Logistic equation with memory

E. Fick, M. Fick, and G. Hausmann
Institute of Solid State Physics, Solid State Theory, Technical University Darmstadt, D-6100 Darmstadt, Germany
(Received 26 November 1990)

A statistical treatment of the macroscopic equation of motion leads to memory functions if Markovian-like approximations are not admissible. A nonlinear discrete model, which is nonlocal in time, may be obtained from the logistic map by replacing its linear term by a convolution with an exponentially decaying memory function $x_{t+1}=a\left[\sum_{t'=0}^{t}\gamma_0\epsilon^{t-t'}x_{t'}-x_t^2\right]$. This one-dimensional discrete map, $R^1\to R^1$: $x_0\to x_1\to x_2\cdots$, possesses the same fixed point $1-1/a$ for all ϵ, if we choose the normalization $\gamma_0=1-\epsilon$. It is the goal of this paper to demonstrate the main features of the map induced by the memory term. Due to the complexity of the problem, most of the results have been derived by a numerical treatment. The range of ϵ lies in the interval $-0.1\le\epsilon<0.3$. For fixed a,ϵ generally two alternative attractors S and S' appear. For small $|\epsilon|\ll 0.02$ only the usual Feigenbaum scenario $S(a)$ exists, slightly modified by ϵ. For $\epsilon\gtrsim 0.02$ the scenario $S(a,\epsilon)$ is shifted, and different windows with the same period coalesce. The ϵ-induced scenarios $S'(a,\epsilon)$ arise suddenly. In the (a,ϵ) plane there exist islands of three-, six-, nine-, or tenfold period-doubling scenarios $S'(a,\epsilon)$. The insertion of these $S'(a,\epsilon)$ discontinuously depends on the starting value.

I. INTRODUCTION

In the microscopic treatment of irreversible thermodynamics the relations between the macroscopic forces and currents are given by dynamical Onsager coefficients determined by memory functions. These relations, retarded in time, are integro-differential equations. Markov-like approximations may be allowed for memory times that are small compared to the correlation times. Differential equations with damping constants and renormalized frequencies result [1].

On the other hand, in the treatment of nonlinear dynamics, usually differential equations of macroscopic motion are the immediate starting point. Due to the complexity of their solutions, nonlinear problems are often reduced to difference equations (discrete maps) by Poincaré sections.

The logistic map represents one of the most important examples of a one-dimensional discrete nonlinear map ($R_1\to R_1$: $x_0\to x_1\to\cdots$),

$$x_{t+1}=a(x_t-x_t^2)\quad (t \text{ an integer, } t\ge 0) \tag{1}$$

with its direct interpretation in the dynamics of population. The control parameter a lies in the interval $0\le a\le 4$. The bifurcation scenario is well known. Chaotic behavior with embedded windows of stable periodic cycles arises beyond the period-doubling interval ($1\le a\le a_\infty=3.5699\ldots$). The universal scaling behavior of the corresponding a_p values has been proved [2,3].

The main object of the present paper is to derive effects caused by the assumption that the first term in Eq. (1) has a memory structure. In the application to population dynamics this means that the population at time $t+1$ no longer depends on the population at time t only but also on former times $t-1,t-2,\ldots$. To simplify matters here we will show only such features that result from a retardation of the linear term in Eq. (1), whereas the quadratic term will not be retarded. Fulinski and Kleszkowski [4] considered several forms of nonlinear maps with memory in a formal way. We will discuss dependences of the bifurcations on memory time in this paper.

II. LOGISTIC MAP WITH LINEAR MEMORY

The modification of the linear term by memory is performed by means of a memory function $\gamma_{t'}=\gamma_0\epsilon^{t'}$. The considered model for a difference equation with memory takes the form

$$x_{t+1}=a\left[\gamma_0\sum_{t'=0}^{t}\epsilon^{t'}x_{t-t'}-x_t^2\right]\quad (t\ge 0)\ . \tag{2}$$

We assume that the (stable or nonstable) fixed point of Eq. (2) is independent of ϵ: $x^*=1-(1/a)$. This yields $\gamma_0=1-\epsilon$. The parameter ϵ ($-0.1\le\epsilon<0.3$) represents the memory time τ_m during which the retardation takes place. It is $\tau_m=-1/\ln|\epsilon|$. Equations (2) are nonautonomous, nonlinear integro- (sum-) difference equations for the series $R_1\to R_1$: $x_0\to x_1\to\cdots$ with an initial value x_0. Their behavior will be numerically analyzed.

