Driving systems with chaotic signals

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We generalize the idea of driving a stable system to the situation when the drive signal is chaotic. This leads to the concept of conditional Lyapunov exponents and also generalizes the usual criteria of the linear stability theorem. We show that driving with chaotic signals can be done in a robust fashion, rather insensitive to changes in system parameters. The calculation of the stability criteria leads naturally to an estimate for the convergence of the driven system to its stable state. We focus on a homogeneous driving situation that leads to the construction of synchronized chaotic subsystems. We apply these ideas to the Lorenz and Rössler systems, as well as to an electronic circuit and its numerical model.

I. INTRODUCTION

The idea of driving a system with a periodic signal is a common theme in nonlinear dynamics [1–5]. It is also the first choice for a drive or timing signal in designing devices. The idea of using a chaotic signal to drive a nonlinear system is rather new and only a few studies of this situation have been attempted. Even so, these early results suggest that there are different, interesting, and useful things to do with a chaotic drive signal. There are also indications that there are times when a chaotic signal would be preferable to a periodic signal as a drive or timing signal in applications.

Recent work by Hubler [6–8] shows that aperiodic signals can be used for various purposes. Nonlinear resonancelike behavior can be induced in the driven system using aperiodic driving derived from a previous transient response, which is then run backwards into the system. This leads to the interesting possibility of using chaos to stimulate particular modes or probe the parameters in a nonlinear system.

We recently showed [9,10] that one can begin with an autonomous chaotic system and construct a set of chaotic systems that will have their common signals synchronized. We develop those ideas further in this paper. We introduce the general theory of driven stable subsystems with the conditional Lyapunov exponent as the test for stability. An estimate for the convergence rates of a driven system to the final attractor falls out of this approach. This requires the use of the eigenvectors of the principal matrix solutions for the variational problem.

We also show the origin of structural stability for synchronization. We demonstrate with particular examples how one can apply these ideas to the construction of homogeneous or heterogeneous driven systems. Along with synchronization of chaotic systems several interesting features of chaotically driven systems emerge.

We note here that the idea of synchronization of chaotic subsystems has also surfaced in studies of coupled systems, as contrasted with driven systems which we report on here. Kaneko [11] has shown that locally coupled maps can display spatial domains of coherence (which is synchronization). Coupling of two phase-locked loops can result in the synchronization of their chaotic behavior, which should be exhibited by certain electrical circuits [12]. Models of laser systems suggest that they can behave chaotically, but synchronously [13,14]. Several biologically motivated studies using flows have also shown this behavior. The work of Kowalski *et al.* [15] shows that even Lorenz systems can be globally coupled and synchronized. The necessary conditions have been worked out for this situation [15]. Work by Strogatz *et al.* [16] shows similar synchronization in large systems of coupled oscillators.

II. THEORY

A. Stability of response systems

By one system driving another we mean that the two systems are coupled so that behavior of the second is dependent on the behavior of the first, but the first is not influenced by the behavior of the second. The first system will be called the *drive* and the second the *response.* Actually, both systems can be combined into one compound dynamical system in which the response subsystem depends on variables from the drive subsystem, but the converse is not true. We can similarly generalize these ideas to include any system which can be divided into two such subsystems. We can also further divide the drive subsystem into those variables that drive the response subsystem and those that do not. This gives a subdivision of the original system into three subsystems which will make analysis easier. Next, we make some definitions of these ideas which will enable us to analyze certain drive-response situations that are of interest. Our notation here is somewhat different than in Ref. [9]. In the Appendix we show that one can be more mathematically elaborate about this.

Suppose our compound dynamical system is *drive decomposable;* that is, it can be divided as mentioned above. Let the dimension of the whole system be n. We use the m-dimensional vector v to represent the drive variables that are used in the response, the k-dimensional vector u to represent the drive variables that are not used in the response, and the l-dimensional vector w to represent the response. Then $n=m+k+l$ and the compound system is divided as follows:

$$\left.\begin{array}{ll}\dot{v}=f(v,u) & (m\,\text{dimensional})\\ \dot{u}=g(v,u) & (k\,\text{dimensional})\end{array}\right\}\text{drive}\,,$$
$$\dot{w}=h(v,w)\quad(l\,\text{dimensional})\}\text{response}\,. \tag{1}$$

