbf{r}_p)\equiv\frac{N!}{(N-p)!}\left\langle\prod_{i=1}^p\delta(\mathbf{r}_i(\mathbf{q}_i-\mathbf{q}_0))\right\rangle\ . \qquad (6.2)$$

An expansion for $G(\lambda)$ follows directly from (6.1) in terms of the Ursell cluster functions [13] $U(\mathbf{r}_1,\ldots,\mathbf{r}_p)$ associated with the correlation functions $g(\mathbf{r}_1,\ldots,\mathbf{r}_p)$,

$$G(\lambda)=\sum_{p=1}^N\frac{1}{p!}\int d\mathbf{r}_1\cdots d\mathbf{r}_p\left[\prod_{i=1}^p\phi(\lambda,\mathbf{r}_i)\right]U(\mathbf{r}_1,\ldots,\mathbf{r}_p)\ . \qquad (6.3)$$

The first few cluster functions are given by

$$U(\mathbf{r}_1)=ng(\mathbf{r}_1)\ ,$$
$$U(\mathbf{r}_1,\mathbf{r}_2)=n^2[g(\mathbf{r}_1,\mathbf{r}_2)-g(\mathbf{r}_1)g(\mathbf{r}_2)]\ . \qquad (6.4)$$

The expansion (6.3) was first given by Baranger and Mozer [14].

The weak-coupling limit is obtained for conditions of small plasma parameter Γ. In the extreme weak-coupling limit all charges become statistically independent on a space scale of the order of the Debye length. However, the large field values of $Q(\varepsilon)$ are determined from the distribution of ions close to the impurity, and it is therefore necessary to have a more accurate account of ion-impurity correlations on this scale. Consequently, we understand the weak-coupling limit to entail the neglect of correlations between ions while retaining correlations between ions and the impurity. In this case, (6.2) becomes

$$g(\mathbf{r}_1,\ldots,\mathbf{r}_p)\rightarrow\equiv\prod_{i=1}^pg(\mathbf{r}_i)\ . \qquad (6.5)$$

It then follows from the cluster property that all $U\rightarrow0$ for $p>1$. Consequently, (6.3) in this weak-coupling limit becomes

$$G(\lambda)\rightarrow n\int d\mathbf{r}\,\phi(\lambda,\mathbf{r})g(r)\ . \qquad (6.6)$$

For the neutral case, (6.6) still applies but with $g(r)=1$.

A similar analysis applies for the generating function for the joint distribution,

$$e^{G(\lambda,\lambda';t)}=\langle e^{i\lambda\cdot\mathbf{E}(t)}e^{i\lambda'\cdot\mathbf{E}}\rangle$$
$$=1+\sum_{p=1}^N\frac{n^p}{p!}\int d\mathbf{r}_1\cdots d\mathbf{r}_p\left[\left[\prod_{i=1}^p\phi(\lambda,\mathbf{r}_i)\right]+\left[\prod_{i=1}^p\phi(\lambda',\mathbf{r}_i)\right]\right]g(\mathbf{r}_1,\ldots,\mathbf{r}_p)$$
$$+\sum_{p=1}^N\frac{1}{p!}\sum_{p'=1}^N\frac{1}{p'!}\int d\mathbf{r}_1\cdots d\mathbf{r}_pd\mathbf{r}'_1\cdots d\mathbf{r}'_{p'}\left[\prod_{i=1}^p\phi(\lambda,\mathbf{r}_i)\right]\left[\prod_{j=1}^p\phi(\lambda',\mathbf{r}'_j)\right]C(\mathbf{r}_1,\ldots,\mathbf{r}_p;\mathbf{r}'_1,\ldots,\mathbf{r}'_{p'};t)\ , \qquad (6.7)$$

with the definition

$$C(\mathbf{r}_1,\ldots,\mathbf{r}_p;\mathbf{r}'_1,\ldots,\mathbf{r}'_{p'};t)\equiv p!q!\sum_{i_1<\cdots<i_p}\cdots\sum\sum_{j_1<\cdots<j_{p'}}\cdots\sum\left\langle\prod_{\alpha=1}^{p'}\delta(\mathbf{r}_\alpha-(\mathbf{q}_{i_\alpha}-\mathbf{q}_0))\prod_{\beta=1}^{p'}\delta(\mathbf{r}_\beta-(\mathbf{q}_{j_\beta}-\mathbf{q}_0))\right\rangle\ . \qquad (6.8)$$

