Fluctuational transitions and related phenomena in a passive all-optical bistable system

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Fluctuational transitions between stable states of an all-optical lumped-parameter bistable system have been observed. The dependences of the transition probabilities, and of the statistical distribution of the transmitted-light intensity, on the intensities of incident light and noise have been investigated. Two types of stochastic modulation, that of the incident-light intensity and that of the device itself, have been used. They correspond to multiplicative and additive noise for the intracavity-phase gain and give rise to qualitatively different pictures of the fluctuations. In particular, for the latter type of noise the logarithm of the statistical distribution of the transmitted-light intensity is described by a multibranch function, and the peaks of the distribution are strongly asymmetric. The onset of a zero-frequency peak in the power spectrum of the transmitted light in the range of the kinetic phase transition has been observed. The experimental data are in good qualitative and quantitative agreement with the theory.

I. INTRODUCTION

Optically bistable systems provide an opportunity to investigate a variety of fluctuation phenomena specific for thermally nonequilibrium systems with coexisting stable states. Many theoretical results and most of the experimental data on fluctuations in optically bistable devices were obtained for lasers, with the bistability corresponding to coexistence of competing modes [1,2], or switched-on and switched-off states [3], and also for hybrid electro-optical devices [4]. For passive all-optical bistable systems the fluctuation phenomenon related to bistability and investigated in most detail is the manifestation of both stable states in the course of switching caused by temporal variation of the parameters in the presence of noise (transient bistability [5]).

In the present paper we report the results of the investigation of fluctuation phenomena arising in a passive all-optical bistable system in stationary conditions. The analyzed effects are related to the noise-induced transitions between the stationary attractors. It was crucial for obtaining and interpretating experimental data that the optical system investigated was a lumped-parameter system and was thus described not by space- and time-dependent variables (fields), but by the dynamical variables depending on time only (in contrast to the majority of optically bistable systems that are continuous). In a lumped-parameter system there are no switching waves (cf. [6] and references therein); it does not split into the sections with different transmission. For sufficiently weak external noise its transmission takes on one or the other value predominantly, i.e., the system predominantly occupies a narrow range in the phase space adjacent to one or the other attractor (the stable state) and from time to time "jumps" from one stable state to another because of noise.

The probabilities W_{ij} of the fluctuational transitions $i \rightarrow j$ $(i,j=1,2)$ for small intensities of driving noise are much smaller than the reciprocal relaxation (switching) time τ. Analogous to the case of a Brownian particle fluctuating in a bistable potential, for almost all values of the parameters of the system the probabilities W_{12} and W_{21} differ drastically from each other and, respectively, the stationary populations of the attractors w_1 and w_2 differ as well (in the absence of noise a bistable system occupies only one stable state, i.e., either $w_1=1$ and $w_2=0$, or $w_1=0$ and $w_2=1$). The most interesting and specific fluctuation phenomena arise in the narrow range of the parameters where the populations w_1 and w_2 are of the same order of magnitude. This range has much in common [7,8] with the range of the first-order phase transition in thermal equilibrium systems where the populations of the phases are of the same order of magnitude. Not only small fluctuations about the stable states, but also large fluctuations due to the system changing from one state to another are substantial here. The characteristic time scale for the latter fluctuations is given by the transition probabilities. Their immediate effect is the onset of a narrow (with a width $\sim W \ll \tau^{-1}$) peak at zero frequency in the susceptibility and power spectrum of the system [8,9]. Such a peak was observed earlier in analog simulations of a Brownian motion in a symmetric double-well potential both in the presence [10(a)] and absence [10(b),10(c)] of the external periodic field. It was also observed for a symmetric ($w_1=w_2=\frac{1}{2}$) bistable laser [11]. It is related immediately [12] to the so-called "stochastic resonance" [11,13], i.e., to the increase of the signal-to-noise ratio with increasing noise intensity in bistable systems. In the present paper the zero-frequency peak has been found, and its shape and intensity have been studied for a passive all-optical bistable device.

The particular system investigated here is a double-cavity membrane system (DCMS) shown in Fig. 1. It consists of a thin semiconductor film (membrane) separat-

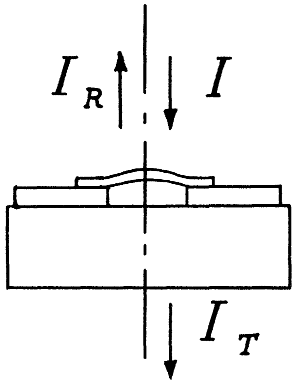

FIG. 1. Sketch of a double-cavity membrane system (DCMS).

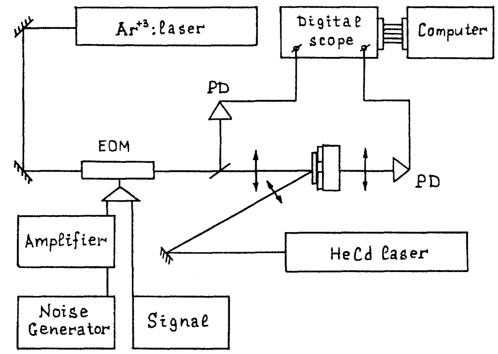

FIG. 2. The experimental setup. EOM is electro-optic modulator; PD is photodiode.

ed from a dielectric mirror with a metal diaphragm. The film itself is the first optical cavity, while the second cavity is an air-spaced gap between the film and the mirror. Laser radiation causes heating of the film. Because of thermal expansion the latter is bent. The bending changes the transmission of the second cavity and thus changes the heating itself. Such thermo-optical nonlinearity can give rise to bistability [14]. It is the smallness of the laser spot as compared with the diaphragm that provides the absence of the spatial effects in the system: the membrane bending is homogeneous within the spot to high accuracy.

Fluctuations in the device are caused by the fluctuations of the laser radiation that heats the membrane. The advantageous feature of the DCMS is the possibility of investigating, for a real physical system, the effects of different types of noisy modulation with easily varying characteristics of noise (intensity, power spectrum). In Sec. II the fluctuations resulting from modulation of the intensity of coherent incident radiation by almost white noise are analyzed. Statistical distribution of the transmitted-light intensity and the dependence of the transition probabilities on the noise intensity are found. The onset of the zero-frequency peak in the power spectrum of the transmitted-light intensity is observed and the shape of the peak is investigated. In Sec. III the effects of the modulation of the optically bistable DCMS by an additional radiation with fluctuating intensity are studied and peculiar features of the statistical distribution of the transmitted-light intensity are revealed. Section IV contains concluding remarks.

II. FLUCTUATION EFFECTS DUE TO NOISY MODULATION OF INCIDENT RADIATION

A. Experiment

The experimental setup is shown in Fig. 2. The membranes in the double-cavity systems investigated were single-crystal GaSe films with a thickness $\sim 1\ \mu$m, the diameter of the diaphragm was $\sim 500\ \mu$m, and an air-spaced gap between the film and the mirror was $\simeq 10\ \mu$m. The incident radiation of the Ar laser ($\lambda = 0.48\ \mu$m) was propagating normally to the mirror. The transmitted-radiation intensity I_T was registered by the oscilloscope. It was also discretized and 2048 instant values of I_T were digitized and recorded in a computer with an eight-bit precision. The bi- and multistability for a stabilized radiation was observed (cf. [14]). The example of a section of the input-output characteristic is shown in Fig. 3.