The exponential function in Eq. (2) allows us to eliminate the sum by writing x_{t+2} and inserting it into Eq. (2) again,

$$x_{t+2}=a(x_{t+1}-x_{t+1}^2)+\epsilon[(1-a)x_{t+1}+ax_t^2]\quad (t\ge 0)\ . \tag{3}$$

The term proportional to ϵ represents the deviation from the usual logistic map.

It is important to note that in our considered one-dimensional map $R_1\to R_1$ [Eq. (2)] the value of $x_1=x_1(x_0)$ is given by Eq. (2) in the form

$$x_1=a(1-\epsilon-x_0)x_0\ . \tag{4}$$

This *precursor* is not contained in Eq. (3), but it completes this equation. Equation (3) alone defines a greater manifold than does Eq. (1). This greater manifold of Eq. (3) is confined to the manifold of Eq. (2) by the precursor (4).

For $t \to \infty$ the ω-limit values x^* of Eq. (2) [of Eqs. (3) and (4), respectively] depend on the parameters a, ϵ, and on the initial value x_0,

$$x^* = x^*(a,\epsilon,x_0) \ . \tag{5}$$

If there exists a limit cycle $x_1^*, \ldots, x_p^*$ of period p, from Eq. (3) the sum formula

$$\sum_{n=1}^{p} x_n^{*2} = [1-(1/a)] \sum_{n=1}^{p} x_n^* \tag{6}$$

easily follows from Eq. (3). This formula is useful for the numerical recognition of a period p.

The fixed point $x^* = 1-(1/a)$ bifurcates into a stable period $p=2$ at the values

$$a_{1\to 2} = \frac{3(1+\epsilon)}{1+3\epsilon} \ . \tag{7}$$

Period 2 bifurcates into a stable period 4 at the values

$$a_{2\to 4} = \frac{u + [u^2 + 5(1+\epsilon^2)(1+\epsilon)^2 v]^{1/2}}{v} \ , \tag{8}$$

with

$$u(\epsilon) = 1 - 2\epsilon - 2\epsilon^3 - 5\epsilon^4 \ ,$$

and

$$v(\epsilon) = 1 + 2\epsilon - 4\epsilon^2 + 6\epsilon^3 - 5\epsilon^4 \ .$$

These equations show that the bifurcation parameters $a_{1\to 2}$ ($a_{2\to 4}$) decrease with increasing memory in the considered range of ϵ (cf. Fig. 4).

III. NUMERICAL RESULTS

The discussion of the bifurcation scenario of Eq. (2) [Eqs. (3) and (4)] leads to systems of algebraic equations. An analytical treatment of periodic orbits has been derived [5] for the Hénon map. This method could be applied to Eq. (3) with greater effort. However, since we are interested in the main structures of the existing attractors as a function of a, ϵ, and x_0, we have only numerically calculated the limit values x^* (ω points). At first we keep fixed the initial value (x_0, say, equal to 0.62).

A. Dependence on *a*

Figure 1 represents the dependence $x^* = x^*(a)$ on a fixed value $\epsilon = 0.123$. Within the usual Feigenbaum scenario S, $p = 4 \to 8 \to 16$, there arises alternatively a six-fold Feigenbaum scenario $S' = S^6$ marked by the number 6 in a box. S^6 begins suddenly with a period 6 and leads to a sixfold chaos via period doubling. The beginning and the ending of S^6 are abrupt. For such values of a for which the attractor S^6 exists the usual scenario S is suppressed.

The abrupt beginning and ending of S^6 do not represent bifurcations. They are caused by a crossing of basin boundaries, a fact which will be further clarified in Secs. III C and III E.