The corresponding expansion for $G(\lambda,\lambda';t)$ is then found to be

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=G(\lambda)+G(\lambda')+\sum_{p=1}^{N}\frac{1}{p!}\sum_{p'=1}^{N}\frac{1}{p'!}\int d\mathbf{r}_1\cdots d\mathbf{r}_p d\mathbf{r}_1'\cdots d\mathbf{r}_{p'}'\left[\prod_{i=1}^{p}\phi(\boldsymbol{\lambda},\mathbf{r}_i)\right]\left[\prod_{j=1}^{p'}\phi(\boldsymbol{\lambda}',\mathbf{r}_j')\right] \times U(\mathbf{r}_1,\ldots,\mathbf{r}_p;\mathbf{r}_1',\ldots,\mathbf{r}_{p'}';t) . \tag{6.9}$$

Here $U(\mathbf{r}_1,\ldots,\mathbf{r}_p;\mathbf{r}_1',\ldots,\mathbf{r}_{p'}';t)$ are cluster functions associated with $C(\mathbf{r}_1,\ldots,\mathbf{r}_p;\mathbf{r}_1',\ldots,\mathbf{r}_{p'}';t)$. The first term of this set corresponds to $p=p'=1$ and is given by

$$U(\mathbf{r}_1;\mathbf{r}_2;t)=C(\mathbf{r}_1;\mathbf{r}_2;t)-n^2g(r_1)g(r_2) =\langle n(\mathbf{r}_1;t)[n(\mathbf{r}_2)-\langle n(\mathbf{r}_2)\rangle]\rangle . \tag{6.10}$$

The order of magnitude of terms in the series (6.9) for weak coupling can be estimated by their values at $t=0$. In this limit the first of Eqs. (2.12) and the weak-coupling limit (6.6) give

$$\begin{aligned}G(\boldsymbol{\lambda},\boldsymbol{\lambda}',0)&=n\int d\mathbf{r}\,\phi(|\boldsymbol{\lambda}+\boldsymbol{\lambda}'|,\mathbf{r})g(r)\\&=n\int d\mathbf{r}\,\phi(\boldsymbol{\lambda},\mathbf{r})g(r)+n\int d\mathbf{r}\,\phi(\boldsymbol{\lambda}',\mathbf{r})g(r)\\&\quad+n\int d\mathbf{r}\phi(\boldsymbol{\lambda},\mathbf{r})\phi(\boldsymbol{\lambda}',\mathbf{r})g(r) .\end{aligned} \tag{6.11}$$

All of the terms on the right-hand side of (6.11) originate from the lowest-order contributions to the three terms on the right-hand side of (6.9). Consequently, the remainder of the series is negligible in the weak-coupling limit. The resulting expression is then simply

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)\to n\int d\mathbf{r}[\phi(\boldsymbol{\lambda},\mathbf{r})+\phi(\boldsymbol{\lambda}',\mathbf{r})]g(r) +\int d\mathbf{r}_1 d\mathbf{r}_2\phi(\boldsymbol{\lambda},\mathbf{r}_1)\phi(\boldsymbol{\lambda}',\mathbf{r}_2)U(\mathbf{r}_1,\mathbf{r}_2;t) . \tag{6.12}$$

It remains to determine the time dependence of $U(\mathbf{r}_1,\mathbf{r}_2;t)$. In the weak-coupling limit it is easily verified that the initial condition for $U(\mathbf{r}_1,\mathbf{r}_2;0)$ is

$$U(\mathbf{r}_1,\mathbf{r}_2;0)=\delta(\mathbf{r}_1-\mathbf{r}_2)g(\mathbf{r}_1) . \tag{6.13}$$