Two types of fluctuations can be investigated immediately with the aid of the setup shown in Fig. 2: fluctuations arising because of modulation of the Ar-laser output by external noise, i.e., because of noise in the radiation causing bistability itself; and fluctuations of the transmitted radiation arising because of modulating the DCMS by an additional light with randomly varying intensity (from an unstabilized He-Cd laser for the setup in Fig. 2). In both cases it is the transmitted radiation of the Ar laser that is registered. Its characteristics are shown below to be strongly different for the two types of noise. In the present section the first type of the noise is analyzed and the results refer to the case when the He-Cd laser is switched off.

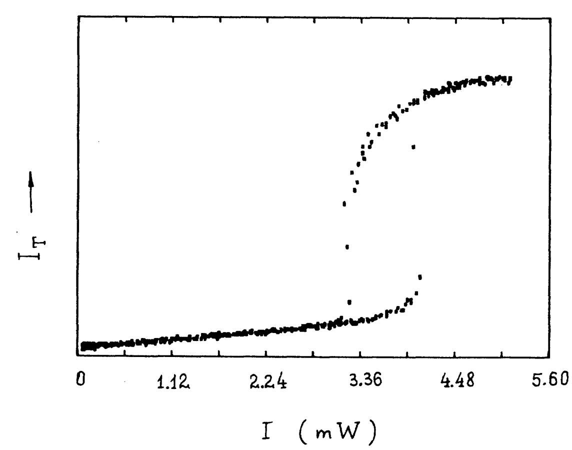

FIG. 3. Oscillogram of the input-output characteristic of the DCMS; I_T and I are the intensities of the transmitted and incident light, respectively (I_T is in arbitrary units).

The dynamics of the DCMS depends on thermoconductance of a membrane, its vibrational eigenfrequencies, damping, etc. (see Ref. [15] for a detailed discussion). For the devices investigated in the present paper the eigenvibrations of the membrane were overdamped because of high losses at its edges (the membrane was attached to the diaphragm with a grease, so that it could move in plane, but with a substantial viscous friction). The "slowest" dynamical time was that of thermal relaxation. It was found from the data on the switching of the DCMS to be ~ 1 msec. Simple estimates show (cf. Ref [16]) that for membranes with typical sizes given above the characteristic reciprocal eigenfrequencies are smaller by a factor of 10 at least, and, since oscillations have not been observed in the devices investigated (in contrast to those studied in Ref. [15]), one can conclude that bending of the film due to thermal expansion follows adiabatically the temperature distribution formed by heating within the laser spot and cooling via the membrane-air and membrane-diaphragm boundaries.

B. Dynamics of the DCMS

The transmission of a DCMS for a narrow (compared with the diaphragm) beam of light is determined by the phase gain ϕ for the light in the air-spaced gap between the membrane and the mirror,

$$I_T = IN(\phi) \ , \tag{1}$$

where I and I_T are the intensities of the incident and transmitted radiation, respectively. The expression for $N(\phi)$ has quite a simple form [15] if a DCMS is assumed a plane-parallel three-layer system with optically homogeneous layers and with the nonlinearity due to bending of the membrane. However, the semiconductor film and the dielectric mirror are obviously not strictly parallel because of the film bending, and this results in a more sophisticated form of the function $N(\phi)$ [and of $M(\phi)$ in (2), see below] than that obtained in Ref. [15].

To a reasonable approximation the "slow" kinetics of the system can be described by a Debye relaxation equation for the phase gain ϕ, which is linear in bending and thus follows adiabatically thermal relaxation of the film:

$$\dot{\phi} + \frac{1}{\tau}\Delta\phi = IM(\phi) \ , \qquad \phi = \phi_0 + \Delta\phi, \quad M(\phi) = M(\phi + 2\pi) \ . \tag{2}$$

Here τ is the relaxation time, ϕ_0 is the phase gain in the "dark," and $\Delta\phi$ is the change of ϕ due to the laser-induced bending. Relaxation in Eq. (2) is linearized in $\Delta\phi$, while the right-hand side is assumed proportional to the incident-radiation intensity I, i.e., proportional to the power absorbed in the film. This proportionality was checked experimentally by modulating I periodically in time at frequencies ω much higher than the reciprocal switching time τ^{-1} (the frequencies used were $\simeq 20$ kHz, while τ was ~ 1 msec). For $\omega\tau \gg 1$ the amplitude of the forced vibrations of ϕ is comparatively small and $M(\phi)$ in (2) can be set constant. Therefore the amplitude of ϕ would be expected to be proportional to the amplitude of oscillations of I. To reveal the small-amplitude vibrations of the membrane, the DCMS transmission of an additional trial radiation (that of the He-Ne laser) has been investigated, and the aforementioned proportionality has been seen to be fulfilled to an accuracy $\sim 10\%$ for actual amplitudes of I.

The form of the function $M(\phi)$ in (2) depends on the mechanism of thermal relaxation, boundary conditions at the edges of a film, etc. It can hardly be obtained explicitly; an approximate expression for $M(\phi)$ was found [15] on the basis of some variational analytic approaches developed in the theory of thermoelasticity of shells. Since thermal relaxation for small bending is practically independent of bending and the laser-induced heating obviously depends on ϕ periodically, the function $M(\phi)$ as a whole is assumed in (2) to be periodic in ϕ. Because of interference in the cavity $M(\phi)$ is strongly nonlinear, and on physical grounds it does not vanish (irradiation should always give rise to heating of the system).

C. Fluctuations of the phase of the DCMS driven by a multiplicative noise

Equations (1) and (2) make it possible to analyze fluctuations for the case of the DCMS driven by a randomly modulated incident radiation:

$$I \equiv I(t) = \bar{I} + \delta I(t), \quad \langle \delta I(t) \rangle = 0 \ , \qquad \langle \delta I(t)\delta I(t') \rangle = \frac{D}{\tau_0}\xi\left[\frac{|t-t'|}{\tau_0}\right], \quad \int_0^\infty \xi(x)dx = 1 \tag{3}$$

[the function $\xi(x)$ is assumed small for $x \gg 1$]. The experimental data below refer to the case when the noise $\delta I(t)$ is almost Gaussian [note, however, that $I(t) \ge 0$] and almost white, i.e., its power spectrum is flat up to frequencies exceeding τ^{-1} substantially (the range of flatness exceeded 40 kHz, while τ was ~ 1 msec). Respectively, the correlation time τ_0 within which the two-time correlator $\langle \delta I(t)\delta I(t') \rangle$ dies down is small compared with τ. The intensity D of the noise could be varied easily in the experiment.