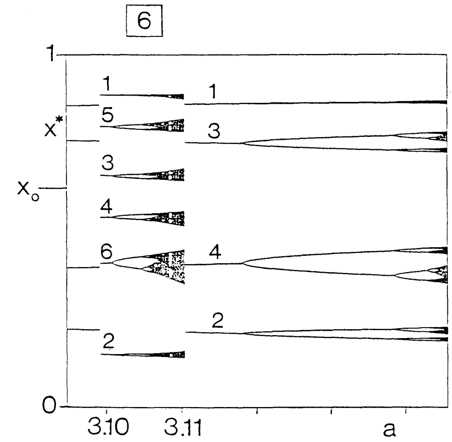

FIG. 1. Dependence of the orbits x^* ($t \to \infty$) on a at fixed ϵ and x_0 ($\epsilon = 0.123$, $x_0 = 0.62$).

B. Dependence on ϵ

Now we take a fixed parameter a ($a = 3.1$) and keep the initial value x_0 ($=0.62$) fixed again. Figure 2 shows the dependence $x^*(\epsilon)$. Within S there is again the alternative scenario $S' = S^6$ with a sudden beginning ($\epsilon_1 = 0.122\,777\,6\ldots$) and ending ($\epsilon_2 = 0.126\,386\ldots$). The value of ϵ_1 (ϵ_2) depends on a and x_0.

C. Transient regions

We consider the time series x_t in the vicinity of the sudden beginning of the alternative period $S' = S^6$. For ϵ values smaller than ϵ_1 we have the period $p = 4$ of S (cf. Fig. 2). If we allow ϵ to approach ϵ_1 ($\epsilon \lesssim \epsilon_1$), there is a

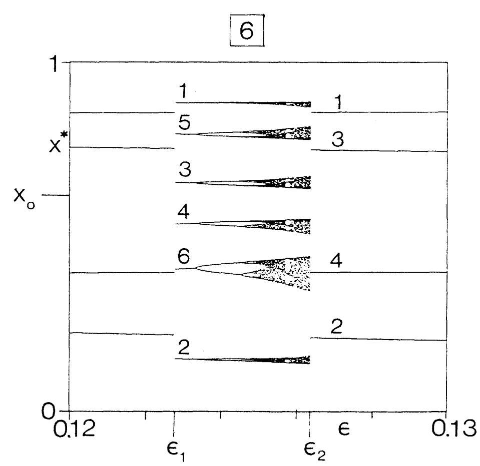

FIG. 2. Dependence of the orbits x^* ($t \to \infty$) on ϵ at fixed a and x_0 ($a = 3.1$, $x_0 = 0.62$).

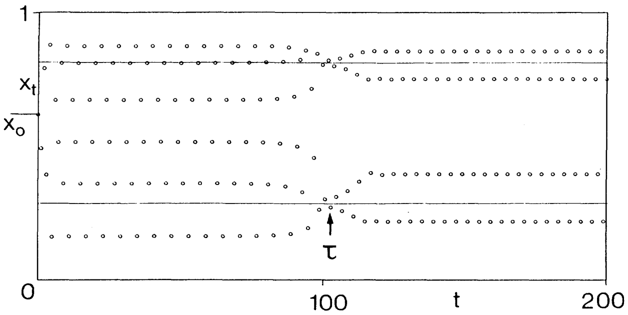

FIG. 3. Transient region x_t for ϵ smaller than ϵ_1 ($a=3.1$, $\epsilon_1-\epsilon=3.8\times10^{-9}$). The two thin lines present the values of the unstable period 2.

long-time laminar region (time τ) (Fig. 3) with an approximately period-6 behavior. At time τ the laminar region ends by crossing an unstable orbit ($p=2$). For times $t>\tau$ the transient has died out and the final stable attractor with period $p=4$ is rapidly reached. With decreasing $\epsilon_1-\epsilon$ the laminar time τ grows. Finally, at $\epsilon\rightarrow\epsilon_1$ the period 4 cannot be built up any longer and the attractor $p=6$ is born. For $\epsilon>\epsilon_1$ the S^6 scenario exists. Our abrupt transitions result with $\tau(\epsilon,a)\rightarrow\infty$ by making a very small change of ϵ or a at fixed x_0. The basins' boundaries of attractions move across the starting point x_0 (cf. Sec. III E) [6].