To calculate the time dependence, $U(\mathbf{r}_1,\mathbf{r}_2;t)$ is first expressed as [using (6.10)],

$$U(\mathbf{r}_1,\mathbf{r}_2;t)=\int dx_0dx_1\delta(\mathbf{r}_1-(\mathbf{q}_1-\mathbf{q}_0))\psi^{(1)}(x_0,x_1;t) , \tag{6.14}$$

where $x_i\leftrightarrow(\mathbf{q}_i,\mathbf{v}_i)$ denotes the position and velocity of the ith particle. It is shown in Appendix C that $\psi^{(1)}(x_0,x_1,;t)$ obeys the following kinetic equation:

$$\left[\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla_{q_0}+\mathbf{v}_1\cdot\nabla_{q_1}-(Z_0/Z)\theta(0,1)\right]\psi^{(1)}(x_0,x_1;t) =n\int dx_2[\theta(1,2)+(Z_0/Z)\theta(0,2)][f_0(v_1)g(\mathbf{q}_1-\mathbf{q}_0)\psi^{(1)}(x_0,x_2;t)+f_0(v_2)g(\mathbf{q}_2-\mathbf{q}_0)\psi^{(1)}(x_0,x_1;t)] , \tag{6.15}$$

where $f_0(v)$ is the Maxwell-Boltzmann distribution, and $\theta(i,j)$ describes the interaction between a pair of ions,

$$\theta(ij)\equiv\nabla_{q_i}V(\mathbf{q}_i-\mathbf{q}_j)\cdot(\nabla_{p_i}-\nabla_{p_j}) . \tag{6.16}$$

The initial condition for Eq. (6.15) is

$$\psi^{(1)}(x_0,x_1;t=0)=f_0(v_1)ng(\mathbf{q}_1-\mathbf{q}_0)\delta(\mathbf{r}_2-(\mathbf{q}_1-\mathbf{q}_0)) . \tag{6.17}$$

Further simplifications occur in the neutral case, for which $\theta(0,i)\to 0$ in Eq. (6.15). The integrations over x_0 can be performed so that (6.14) simplifies to

$$U(\mathbf{r}_1,\mathbf{r}_2;t)=U(\mathbf{r}_1-\mathbf{r}_2;t)=\int d\mathbf{v}\,\psi(\mathbf{r}_1-\mathbf{r}_2,\mathbf{v};t) , \tag{6.18}$$

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla_r\right]\psi(\mathbf{r},\mathbf{v};t)=n\int dx_2\theta(1,2)f_0(v_1)\psi(\mathbf{r}_2,\mathbf{v}_2;t) , \tag{6.19}$$

$$\psi(\mathbf{r},\mathbf{v};0)=f_0(v_1)n\delta(\mathbf{r}_2-(\mathbf{q}_1-\mathbf{q}_0)) . \tag{6.20}$$

Equation (6.19) is the linearized Vlasov equation, which can be solved easily by Fourier transformation. Also for the neutral case, the first term on the right-hand side of (6.12) can be evaluated analytically so that the entire weak-coupling limit becomes

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=c(\lambda e_0)^{3/2}+c(\lambda'e_0)^{3/2} +\int d\mathbf{k}\,\tilde{\phi}(\boldsymbol{\lambda},\mathbf{k})\tilde{\phi}(\boldsymbol{\lambda}',-\mathbf{k})S(\mathbf{k};t) , \tag{6.21}$$

$$\begin{aligned}S(k;t)&\equiv\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}U(\mathbf{r};t) ,\\ \tilde{\phi}(\boldsymbol{\lambda},\mathbf{k})&\equiv\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}\phi(\boldsymbol{\lambda},\mathbf{r}) .\end{aligned} \tag{6.22}$$

The dynamic structure factor $S(k;t)$ is related to the OCP dielectric function by

$$S(k;t)=n((k/\kappa)^2+1)\int_{-\infty}^{\infty}d\omega(\pi\omega)^{-1}e^{i\omega kut}\mathrm{Im}\epsilon^{-1}(k,\omega) . \tag{6.23}$$

Here it is determined from the Vlasov equation that

$$\varepsilon(k,\omega)\equiv 1+(\kappa/k)^2[1+\omega\psi(\omega)] , \tag{6.24}$$

$$\psi(\omega)\equiv(2\pi)^{-1/2}\int_{-\infty}^{\infty}dx\,e^{-x^2}(x-\omega+i\eta)^{-1} , \tag{6.25}$$

with $\kappa^2\equiv 4\pi ne^2/k_BT$, and $\varepsilon(k;\omega)$ is the weak-coupling expression for the dielectric function of a one-component plasma.

VII. DISCUSSION

The objective has been to explore electric-field dynamics at a neutral or charged point in terms of the joint probability density for two fields at two times. The limits explored here yield manageable expressions in the sense that they are determined from the structure and dynamics of fluctuations in the density of ions near the impurity. For the neutral-point case there exist quite good models for this ion density autocorrelation function. The charged-point case is somewhat more difficult, since the corresponding correlation function involves ion correlations in the presence of a third charged particle (the impurity). The results obtained are summarized as follows.