Equations (2) and (3) describe the stochastic motion of the phase ϕ driven by a multiplicative noise. Allowing for the condition $M(\phi) \neq 0$, it is convenient following the standard procedure [17] to rewrite (2) and (3) in the form

$$\dot{x} + U'(x) = \delta I(t), \quad x = \int_0^{\phi(x)} d\phi\, M^{-1}(\phi) \ , \qquad U(x) \equiv U(x,\bar{I}) = -\bar{I}x + \frac{1}{\tau}\int_0^{\phi(x)} d\phi(\phi - \phi_0)M^{-2}(\phi) \ . \tag{4}$$

For the intensity of the incident light $\bar{I} \equiv \langle I(t) \rangle$ lying in the range of optical bistability, i.e., for $\bar{I}$ such that Eq. (2) with $I = \bar{I}$ has two stable solutions ϕ_1 and ϕ_2, the potential $U(x)$ obviously, from (2) and (4), has two minima. Their positions x_1 and x_2 are given by $x_i = x(\phi_i)$, $i = 1,2$. The character of the random motion of the system depends substantially on the ratio of the depths of the corresponding potential wells to the noise intensity. In what follows we consider the most interesting case of "deep" wells or, equivalently, "weak" noise

$$\Delta U_i \gg D, \quad \Delta U_i = U(x_s) - U(x_i), \quad i=1,2 \tag{5}$$

[x_s is the position of the local maximum of the potential that lies between x_1 and x_2; it corresponds to the unstable stationary solution of Eq. (2) for $I=\bar{I}$].

If (5) is fulfilled, the probability W_{ij} of an $i \to j$ transition is a well-defined quantity. The transition probabilities are exponentially small, $W_{ij} \propto \exp(-\Delta U_i/D)$ [18] (see also [8,19] for the case of Gaussian noise with arbitrary power spectrum),

$$W_{ij} \ll \tau^{-1} . \tag{6}$$

The stationary distribution $p(\phi)$ over the phase for the system (1)–(4) is given by

$$p(\phi) \equiv p(\phi,\bar{I}) = Z^{-1} M^{-1}(\phi) \exp[-U(x(\phi),\bar{I})/D] ,$$
$$Z = \int_{-\infty}^{+\infty} dx \exp[-U(x,\bar{I})/D] . \tag{7}$$

The distribution (7) has sharp maxima at the values ϕ_1 and ϕ_2 of the phase in the stable states, so that one can say about the populations $w_{1,2}$ of the stable states 1,2 separately

$$w_i \simeq Z^{-1}[2\pi D/U''(x_i)]^{1/2} \exp[-U(x_i)/D] , \quad i=1,2$$
$$w_1/w_2 = W_{21}/W_{12} \propto \exp\{[U(x_2)-U(x_1)]/D\} . \tag{8}$$

For small D the ratio w_1/w_2 is seen from (5) and (8) to be exponentially large or small for almost all values of the average intensity $\bar{I}$ of the incident light that determines the form of the potential $U(x)$. Only in a narrow range $|\bar{I}-I_c| \ll I_c$, where I_c is the solution of the equation

$$U(x_1,I_c) = U(x_2,I_c) \tag{9}$$

are w_1 and w_2 of the same order of magnitude. As it has been explained, this is the range of a "kinetic" phase transition.

D. Discussion of the experimental results

1. Statistical distribution

The statistical distribution $P(I_T)$ of the transmitted-light intensity obtained experimentally for a noise-modulated incident radiation is shown in Fig. 4. The value of $\bar{I} = \langle I \rangle$ was chosen in such a way that it lay in the range of bistable transmission of the DCMS (cf. Fig. 3) and was close to the critical value I_c given by (9). Therefore, in agreement with (8), there are two prominent peaks in $P(I_T)$, and their areas are close in order of magnitude. To describe $P(I_T)$ we note that the fluctuations of the incident light influence the transmitted light both "directly," i.e., irrespective of the change of the transmission of the DCMS, and via fluctuations of the latter. The respective contributions to the fluctuations of I_T are characterized by the correlation times τ_0 and τ, which are substantially different. For $\tau_0 \ll \tau \ll W_{ij}^{-1}$ the distribution $P(I_T)$ is formed over three stages. Within a time $\sim\tau_0$ there is formed the distribution for a given value of the transmission $N(\phi)$ [see Eq. (1)]; it reproduces the distribution $P_{\text{in}}(I)$ of the incident-light intensity $P(I_T|\phi) = N^{-1}(\phi) P_{\text{in}}(I_T N^{-1}(\phi))$. Within a time $\sim\tau$ this distribution becomes smeared over the values of ϕ close to ϕ_i, with i corresponding to the stable state occupied initially. Finally, over the time $\sim W_{ij}^{-1}$ the smearing over both ranges of ϕ, those close to ϕ_1 and ϕ_2, becomes substantial, and the distribution takes on the form of the convolution of $P(I_T|\phi)$ and $p(\phi)$ (7). Assuming $P_{in}(I)$ Gaussian we arrive at the expression

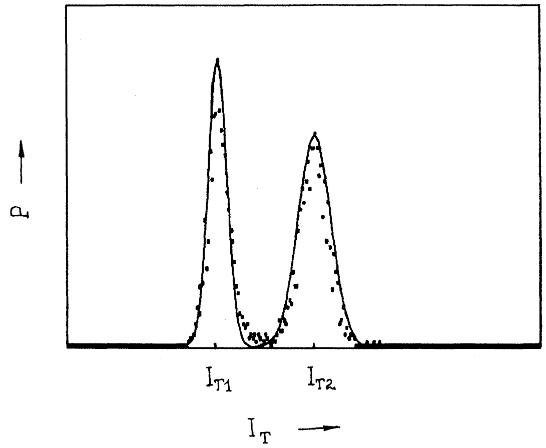

FIG. 4. Statistical distribution $P(I_T)$ of the transmitted-light intensity I_T measured (data points) and calculated (solid curve) for a noise-modulated incident radiation.

$$P(I_T) = \frac{1}{\sqrt{\pi\sigma}} \int d\phi\, p(\phi) N^{-1}(\phi) \times \exp\left[-\frac{1}{\sigma}[I_T N^{-1}(\phi) - \bar{I}]^2\right] , \tag{10}$$
$$\sigma = 2D\tau_0^{-1}\xi(0) .$$

For small noise intensities D [see (5)] the factor $p(\phi)$ as given by (7) has two sharp maxima at $\phi=\phi_1$ and ϕ_2 with the half widths $\sim(D/\tau)^{1/2}\tau M(\phi_i)[1-\bar{I}\tau M'(\phi_i)]^{-1/2}$ ($i=1,2$). The second integral in (10) is much more smooth, since $\sigma \propto D/\tau_0 \gg D/\tau$ (in the limit of the white-noise modulation $\sigma \to \infty$). Therefore, to a good approximation

$$P(I_T) = \sum_{i=1,2} w_i(\pi\sigma_i)^{-1/2} \exp\left[-\frac{1}{\sigma_i}(I_T - I_{Ti})^2\right] ,$$
$$\sigma_i = \sigma N^2(\phi_i), \quad I_{Ti} = \bar{I} N(\phi_i) . \tag{11}$$

The distribution (11) has two peaks lying at $I_T = I_{T1,2}$, i.e., at the stable values of I_T in the absence of noisy modulation of the incident radiation. The areas of the peaks at I_{T1} and I_{T2} are equal to w_1 and w_2, respectively; their characteristic widths are given by $N(\phi_{1,2})\sqrt{\sigma}$, and the ratio of the widths to the positions of the maxima is $\sqrt{\sigma}/\bar{I}$ and equals that for the peak in the distribution $P_{\text{in}}(I)$ of the incident radiation. This makes it possible to describe the observed distribution $P(I_T)$ in Fig. 4 quantitatively basing upon the independent experimental data for $I_{T1,2}$, $\sqrt{\sigma}/\bar{I}$, and w_1/w_2. The latter ratio is given [see (8)] by the ratio of the transition probabilities which has been obtained immediately (see Sec. II D 2). The agreement between the corresponding simple calculations and the immediate data on $P(I_T)$ is seen from Fig. 4 to be good. We stress that the theory does not contain any adjustable parameter, and the above relations are valid for any particular form of the transmission $N(\phi)$. The expression (11) is based, in essence, on two assumptions only, namely, that (i) the shape of the distribution of the incident-radiation intensity is Gaussian, and (ii) the ratio τ_0/τ is small.