A similar behavior occurs at ϵ_2 (end of S^6), where a sudden change of the six-fold chaotic motion of S^6 into period $p=4(S)$ takes place again. The mentioned sudden change of an attractor caused by crossing a repeller possesses some similarity with a "crisis" [7]. But in our case it is *not necessary* that a chaotic attractor be involved. Our sudden changes may happen even with periodic orbits.

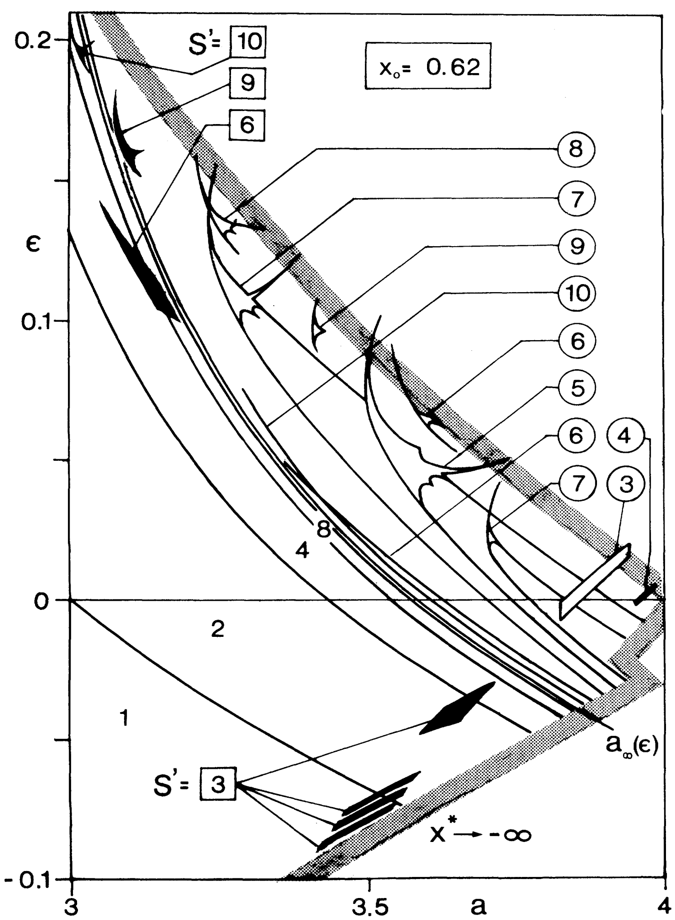

FIG. 4. The (a,ϵ) plane with regions of period p and windows (circled p) of the Feigenbaum scenarios S (fixed $x_0=0.62$). The regions of the alternative p-fold Feigenbaum scenarios S' (boxed p) are marked black (cf. Figs. 1 and 2). On the hatched boundary x^* diverges.

D. Dependence on a and ϵ

In Fig. 4 we find the domains of important periods in the a,ϵ plane for fixed x_0 ($=0.62$). For $\epsilon=0$ we see the usual Feigenbaum scenario S ($\epsilon=0$) of Eq. (1). With increasing ϵ the period doubling of S tends to smaller values of a in most cases. The line $a_\infty(\epsilon)$ of the points, where the period doubling of $S(\epsilon)$ tends to infinity, possesses this behavior too, as is shown in Fig. 4. However, it is remarkable that the periods 3 and 4 of S shift quite in the other direction. Two of the $p=7$ windows of S coalesce with increasing ϵ (Figs. 5 and 6). The same fact is valid for $p=5$ windows of S.