(i) The generating function $G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)$ is the appropriate quantity for theoretical study and is constrained by the symmetries and related properties of the equilibrium ensemble. The dominant constraints to be preserved are given in Eqs. (2.11)–(2.14). In particular, the short- and long-time limits are determined from the generating function $G(\lambda)$ for the electric-field magnitudes.

(ii) The short-time limit of $G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)$ has a leading time dependence of order t^2, with a coefficient that is determined by the constrained correlation functions $\tilde{g}(\mathbf{r};\boldsymbol{\lambda})$ and $\tilde{g}(\mathbf{r},\mathbf{r}';\boldsymbol{\lambda})$. These are related by Fourier transformation to the probability density for an ion at $\mathbf{r}$, or two ions at $\mathbf{r}$ and $\mathbf{r}'$, relative to the impurity, in the presence of a field $\boldsymbol{\varepsilon}$ due to all ions [see Eqs. (3.13) and (3.14)]. These quantities have been studied recently [9] and shown to be determined quite accurately from the usual pair and triplet correlation functions $g(r)$ and $g(\mathbf{r},\mathbf{r}')$ and the static electric-field distribution $Q(\varepsilon)$. Since good approximations exist for these, the short-time expression can be determined. The joint distribution $Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)$ is found to be Gaussian with respect to the field difference $(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}')$, but with coefficients depending on the average field value $(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}')/2$.

(iii) A different Gaussian form is obtained for the charged-point case when the parameter $Z_0\Gamma/Z>>1$. This characterizes conditions such that most ions are sufficiently far from the impurity that the associated fields are weak ($E/Ne_0 \ll 1$). The distribution $Q(\epsilon)$ is then Gaussian in ε, with a half-width determined from $\langle E^2\rangle$. The joint distribution, $Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)$, is also Gaussian in the variable $[\boldsymbol{\varepsilon}-\alpha(t)\boldsymbol{\varepsilon}']$, with half-width determined from $\alpha(t)$. Here, $\alpha(t)$ is the normalized electric-field autocorrelation function, $\langle \mathbf{E}(t)\cdot\mathbf{E}\rangle/\langle E^2\rangle$. The latter is accurately determined from coefficients in its short-time expansion and its relationship to the self-diffusion coefficient. If changes sign as t increases, so the approach of $Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)$ to its asymptotic value is not monotonic.

(iv) The distribution $Q(\varepsilon)$ at large field values is determined simply from the radial distribution function $g(r)$ characterizing ion configurations near the impurity [Eq. (5.5)]. The joint distribution $Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)$ at large $\boldsymbol{\varepsilon}$ is given by (5.8), which expresses the contribution to $\boldsymbol{\varepsilon}'$ as arising from all particles, but that to $\boldsymbol{\varepsilon}$ as due to a single particle. Similarly, if both fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are large, then (5.11) shows that each is due to a single particle. Finally, at short times and large fields, both fields are due to the same single particle.

(v) For weak coupling, the Baranger-Moser expansion of $G(\lambda)$ and its extension for $G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)$ truncates after the first and second terms, respectively. The resulting expressions are determined for all fields and all times in terms of the pair correlation function $g(r)$ and the density autocorrelation function $\langle n(\mathbf{r},t)[n(\mathbf{r}')-\langle n(\mathbf{r}')\rangle]\rangle$. For the neutral-point case the autocorrelation function is easily calculated from the Vlasov equation.

The distribution of fields at two times allows calculation of all equilibrium time-correlation functions in the form

$$\langle A(\mathbf{E}(t))B(\mathbf{E})\rangle=\int d\boldsymbol{\varepsilon}\int d\boldsymbol{\varepsilon}'\,A(\boldsymbol{\varepsilon})B(\boldsymbol{\varepsilon}')Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)\ ,$$

where $A(\mathbf{E})$ and $B(\mathbf{E})$ are arbitrary functions of $\mathbf{E}$. The information contained in $Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)$ completely characterizes two time properties. For a more transparent description of the field dynamics it is useful to focus on a more restricted property. An important example is the conditional electric field, or the average field at time t, given a specified value at $t=0$,

$$\begin{aligned}\langle \mathbf{E}(t);\boldsymbol{\varepsilon}\rangle &\equiv \langle \mathbf{E}(t)\delta(\mathbf{E}-\boldsymbol{\varepsilon})\rangle/Q(\varepsilon)\\ &=\int d\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1 Q(\boldsymbol{\varepsilon}_1,t;\boldsymbol{\varepsilon},0)/Q(\varepsilon)\ .\end{aligned}$$

A simple model for this quantity has been discussed elsewhere [15] using exact coefficients in a short-time expansion. A more controlled evaluation of this and related dynamical properties, based on the results of the present paper, is planned to be given elsewhere [4].