2. *Transition probabilities*

The sequence of the discretized values of the transmitted-light intensity $I_T(t)$ that displays explicitly the fluctuations and the fluctuational transitions in particular is shown in Fig. 5(a). The interval Δt between sequential records of $I_T(t)$ [sequential points in Fig. 5(a)] was chosen in such a way that

$$W_{ij}^{-1} \gg \Delta t \gg \tau \gg \tau_0 \qquad (12)$$

[for the data in Fig. 5(a) $\tau \simeq 1$ msec, $W_{ij}^{-1} \simeq 1$ sec, $\tau_0 \simeq 10$

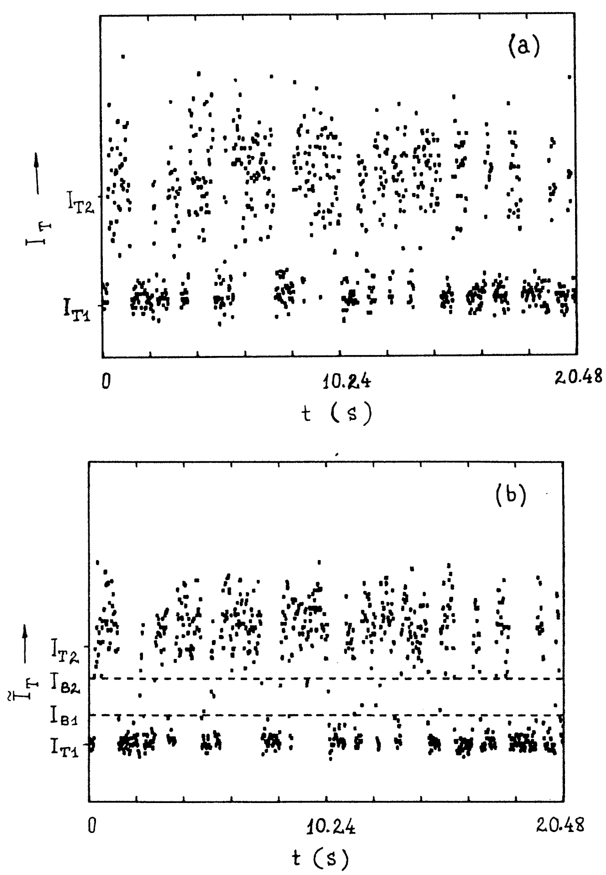

FIG. 5. (a) Sequential values of the transmitted-light intensity $I_T(n\Delta t)$, $\Delta t = 20$ msec, for a noise-modulated incident radiation; (b) "filtered" intensity $\tilde{I}_T(n\Delta t)$. The filtration time $\tau_f = 60$ msec. The "boundaries" $I_{B1,2}$ are shown dotted.

μsec, and $\Delta t = 20$ msec]. The condition (12) means that the sequential $I_T(t_n)(t_n - t_{n-1} \equiv \Delta t)$ refer with the overwhelming probability to the same stable state of the DCMS: the probabilities of transitions between the stable states within the interval Δt are $W_{ij}\Delta t \ll 1$. Therefore sequential $I_T(t_n)$ would be expected to lie mostly in the vicinity of the same value I_{Ti} of the transmitted-light intensity ($i = 1,2$). This is seen from Fig. 5(a) to be true. However, there is a relatively large amount of sequential values of I_T that differ noticeably one from another and could be related to different stable states. As a rule, such "outbursts" have no "postaction," i.e., if several sequential $I_T(t_{n-k})$ ($k \geq 0$) lie close to I_{T1}, e.g., and $I_T(t_{n+1})$ lies rather far from I_{T1}, then the values of $I_T(t_{n+k})$ for several $k > 1$ will lie most probably in the vicinity of I_{T1} again. It is natural to associate such outbursts of I_T with the short outbursts of the incident-radiation intensity $I(t)$ which do not result in the transitions of the DCMS's phase ϕ from one stable state to another, but form the wings of the relatively broad peaks of the distribution $P(I_T)$ (11). If for a given Δt the relative areas of these wings taken conventionally for each peak over the range of the maximum of the other peak exceed substantially $W_{ij}\Delta t$, the probabilities of the above "short" outbursts of I_T ("false" transitions) are obvious to exceed these of the "true" transitions.

The true transitions can be singled out by coarse-graining the data in Fig. 5(a) over time. This can be done, e.g., by changing form $I_T(t_n)$ to the "filtered" quantity

$$\tilde{I}_T(t_n) = (1-q) \sum_{m=-\infty}^{n} q^{n-m} I_T(t_m) \ . \qquad (13)$$

The procedure (13) is a discrete analog of a standard filtration described by the equation $d\tilde{I}_T/dt + \tilde{I}_T/\tau_f = I_T/\tau_f$ with the time constant of the filtration

$$\tau_f = \Delta t / |\ln q| \ . \qquad (14)$$

The filtration (13) averages out the registered transmitted-light intensity and thus reduces the straggling of the data points. As one can easily see for $\tau_f \gg \tau_0$ the distribution of $\tilde{I}_T$ is given by Eq. (11) with σ having been replaced by $\sigma(1-q)/(1+q)$. The peaks of this distribution are therefore narrower than those of (11) and the probabilities of large, short outbursts of $\tilde{I}_T$ are exponentially smaller than those of I_T; e.g., the probability density of reaching the value I_{T2} when the DCMS's phase ϕ occupies the state 1 is proportional to $\exp[-(I_{T2}-I_{T1})^2/\sigma_1]$ for I_T and $\exp[-(I_{T2}-I_{T1})^2(1+q)/(1-q)\sigma_1]$ for $\tilde{I}_T$.

To determine the transition probabilities we introduce some auxiliary boundaries I_{B1} and I_{B2} of the ranges that enclose the values I_{T1} and I_{T2} and the ranges of the small fluctuations about I_{T1} and I_{T2} [in what follows we assume $I_{T1} < I_{B1} < I_{B2} < I_{T2}$; see Fig. 5(b)]. We suppose that one transition $1 \rightarrow 2$ has occurred within the interval $t_m < t < t_n$, if $\tilde{I}_T(t_n) \geq I_{B2}$, $\tilde{I}_T(t_m) \leq I_{B1}$, and $I_{B1} < \tilde{I}_T(t_k) < I_{B2}$ for $t_m < t_k < t_n$. The transition $2 \rightarrow 1$ is defined similarly. The above procedure is reasonable provided the resulting values of the transition probabilities W_{ij} are independent both of the boundaries I_{B2} and I_{B1} and of the filtration time τ_f (14) in quite broad ranges of these quantities.