The black areas in Fig. 4 ($\epsilon>0$) describe the embedded domains of the alternative six-, nine-, and tenfold Feigenbaum scenarios S' (boxed numbers). For negative ϵ, i.e., an oscillating memory function, there are several black areas that show threefold Feigenbaum scenarios S' (boxed number 3). Each of the scenarios S' possesses period-doubling and chaotic domains with windows, intermittency and so on within the black areas.

E. Dependence on x_0

In Fig. 7 we see the dependence of x^* on the initial value x_0 for fixed a ($=3.1$) and ϵ ($=0.123$). There exist two basins $\{x_0\}$ and $\{x_0'\}$ of the alternative attractors S:

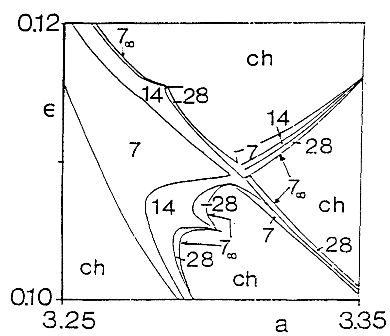

FIG. 5. Enlarged sector of Fig. 4. Details of the coalescence of two $p=7$ windows in S. The boundaries $7_\infty-ch$ mark transitions from period doubling to chaos. At the lines $7-ch$ intermittency appears.

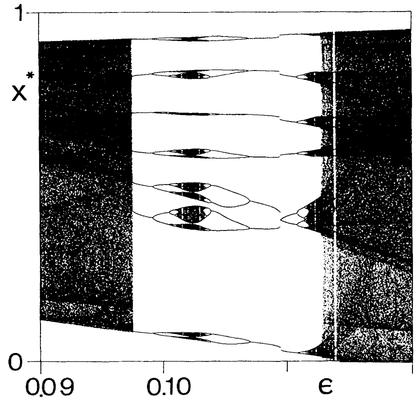

FIG. 6. Dependence of the orbits x^* on ϵ in the region of two coalescent $p=7$ windows in S.

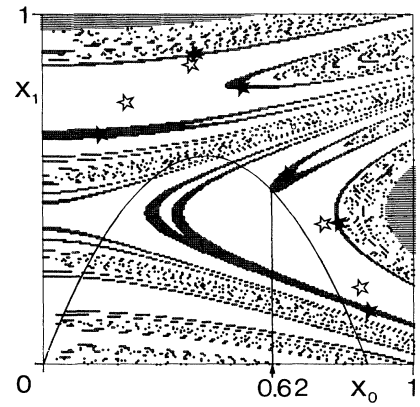

FIG. 8. In the $(x_0, x_1=y_0)$ plane the basin (white) of period 4 (white star) and the basin (black dots) of period 6 (black star), [Eq. (8)]. The parabola presents the precursor [Eq. (4)] ($a=3.1$, $\epsilon=0.123$) order to find the attractor for an initial value x_0 the vertical section within the parabola is to be used.

$p=4$ and $S'=S^6$: $p=6$. This behavior may be better understood when we consider the equivalent two-dimensional map [8]

$$F:\ \begin{cases} x_{t+1}=y_t \\ y_{t+1}=a(y_t-y_t^2)+\epsilon[(1-a)y_t+ax_t^2] \end{cases} \qquad (t\geq 0) \qquad (9)$$

instead of Eq. (3). In Fig. 8 the $(x_0, x_1=y_0)$ plane with the basins for $p=4$ and 6 is drawn. Each white point in the plane represents the basin for period $p=4$ (white star x^*), each black dot in the plane leads to period $p=6$ (black star x^*).

The precursor (4) has the form of a parabola in the (x_0, x_1) plane. Starting with an initial value x_0 the cut of the parabola with a white dot yields a period $p=4$ (white star), and the cut with a black dot yields a period $p=6$ (black star). Running along the parabola in Fig. 8 the dependence $x^*(x_0)$ of Fig. 7 results. This is the essence of the integro-differential equation for $R_1 \to R_1$ in the enlarged two-dimensional auxiliary plane (x_0, x_1).