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APPENDIX A: FIELDS AT A NEUTRAL POINT DUE TO NONINTERACTING PARTICLES

If the interactions among ions are neglected, Eqs. (2.9) and (2.10) simplify to

$$\begin{aligned}G(\lambda)=\ln(\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle)=\ln\left[\prod_{i=1}^{N}V^{-1}\int d\mathbf{q}_i e^{i\boldsymbol{\lambda}\cdot\mathbf{e}(\mathbf{q}_i)}\right]&=N\ln\left[1+V^{-1}\int d\mathbf{q}(e^{i\boldsymbol{\lambda}\cdot\mathbf{e}(\mathbf{q})}-1)\right]\\ &\to n\int d\mathbf{q}\,\phi(\boldsymbol{\lambda},\mathbf{q})\ ,\end{aligned}\tag{A1}$$

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=\ln(\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}(t)}e^{i\boldsymbol{\lambda}'\cdot\mathbf{E}}\rangle)=\ln\left[\prod_{i=1}^{N}V^{-1}\int d\mathbf{v}_i f_0(v_i)\int d\mathbf{q}_i e^{i\boldsymbol{\lambda}\cdot\mathbf{e}(\mathbf{q}_i+\mathbf{v}_it)+i\boldsymbol{\lambda}'\cdot\mathbf{e}(\mathbf{q}_i)}\right]$$
$$=N\ln\left[1+V^{-1}\int d\mathbf{v}\,f_0(v)\int d\mathbf{q}(e^{i\boldsymbol{\lambda}\cdot\mathbf{e}(\mathbf{q}+\mathbf{v}t)+i\boldsymbol{\lambda}'\cdot\mathbf{e}(\mathbf{q})}-1)\right]$$
$$\to n\int d\mathbf{v}f_0(v)\int d\mathbf{q}(e^{i\boldsymbol{\lambda}\cdot\mathbf{e}(\mathbf{q}+\mathbf{v}t)+i\boldsymbol{\lambda}'\cdot\mathbf{e}(\mathbf{q})}-1)\;. \tag{A2}$$

This can be rearranged to have the form of the weak-coupling limit obtained in Sec. VI,

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=n\int d\mathbf{q}[\phi(\boldsymbol{\lambda},\mathbf{q})+\phi(\boldsymbol{\lambda}',\mathbf{q})]+n\int d\mathbf{q}\,d\mathbf{v}f_0(v)\phi(\boldsymbol{\lambda},\mathbf{q}+\mathbf{v}t)\phi(\boldsymbol{\lambda}',\mathbf{q})\;. \tag{A3}$$

Here f_0 is the Maxwell-Boltzmann distribution and $\phi(\boldsymbol{\lambda},\mathbf{q})$ is defined by Eq. (5.2). Performing the coordinate integral in (A2) and the velocity integral in (A3) leads to the final results

$$G(\lambda)=-\tfrac{2}{5}(2\pi)^{1/2}c_1\lambda^{3/2}\;, \tag{A4}$$

$$G(\boldsymbol{\lambda},\boldsymbol{\lambda}';t)=G(\lambda)+G(\lambda')+n\int d\mathbf{r}_1d\mathbf{r}_2\phi(\boldsymbol{\lambda},\mathbf{r}_1)\phi(\boldsymbol{\lambda}',\mathbf{r}_2)C(\mathbf{r}_1-\mathbf{r}_2,t)\;. \tag{A5}$$

This agrees with the weak-coupling expressions (6.4) and (6.12), except that here $C(\mathbf{r}_1-\mathbf{r}_2,t)$ is the density autocorrelation function for an ideal gas,

$$C(\mathbf{r}_1-\mathbf{r}_2,t)=n(\pi u^2t^2)^{-3/2}\exp[-(\mathbf{r}_1-\mathbf{r}_2)^2/(ut)^2]\;. \tag{A6}$$