The limitations on I_{B2} and I_{B1} follow from the condition that they should obviously lie in the gap between the ranges of the small fluctuations of I_T about I_{T1} and I_{T2}. The lower limit on τ_f (obviously $\tau_f > \Delta t$) is imposed by the effectiveness of the filtration: the probability of a false transition of I_T in the course of a short outburst of noise should be small compared to that of a true transition, i.e., of a transition of the phase ϕ from one state to the other. The upper limit on τ_f is imposed by the condition that the time necessary for $\tilde{I}(t)$ to cross the boundary I_{Bi} $(i=1,2)$ after the phase has changed from the state 3-i to the state i should be small as compared with W_{ij}^{-1}. Allowing for (13) and (14), this means that

$$W_{ij}\tau_f \ll 1/\ln|(I_{T1}-I_{T2})/(I_{Ti}-I_{Bi})| \tag{15}$$

(we see, in particular, that the limits on τ_f and $I_{B1,2}$ are interrelated).

The existence of the range of τ_f and $I_{B1,2}$, where W_{ij} as determined following the above procedure are independent of these quantities, has been shown for the DCMS experimentally. The values of the transition probabilities W_{12} and W_{21} were obtained from the relation $W_{ij}=\langle T_i\rangle^{-1}$ where $\langle T_i\rangle$ was the average time the system spent in the state i. The dependence of W_{12} on the noise intensity D is shown in Fig. 6 (the data refer to the phase-transition range where $W_{12}\simeq W_{21}$). As it has been expected this dependence is of the activation type to a good accuracy.

3. *Zero-frequency peak in the power spectrum*

The very general feature of fluctuating bistable systems is the onset of narrow zero-frequency spectral peaks for the parameter values lying in the range of the kinetic phase transition where the stationary populations w_1 and w_2 of the stable states are close in order of magnitude [8,9]. To reveal such a peak we have investigated the power spectrum of the intensity of the light transmitted by the DCMS,

$$Q(\omega)=\frac{1}{4\pi t_0}\left|\int_{-t_0}^{t_0} dt\, e^{i\omega t}[I_T(t)-\langle I_T\rangle]\right|^2 \quad (t_0 \gg W_{ij}^{-1},\tau,\tau_0)\ . \tag{16}$$

At weak noise for almost all values of the average incident-light intensity $\bar{I}$, the spectrum $Q(\omega)$ was practically flat for frequencies $\omega/2\pi \le 10$ kHz. A narrow peak at zero frequency arose indeed only in a narrow interval of $\bar{I}$ where the transition probabilities W_{12} and W_{21} were close to each other. It is shown in Fig. 7(a).

The shape of the zero-frequency peak is insensitive to particular details of the kinetics of the system and the characteristics of noise (in particular, its power spectrum). If noise is sufficiently weak, the peak is described by the expression

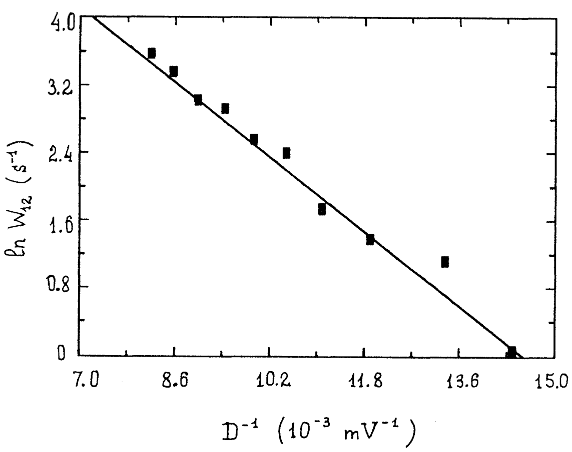

FIG. 6. Logarithm of the transition probability W_{12} vs the reciprocal noise intensity D^{-1} for noise-modulated incident radiation in the range of the kinetic phase transition. The solid straight line is the result of the method of least squares.

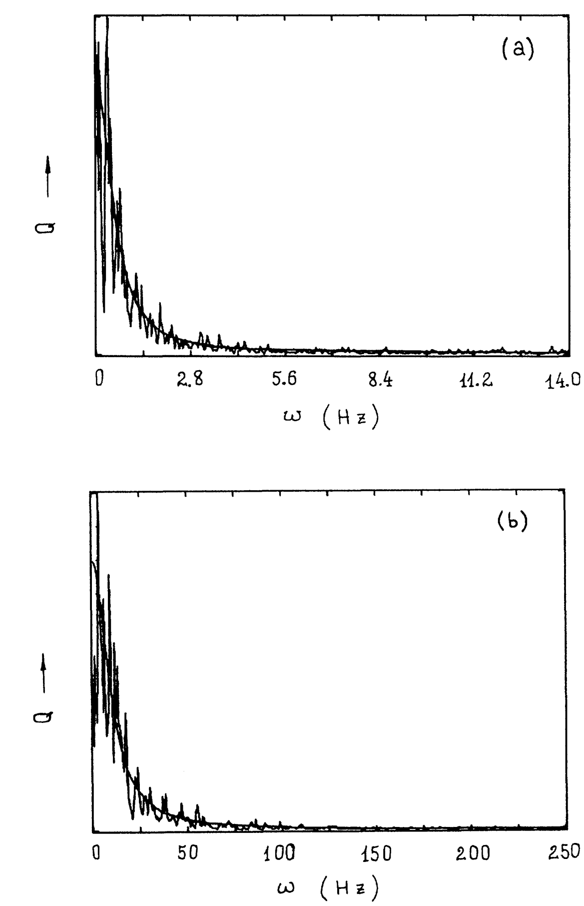

FIG. 7. Low-frequency power spectrum of the transmitted-radiation intensity in the kinetic phase transition range with a distinct zero-frequency peak for (a) noise-driven incident radiation, (b) DCMS driven by an additional fluctuating light beam. Solid lines correspond to Eq. (17) [with an added background in (a)].

$$Q_0(\omega)=\frac{w_1 w_2}{\pi}(I_{T1}-I_{T2})^2 \times (W_{12}+W_{21})/[\omega^2+(W_{12}+W_{21})^2] \ . \quad (17)$$

It is Lorentzian in shape. Its half width equals to the sum of the transition probabilities. Its integral intensity is seen from (8) to be proportional to $\exp[-|U(x_1,\bar{I})-U(x_2,\bar{I})|/D]$ and thus it depends exponentially sharply on the difference between the given $\bar{I}$ and the critical value I_c (9) corresponding to the phase-transition point (similar exponential dependence was observed in Ref. [20] for a "high-frequency replica" [8] of the zero-frequency peak in the periodically driven bistable system).

Equation (17) has been shown to describe the observed zero-frequency peak quite well (see Fig. 7). Note that there are no adjustable parameters in the theory, since the values of W_{12} and W_{21}, and I_{T1} and I_{T2} have been determined from other experiments.