If we especially take one value of the ω-limit set (black star) as starting point, Fig. 8 shows that we are led either to the same ω-limit set (black star) again or to the other one (white star), and vice versa. This behavior, which does not occur in equations without memory, is a consequence of the precursor.

The same considerations are valid for other values of a and ϵ, yielding corresponding results. In Fig. 9 the dependence of x^* on the initial value x_0 is given for $a=3.1$ and $\epsilon=0.1505$. In this case an additional alternative S^9 with period 9 results within the usual chaotic region of the Feigenbaum S.

IV. CONCLUSIONS

The introduction of a retardation ϵ in the logistic equation leads to a modification of the Feigenbaum scenario S. The periodic orbits are shifted. A coalescence of some

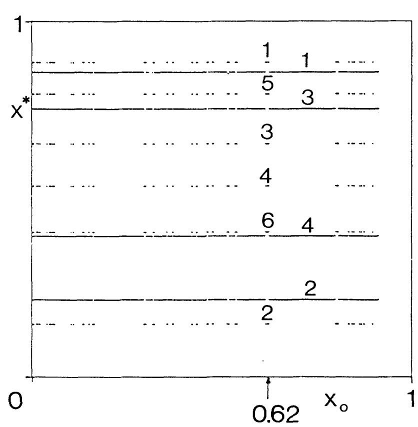

FIG. 7. Dependence of the orbit x^* on the initial value x_0 at fixed a ($=3.1$) and ϵ ($=0.123$). Basins $\{x_0\}$ and $\{x_0'\}$ of the attractors $S(p=4)$ and $S'(p=6)$.

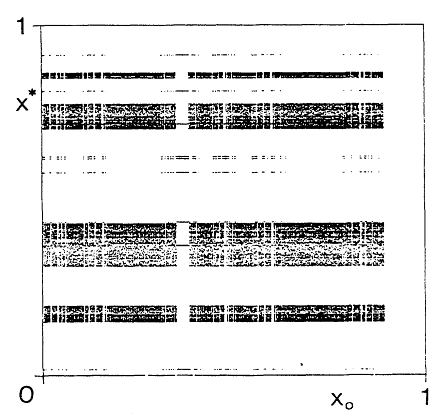

FIG. 9. Alternative period 9 of S' within the chaotic regime of S ($a=3.1$, $\epsilon=0.1505$).

windows with increasing ϵ is found. In addition to S alternative attractors, S'—with basins $\{x_0\}$ and $\{x'_0\}$—appear suddenly if the parameters a or ϵ are continuously changed. This effect is caused during the transient region by crossing an unstable orbit. The alternative attractors S' appear for $\epsilon \gtrsim 0.1$, e.g., for memory times $\tau_m \gtrsim 0.6$ in unit 1 of the time scale ($\Delta t = 1$). It represents the expected result that in these cases a Markov-like approximation no longer holds.

The interpretation of the one-dimensional map with memory in a two-dimensional auxiliary (x_0, x_1) plane requires a precursor. It implies the fact that a trajectory that starts from a point of the ω-limit set does not generally return to the same set, in contrast to difference equations without memory.

This work was performed within a program of the Sonderforschungsbereich 185 Nichtlineare Dynamik Darmstadt-Frankfurt-Marburg, Germany,

---

[1] E. Fick, and G. Sauermann, *The Quantum Statistics of Dynamic Processes* (Springer, Berlin, 1990).

[2] S. Grossmann and S. Thomae, Z. Naturforsch. **32A**, 1353 (1977).

[3] M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); Los Alamos Sci. **1**, 49 (1980).

[4] A. Fulinski and A. S. Kleszkowski, Phys. Scr. **35**, 119 (1987).

[5] Huang Yung-Nien, Chin. Phys. Lett. **2**, 97 (1985); Sci. Sin. A **29**, 1302 (1986).

[6] It should be noted that the precursor [Eq. (4)] also depends on a and ϵ.

[7] C. Grebogi, E. Ott, and J. A. Yorke, Physica **7D**, 181 (1983).

[8] Its Jacobian $DF = -2\epsilon a x_t$ is not constant, in contrast to the Hénon map.