APPENDIX B: COEFFICIENTS IN SHORT-TIME EXPANSION

In this appendix the coefficients in the short-time expansion are simplified for numerical evaluation. Consider first the evaluation of (3.2),

$$F_{ij}(\boldsymbol{\lambda})\equiv\langle\dot{E}_i\dot{E}_je^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle/\langle e^{i\boldsymbol{\lambda}\cdot\mathbf{E}}\rangle\;, \tag{B1}$$

where $\dot{E}_i$ is given by the time derivative of (2.1),

$$\dot{E}_i=\sum_{\alpha=1}^{N}[(\mathbf{p}_0/m_0)-(\mathbf{p}_\alpha/m_\alpha)]\cdot\nabla_{q_0}e_i(\mathbf{r}_\alpha)+(\mathbf{p}_0/m_0)\cdot\nabla_{q_0}e_{bi}(\mathbf{q}_0)\;, \tag{B2}$$

$$\mathbf{e}_b(\mathbf{q}_0)\equiv-\nabla_{q_0}\int d\mathbf{r}\,\rho\Big/|\mathbf{q}_0-\mathbf{r}|\;, \tag{B3}$$

and $\rho\equiv Zen$ is the charge density for the uniform background. The background field obeys the identity

$$\frac{\partial}{\partial q_{0j}}e_{bi}(\mathbf{q}_0)=\frac{\partial^2}{\partial q_{0i}\partial q_{0j}}\int d\mathbf{r}\,\rho\Big/|\mathbf{q}_0-\mathbf{r}|=-\delta_{ij}(4\pi Zen/3)\;. \tag{B4}$$

Use of (B2) and (B4) in (B1) gives directly the result (3.3),

$$F_{ij}(\boldsymbol{\lambda})=(e_0^2/\beta m_0r_0^2)\delta_{ij}+(nk_BT/\mu)\int d\mathbf{r}\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r})}{\partial r_l}\right]\tilde{g}(\mathbf{r};\boldsymbol{\lambda})+(n^2k_BT/m_0)\int d\mathbf{r}\,d\mathbf{r}'\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r}')}{\partial r_l'}\right]\tilde{g}(\mathbf{r},\mathbf{r}';\boldsymbol{\lambda})\;, \tag{B5}$$

with the correlation functions defined in (3.4) and (3.5). The corresponding expression for $A_{ij}(\mathbf{a})$ is now obtained directly from (3.10),

$$A_{ij}(\mathbf{a})=(e_0^2/\beta m_0r_0^2)\delta_{ij}+(nk_BT/\mu)\int d\mathbf{r}\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r})}{\partial r_l}\right]g(\mathbf{r};\mathbf{a})+(n^2k_BT/m_0)\int d\mathbf{r}\,d\mathbf{r}'\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r}')}{\partial r_l'}\right]g(\mathbf{r},\mathbf{r}';\mathbf{a})\;. \tag{B6}$$

These expressions apply for the charged-point case, as they include fluctuations in $\dot{E}_i$ due to absolute motion of the impurity relative to the background [the second term of (B2)]. However, for a neutral point the impurity has constant velocity, which can be transformed to zero by a suitable Galilean transformation in the definition of $Q(\boldsymbol{\varepsilon},t;\boldsymbol{\varepsilon}',0)$, so that the latter is completely independent of the properties of the impurity. The correct result for the neutral point case can be obtained from (B6) by taking the limit $m_0\to\infty$ to suppress these absolute motion contributions,

$$A_{ij}(\mathbf{a})\to(nk_BT/m)\int d\mathbf{r}\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r})}{\partial r_l}\right]g(\mathbf{r};\mathbf{a})\;. \tag{B7}$$

In the following, we restrict further attention to this neutral-point case.