We note in concluding this section that the advantage of the above way of driving optically bistable systems by noise is the possibility of varying easily the power spectrum and intensity of noise while retaining the other characteristics constant. The detailed data on the effects of colored noise will be given elsewhere. Here we mention only that in agreement with the above analysis the change in the power spectrum that did not break the condition $\tau_0 \ll \tau$ influenced immediately the transition probabilities only. The statistical distribution $P(I_T)$ and the zero-frequency peak were influenced only via the change of W_{ij}.

III. EFFECTS OF MODULATION OF THE TRANSMISSION BY AN ADDITIONAL LIGHT BEAM WITH RANDOMLY VARYING INTENSITY

In the present section we analyze the effects of modulation of an optically bistable DCMS by radiation from an additional (to that causing bistability) unstabilized laser (the He-Cd laser in Fig. 2). The latter caused fluctuating heating of the semiconductor film. The intracavity interference of the modulating radiation was negligible because the beam was inclined with respect to the cavity axis. Therefore heating was proportional to the modulating-light intensity $I_m(t)$.

It follows from the above model of the dynamics of the DCMS that the transmission of the coherent radiation of the Ar laser in the presence of the modulating radiation is described by Eq. (1), while Eq. (2) for the phase gain in the DCMS takes the form

$$\dot{\phi}+\frac{1}{\tau}\Delta\phi=IM(\phi)+I_m(t) \quad (18)$$

[we assume $I_m(t)$ to be properly normalized].

The direct measurements of $I_m(t)$ showed its spectral density of fluctuations to be flat up to frequencies $\simeq 20$ kHz. Therefore in (18) we set the function $I_m(t)$ equal to the sum of the regular term $\bar{I}_m$ and the white Gaussian noise $\delta I_m(t)$:

$$I_m(t)=\bar{I}_m+\delta I_m(t), \quad \langle \delta I_m(t)\rangle=0 \ , \quad \langle \delta I_m(t)\delta I_m(t')\rangle=2D_m\delta(t-t') \ . \quad (19)$$

We note that $I_m(t)$ as a whole could be changed with the aid of neutral filters, but the noise intensity D_m and $\bar{I}_m$ are interrelated and could not be varied independently; to a good accuracy $D_m \propto \bar{I}_m^2$.

Just as in the case of noisy modulation of the coherent incident radiation, in the present case of noisy modulation of the DCMS by additional radiation both the fluctuations of the transmitted-light intensity I_T about its stable values and the fluctuational transitions have been observed. They are seen from Fig. 8 where a sample of the sequential data recorded with the time interval $\Delta t = 10$ msec is shown. These data made it possible to find immediately the stationary distribution $P(I_T)$ of the transmitted-light intensity and to determine the transition probabilities. The evolution of the obtained distribution $P(I_T)$ with the varying intensity I of the coherent incident light for a given $\bar{I}_m$ is seen from Fig. 9(a). It is two peaked in the range of optical bistability. The specific feature obvious from Fig. 9(a) is strong asymmetry of the peaks.

A. Theory of the shape of the stationary distribution

The noise under consideration is additive for the phase of the DCMS [see (18) and (19)], and in contrast to the case considered in Sec. II and described by Eqs. (2) and (3) the fluctuations of I_T are due to the phase fluctuations only. The statistical distribution of the phase $p_m(\phi)$ is of the form

$$p_m(\phi)=Z_m^{-1}\exp[-U_m(\phi)/D_m],$$
$$Z_m=\int d\phi \exp\left[-\frac{U_m(\phi)}{D_m}\right], \quad (20)$$
$$U_m(\phi)=-I\int_0^{\phi} d\phi' M(\phi')+\frac{1}{2\tau}\phi^2-\frac{1}{\tau}(\phi_0+\bar{I}_m\tau)\phi$$

(we note that just as for the potential $U(x(\phi))$ (4), the

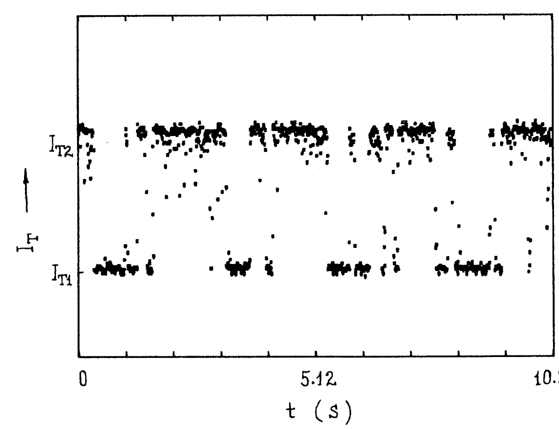

FIG. 8. Sequential values of the transmitted-light intensity $I_T(n\Delta t)$, $\Delta t = 10$ msec, for the DCMS driven by an additional fluctuating light beam.

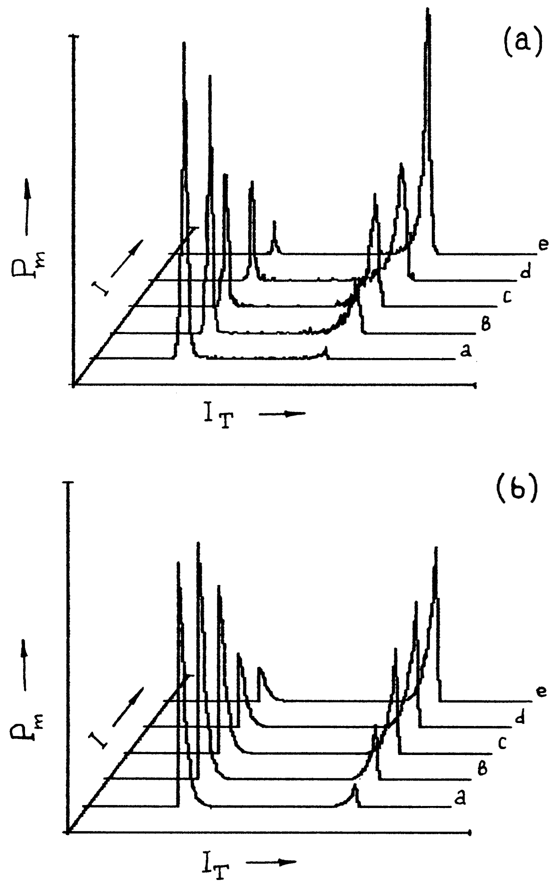

FIG. 9. (a) Experimentally measured statistical distribution $P_m(I_T)$ for the DCMS driven by an additional fluctuating light beam for different values of the intensity of the incident light (in arbitrary units): $I=8.4$, 8.6, 8.8, 9.0, 9.2 for the curves $a-e$. (b) $P_m(I_T)$ for a Fabry-Pérot cavity described by Eqs. (1), (21), and (26) (the finesse $F=0.9$) for different values of the parameter I in Eq. (18): $I=2.56$, 2.57, 2.58, 2.59, 2.60 for the curves $a-e$.

potential $U_m(\phi)$ is proportional to ϕ^2 for $|\phi|\gg 1$, therefore $\exp[-U_m(\phi)/D_m]$ falls down rapidly for the large $|\phi|$). In the range of optical bistability the potential $U_m(\phi)$ has two minima lying at $\phi=\phi_{1,2}$ where $\phi_{1,2}$ are the stable solutions of Eq. (18) with $I_m(t)=\bar{I}_m$.