To evaluate (B7) it is necessary to specify the distribution of charges $g(\mathbf{r};\mathbf{a})$. It has been shown elsewhere [10] that this distribution can be obtained from a suitable functional derivative of the generating function for $Q(\varepsilon)$. In particular, if $Q(\varepsilon)$ is given by the adjustable-parameter exponential (APEX) approximation [11], then $g(\mathbf{r};\mathbf{a})$ becomes

$$g(\mathbf{r};\mathbf{a})\rightarrow Q(|\mathbf{a}-\mathbf{e}^*(\mathbf{r})|)/Q(a)\ ,\qquad \mathbf{e}^*(\mathbf{r})\equiv\mathbf{e}(\mathbf{r})(1+\alpha r)e^{-\alpha r}\ , \tag{B8}$$

where $\alpha=-2\beta U_{\rm ex}(\Gamma)/r_0\Gamma$, and $U_{\rm ex}$ is the excess internal energy for the OCP. Equation (B7) simplifies to

$$A_{ij}(\mathbf{a})\rightarrow(nk_BT/m)\int d\mathbf{r}\left[\frac{\partial E_i(\mathbf{r})}{\partial r_l}\right]\left[\frac{\partial E_j(\mathbf{r})}{\partial r_l}\right]\times Q(|\mathbf{a}-\mathbf{e}^*(\mathbf{r})|)/Q(a)\ . \tag{B9}$$

The coefficients $A_1(a)$ and $A_2(a)$ are then calculated from Eq. (3.9),

$$\begin{aligned}Q(a)A_1(a)&=(nZ^2e^2k_BT/2m)\int d\mathbf{r}\,r^{-6}[5-3(\hat{\mathbf{a}}\cdot\hat{\mathbf{r}})^2]Q(|\mathbf{a}-\mathbf{e}^*(\mathbf{r})|)\\&=(nZ^2e^2k_BT/2m)\int_0^\infty dr\,r^{-4}(2\pi)^{-3}\int_0^\infty d\lambda\,\lambda^2e^{G(\lambda)}\int d\Omega_r d\Omega_\lambda[5-3(\hat{\mathbf{a}}\cdot\hat{\mathbf{r}})^2]e^{i\boldsymbol{\lambda}\cdot[\mathbf{a}-\mathbf{e}^*(r)]}\\&=(4nZ^2e^2k_BT/m\pi)\int_0^\infty dr\,r^{-4}\int_0^\infty d\lambda\,\lambda^2e^{G(\lambda)}[j_0(a\lambda)j_0(\lambda e^*)-\tfrac{1}{2}j_2(a\lambda)j_2(\lambda e^*)]\\&=(4nZ^2e^2k_BT/m\pi)\int_0^\infty d\lambda\,\lambda^2e^{G(\lambda)}[j_0(a\lambda)G_0(\lambda)-\tfrac{1}{2}j_2(a\lambda)G_2(\lambda)]\ ,\end{aligned} \tag{B10}$$

where j_l are the spherical Bessel functions, and $G_0(\lambda)$ and $G_2(\lambda)$ are

$$G_0(\lambda)=\int_0^\infty dr\,r^{-4}j_0(\lambda e^*)\ , \tag{B11}$$

$$G_2(\lambda)=\int_0^\infty dr\,r^{-4}j_2(\lambda e^*)\ . \tag{B12}$$

Similarly, $A_2(a)$ is found to be

$$\begin{aligned}Q(a)A_2(a)&=(nZ^2e^2k_BT/m)\int d\mathbf{r}\,r^{-6}[1+3(\hat{\mathbf{a}}\cdot\hat{\mathbf{r}})^2]Q(|\mathbf{a}-\mathbf{e}^*(\mathbf{r})|)\\&=(4nZ^2e^2k_BT/m\pi)\int_0^\infty d\lambda\,\lambda^2e^{G(\lambda)}[j_0(a\lambda)G_0(\lambda)+j_2(a\lambda)G_2(\lambda)]\ .\end{aligned} \tag{B13}$$

The generating function $G(\lambda)$ in the APEX approximation is given by

$$G(\lambda)=4\pi n\int_0^\infty dr\,r^2R(r)[j_0(\lambda e^*)-1]\ , \tag{B14}$$

$$R(r)\equiv e(r)/e^*(r)\ . \tag{B15}$$

For the calculation in Table I at $\Gamma=1$ the parameter α in (B8) has the value $\alpha=1.14/r_0$.