The quantity investigated experimentally is not $p_m(\phi)$ but the distribution of the transmitted-light intensity $P_m(I_T)$. It is substantial for the noisy modulation described by (18) and (19) that $P_m(I_T)$ as determined from the experiment [and equal by definition to the relative number of the discretized $I_T(t)$ lying between $I_T-\Delta I_T$ and $I_T+\Delta I_T$] is the coarse-grained "bare" distribution $P(I_T)$:

$$P_m(I_T)=\frac{1}{2\Delta I_T}\int_{I_T-\Delta I_T}^{I_T+\Delta I_T} P(I_T')dI_T' \ , \tag{21}$$

where

$$P(I_T)=K(I_T)\exp[-V(I_T)/D_m] \ , \quad V(I_T)=U_m(\phi(I_T))$$
$$K(I_T)=I_T^{-1}Z_m^{-1}\left|\frac{d\ln N(\phi)}{d\phi}\right|^{-1}_{\phi=\phi(I_T)} \ , \quad I_T=IN(\phi(I_T)) \ . \tag{22}$$

Both the exponential and the prefactor in (22) are singular. Since on physical grounds the function $N(\phi)$ is periodic, there are at least two extreme points $\phi=\phi^{(e)}$ in each 2π interval of ϕ where $dN/d\phi$ vanishes. The function $K(I_T)$ diverges for $I_T=I_T^{(e)}\equiv IN(\phi^{(e)})$ corresponding to the extrema of $N(\phi)$. In the general case of $N(\phi)$ parabolic near the extrema $K(I_T)\propto|I_T-I_T^{(e)}|^{-1/2}$. Obviously, $K(I_T)$ vanishes identically for I_T lying outside the interval between the minimum and maximum of $IN(\phi)$. (We note that both for standard models of a bistable Fabry-Pérot cavity [6] (see also below) and for the model of the DCMS as a plane-parallel three-layer system [15] the function $N(\phi)$ has only two extrema, the absolute minimum and maximum, within a period.)

The values $I_T=I_T^{(e)}$ are also singular (branch) points of the potential $V(I_T)$. As a whole $V(I_T)$ is a multibranch function since the ϕ- axis is mapped by the relation $I_T=IN(\phi)$ onto the finite interval between the minimum and maximum values of $I_T^{(e)}$. The "sequential branches" of $V(I_T)$ correspond to the sections of $U_m(\phi)$ for successive intervals of ϕ. For the optically bistable DCMS $V(I_T)$ has two minima and one local maximum. We stress that the extrema can lie on different branches of $V(I_T)$ (see Fig. 10).

The distribution $P(I_T)$ for sufficiently small noise intensity D_m has pronounced maxima for $I_T=I_{T1},I_{T2}$, corresponding to the minima of $V(I_T)$, and it also has singularities for $I_T=I_T^{(e)}$. The structure of the physically observable coarse-grained distribution $P_m(I_T)$ (21) depends on the interrelation between the interval of the coarse-graining ΔI_T, the distances $|I_T^{(e)}-I_{Ti}|$ $(i=1,2)$ and $\sqrt{D_m}$. The most simple structure arises for

$$\Delta I_T \ll \sqrt{D_m} \ll I_{T2}-I_{T1}, |I_{T1,2}-I_{Tk}^{(e)}| \tag{23}$$

(the subscript k in $I_{Tk}^{(e)}$ numbers the different values of $I_T^{(e)}$). In the case (23) $P_m(I_T)$ has intense peaks at I_{T1} and/or I_{T2}. The shape of the ith peak near the maximum is Gaussian [cf. Eq. (11)] with the characteristic width

$$\sigma_i=2D_m\left[\frac{d^2V}{dI_T^2}\right]^{-1}_{I_{Ti}} \ . \tag{24}$$

The expression (24) is simplified in the important case when in (1) and (2) $M(\phi)=\alpha N(\phi)$ (α=const), i.e., when the power absorbed in the semiconductor film is simply proportional to the transmitted-light intensity:

$$\sigma_i=\alpha^2\tau\rho_i(\rho_i-1)^{-2}, \quad \rho_i=\left[\frac{d\ln I_T}{d\ln I}\right]_{I_{Ti}} ,$$
$$M(\phi)=\alpha N(\phi) \ . \tag{25}$$

Here ρ_i is determined by the transmission in the absence of noise, i.e., by the input-output characteristic $I_T(I)$ of

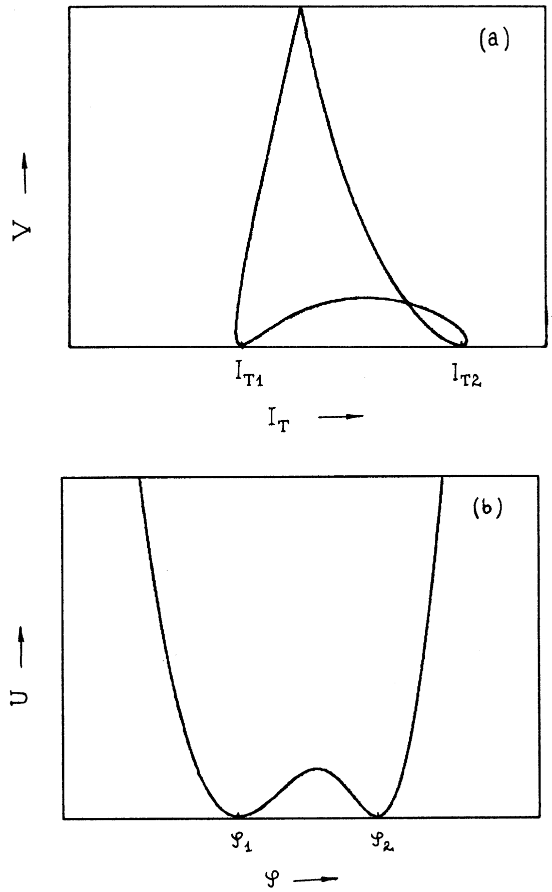

FIG. 10. (a) Multibranch effective potential $V(I_T)$ for the distribution of the intensity of light transmitted by a nonlinear Fabry-Pérot cavity described by Eqs. (1), (21), and (26). (b) The corresponding effective potential $U_m(\phi)$ for the distribution over the phase.

the DCMS, and can be thus measured in the independent experiment [although this is not trivial in view of the shift of the input-output characteristic due to the finite average heating of the film ($\propto \bar{I}_m$) by the noisy radiation].

In addition to the Gaussian peak (or two peaks) at I_{Ti} with the area close to unity, there arise also strongly asymmetric peaks in $P_m(I_T)$ at $I_{Tk}^{(e)}$ with the shapes described by the function $\Theta(x+\Delta)\sqrt{x+\Delta} - \Theta(x-\Delta)\sqrt{x-\Delta}$, where $\Delta=\Delta I_T$ and $x=\pm[I_T-I_{Tk}^{(e)}]$ [the signs $+$ and $-$ refer to the peaks that correspond to the minima and maxima of $N(\phi)$, respectively]. The heights of these peaks are proportional to the products of the large factor $(\Delta I_T)^{-1/2}$ and the exponentially small factors proportional to $\exp\{-[V(I_{Tk}^{(e)})-V_{\min}]/D_m\}$, where $V_{\min}=\min(V(I_{T1}), V(I_{T2}))$, and their areas are small.

If $D_m^{1/2}$ exceeds or is of the order of $|I_{T1,2}-I_{Tk}^{(e)}|$, the peaks of the two types overlap and the resulting distribution $P_m(I_T)$ has one distinct peak (or two distinct peaks if the system is in the range of the kinetic phase transition). For high-quality bistable cavities the values of I_{T1} and I_{T2} often lie close to $I_T^{(\min)}$ and $I_T^{(\max)}$, respectively ($I_T^{(\min)}$ and $I_T^{(\max)}$ are the smallest and the largest values of $I_T^{(e)}$), and these are the respective peaks of $P_m(I_T)$ which would be expected to merge pairwise for not too small noise intensities D_m. The resulting peaks would be noticeably asymmetric, with their "external" wings, i.e., those on the side of the nearest edge of the distribution [$P_m(I_T)=0$ for $I_T < I_T^{(\min)}-\Delta I_T$ or $I_T > I_T^{(\max)}+\Delta I_T$], being much more steep than the "internal" ones.

B. Discussion of results

The above qualitative picture agrees with the experimental results displayed in Fig. 9(a). We have observed two distinct peaks in $P(I_T)$. The ratio of their intensities depended crucially on the intensity of the coherent incident radiation I: for relatively small I the peak near I_{T1} was dominating, while for relatively large I, vice versa, the dominating peak was that near I_{T2}. In the intermediate range (it was not too narrow because the noise intensity D_m was not very small) both peaks were seen well. The external wings of the peaks are very steep indeed, and the peaks corresponding to the singularities of $K(I_T)$ are not resolved.

It is seen from Fig. 9(a) that the widths of the peaks of $P_m(I_T)$ corresponding to higher transmission exceed these of the lower-transmission peaks by several times. This is in qualitative agreement with the expression (25), since for the actual input-output characteristic ρ_2 was noticeably higher than ρ_1 (both ρ_1 and ρ_2 were less than 1 for almost all I). Also in qualitative agreement with Eq. (25) the widths of the peaks increased with I approaching the value where the corresponding stable state (and the peak itself) disappeared.

Since we do not have a quantitative model of the DCMS, we could not give a detailed theoretical description of the curves in Fig. 9(a). To illustrate the evolution of $P_m(I_T)$ the calculations have been performed for a model nonlinear Fabry-Pérot cavity. Such calculations are interesting in particular because of their immediate relevance to many actual optically bistable devices. In the model used the nonlinearity of the Fabry-Pérot cavity has been assumed to result from the linear dependence of the intracavity reflection on light intensity, and relaxation of the intracavity phase gain has been described by the Debye equation (2). The functions $M(\phi)$ and $N(\phi)$ in (1) and (2) for such a model take the form [6]

$$M(\phi)=A_M[1+F\sin^2(\phi/2)]^{-1},$$
$$N(\phi)=A_N[1+F\sin^2(\phi/2)]^{-1}, \qquad (26)$$

where F is finesse.

The results obtained for the model (26) are shown in Fig. 9(b). The multibranch potential is shown in Fig. 10(a). The theoretical and experimental curves in Fig. 9 are obvious to be very similar in structure, and the evolutions of the theoretical and experimental distributions are similar as well. This demonstrates the universality of the observed qualitative feature of the statistical distribution of I_T for noise-driven bistable cavities.

It is obvious that the activation energies ΔU_{m1} and

ΔU_{m2} of the transitions $1 \rightarrow 2$ and $2 \rightarrow 1$ (the states 1 and 2 correspond to the lower and higher transmission) would decrease and increase with the incident-light intensity I, respectively, and tend to zero for I approaching the upper and lower boundaries of the range of bistability. These arguments are confirmed by the data on $\ln W_{12}$ and $\ln W_{21}$ shown in Fig. 11.

The activation energy ΔU_{mi} is related directly to the ratio of the stationary distribution over the phase $p_m(\phi)$ in its local minimum ϕ_s to $p_m(\phi_i)$. It is the feature of the branching potential $V(I_T)$ that the latter is not the case for $P_m(I_T)$: the logarithm of the ratio of $P_m(I_{Ti})$ to $P_m(I_T)$ in its local minimum is much less than the logarithm of W_{ij}^{-1}. The former logarithm is not equal to $\Delta U_{mi}/D_m$, but approximately equals $[V(I_T^{(cr)}) - V(I_{Ti})]/D_m \ll \Delta U_{mi}/D_m$, where $I_T^{(cr)}$ is the value of the transmitted-light intensity for which the lowest branches of the potential $V(I_T)$ cross one another [see Fig. 10(a)].

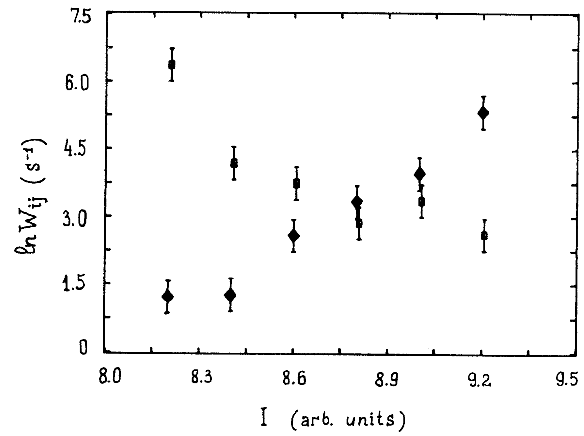

FIG. 11. Measured values of $\ln W_{12}$ (solid squares) and $\ln W_{21}$ (open squares) vs the incident-light intensity I.

IV. CONCLUSION

It follows from the above results that the double-cavity membrane systems provide an opportunity of the experimental investigation of noise-driven dynamics in real physical objects: passive optically bistable systems. The effects of various types of noise with easily varying intensity and power spectrum can be studied here. Such universal bistable systems phenomena as fluctuational transitions between the stable states, double-peak statistical distribution, and the onset of zero-frequency peak in the power spectrum of the transmitted-light intensity in the range of the kinetic phase transition cannot only be observed, but can also be fully investigated. On the other hand, the investigation of fluctuation phenomena and the comparison with theory have demonstrated the possibility of gaining a deeper insight into the underlying physics of the system and revealed a lot of specific features (and sometimes quite unusual ones, such as, e.g., the multibranch "potential" for the statistical distribution) of fluctuations under optical bistability.

We note in conclusion that, for passive optically bistable devices, it is their "internal" variables that fluctuate because of the external noise, while the quantities investigated directly are the characteristics of the transmitted radiation. It has been demonstrated above, however, that there exists a set of complementary experiments which makes it possible to avoid ambiguity and to reveal the characteristics of the system itself.

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