APPENDIX C: WEAK-COUPLING LIMIT

In this appendix the weak-coupling kinetic equation for $U(\mathbf{r}_1,\mathbf{r}_2;t)$ is obtained. From the definition, (6.10),

$$U(\mathbf{r}_1;\mathbf{r}_2;t)=\langle n(\mathbf{r}_1;t)[n(\mathbf{r}_2)-\langle n(\mathbf{r}_2)\rangle]\rangle=\int dx_0dx_1\delta(\mathbf{r}_1-(\mathbf{q}_1-\mathbf{q}_0))\psi^{(1)}(x_0,x_1;t)\ , \tag{C1}$$

$$\psi^{(1)}(x_0,x_1;t)\equiv N\int dx_2,\ldots,dx_N\rho_N[n(\mathbf{r}_2,t)-\langle n(\mathbf{r}_2,t)\rangle]\ , \tag{C2}$$

where x_i denotes the position and momentum of the ith particle, and ρ_n is the N-particle equilibrium distribution function. The reduced distribution function $\psi^{(1)}(x_0,x_1;t)$ satisfies the second Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy equation [16]

$$\left[\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla_{q_0}+\mathbf{v}_1\cdot\nabla_{q_1}-(Z_0/Z)\theta(0,1)\right]\psi^{(1)}(x_0,x_1;t)=\int dx_2[\theta(1,2)+(Z_0/Z)\theta(0,2)]\psi^{(2)}(x_0,x_1,x_2;t)\ , \tag{C3}$$

where the operators $\theta(ij)$ are defined in (6.15), and $\psi^{(2)}(x_0,x_1,x_2;t)$ is defined by

$$\psi^{(2)}(x_0,x_1,x_2;t)\equiv N(N-1)\int dx_3\cdots dx_N\rho_N[n(\mathbf{r}_2,t)-\langle n(\mathbf{r}_2,t)\rangle]\ . \tag{C4}$$

A kinetic equation is obtained by expressing $\psi^{(2)}$ in terms of $\psi^{(1)}$ so that (C3) becomes a closed equation. As in Sec. VI, we use the initial conditions to estimate the weak-coupling form

$$\psi^{(1)}(x_0,x_1;0)=f_0(v)f_0(v_1)n\Big[\delta(\mathbf{r}_2-(\mathbf{q}_1-\mathbf{q}_0))g(\mathbf{q}_1-\mathbf{q}_0)$$
$$+n\int d\mathbf{q}_2\delta(\mathbf{r}_2-(\mathbf{q}_2-\mathbf{q}_0))[g(\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)-g(\mathbf{q}_2-\mathbf{q}_0)g(\mathbf{q}_1-\mathbf{q}_0)]\Big] , \tag{C5}$$

$$\psi^{(2)}(x_0,x_1,x_2;0)=f_0(v)f_0(v_1)f_0(v_2)n^2\Big[\delta(\mathbf{r}_2-(\mathbf{q}_1-\mathbf{q}_0))g(\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)+\delta(\mathbf{r}_2-(\mathbf{q}_2-\mathbf{q}_0))g(\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)$$
$$+n\int d\mathbf{q}_3\delta(\mathbf{r}_2-(\mathbf{q}_3-\mathbf{q}_0))[g(\mathbf{q}_3-\mathbf{q}_0,\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)$$
$$-g(\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)g(\mathbf{q}_3-\mathbf{q}_0)]\Big] . \tag{C6}$$

These results are still exact. In the weak-coupling limit, as described in Sec. VI, the correlation functions simplify to

$$g(\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)\rightarrow g(\mathbf{q}_2-\mathbf{q}_0)g(\mathbf{q}_1-\mathbf{q}_0) , \tag{C7}$$

$$g(\mathbf{q}_3-\mathbf{q}_0,\mathbf{q}_2-\mathbf{q}_0,\mathbf{q}_1-\mathbf{q}_0)\rightarrow g(\mathbf{q}_3-\mathbf{q}_0)g(\mathbf{q}_2-\mathbf{q}_0)g(\mathbf{q}_1-\mathbf{q}_0) . \tag{C8}$$

With these results, (C6) can be written as

$$\psi^{(2)}(x_0,x_1,x_2;0)=f_0(v_1)n\psi^{(1)}(x_0,x_2;0)+f_0(v_2)n\psi^{(1)}(x_0,x_1;0) . \tag{C9}$$

Equation (C9) is the correct weak-coupling functional relationship at $t=0$. We assume that the dynamics preserves this relationship at finite t and write

$$\psi^{(2)}(x_0,x_1,x_2;t)\rightarrow f_0(v_1)n\psi^{(1)}(x_0,x_2;t)+f_0(v_2)n\psi^{(1)}(x_0,x_1;t) . \tag{C10}$$

Use of (C10) in the hierarchy equation (C3) gives the result (6.14) of the text.

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