An example of this type of decomposition is a sinusoidally driven system, such as a pendulum. The dimension of the composite system is $n=4$. The sinusoidal drive comes from a two-dimensional system, so $m+k=2$. The pendulum is the response system with $l=2$ driven by one component of the drive, so $m=1$ (and, therefore, $k=1$). The equations of motion would be

$$\left.\begin{array}{l}\dot{v}=f=\omega u\\ \dot{u}=g=-\omega v\end{array}\right\}\text{drive}\,,$$
$$\left.\begin{array}{r}\dot{w}_1=h_1=w_2\\ \dot{w}_2=h_2=-\gamma w_2-\sin(w_1)+v\end{array}\right\}\text{response}\,. \tag{2}$$

The central question now becomes: When is the response system a "stable" subsystem? That is, when is its trajectory $w(t)$ immune to perturbations? This would guarantee that for a fixed set of drive initial conditions we would know that wherever $w(t)$ starts, it will always converge to the same trajectory and at each point in time always be at the same predictable place on that trajectory. We are ignoring the possibility of other basins of attraction for now.

This question immediately leads to a variational equation of motion. We have a trajectory $w(t)$ (whose stability we want to determine) with initial condition $w(0)$. Consider a trajectory started at a nearby point $w'(0)$ at $t=0$. The drive signal is the same for both response systems. Under what conditions does $\Delta w(t)=|w'(t)-w(t)|\to 0$? In terms of the vector fields,

$$\begin{aligned}\Delta\dot{w}&=h(v,w')-h(v,w)\\ &=D_w h(v,w)\Delta w+o(w,v)\,,\end{aligned} \tag{3}$$

where $D_w f$ is the Jacobian of the response vector field with respect to the response variables and $o(w)$ represents the higher-order terms. In the limit of small Δw we get the equation

$$\Delta\dot{w}=D_w h(v,w)w\,. \tag{4}$$

Normally, when $w(t)$ is constant (fixed point) or $w(t)$ represents a periodic orbit we could determine the eigenvalues of $D_w f$ or the Floquet multipliers of Eq. (4) and thereby determine the stability of $w(t)$ [2,17]. However, we want to drive w with chaotic v signals and these simpler approaches will not work.

A solution to this can be found by calculating certain Lyapunov exponents. These quantities are the exponents that describe the expansion or shrinkage of small displacements along the trajectory averaged over the attractor. In other words, they tell if small displacements of the trajectory are along stable or unstable directions. If we are looking for a stable subsystem, then we would want all the exponents to be negative so that all small perturbations will exponentially decay to zero. This calculation can be done by integrating Eq. (4) along with the system (v,u,w). But there is a more general approach that does not depend on the initial choice of a displacement.

Use a matrix Z in place of w in Eq. (4) such that $Z(0)$ is equal to the identity matrix. Thus

$$\Delta\dot{Z}=D_w h(v,w)Z\,. \tag{5}$$

The solutions $Z(t)$ is often referred to as the transfer function [17] or the principal matrix solution [18]. $Z(t)$ will determine whether perturbations will grow or shrink in any particular directions. One can think of $Z(t)$ as representing the evolution of a coordinate system which begins by being Cartesian, but becomes stretched, compressed, and distorted in time in the various directions (not necessarily orthogonal) corresponding to the Lyapunov exponents. The matrix $Z(t)$ as $t\to\infty$ can be used to determine the Lyapunov exponents [2,19] for the w subsystem for the particular drive trajectory $(v,u)(t)$. If all the Lyapunov exponents are negative, $w(t)$ is an asymptotically stable trajectory. These exponents depend on $v(t)$ and are only a measure of the w subsystem stability. In general, they are not simply a subset of the Lyapunov exponents of the composite system (v,u,w). In an earlier paper [9] we referred to them as sub-Lyapunov exponents, but because of their dependence on $v(t)$, it has been suggested [20] that they be called *conditional Lyapunov exponents.* We do so in this paper.

The negativity of the conditional Lyapunov exponents for the w system is obviously a necessary condition for the stability of the response. The fundamental linear-stability theorem shows that it is also a sufficient condition for many dynamical systems [17].

Theorem: The null solution of the nonlinear nonstationary system

$$\dot{x}=A(t)x+o(x,t)\,,$$

with $o(0,t)=0$ for all t, is uniformly asymptotically stable if

(i) $\lim_{\|x\|\to 0}\|o(x,t)\|/\|x\|=0$ holds uniformly with respect to t; (ii) $A(t)$ is bounded for all t; and (iii) the null solution of the linear system, $\dot{x}=A(t)x$, is uniformly asymptotically stable. Criteria (i) and (ii) hold for most systems in which parameter values are the same (see below) and (iii) is guaranteed when the conditional Lyapunov exponents are all negative.

This theorem guarantees that there will be a nonempty set of initial conditions $w'(0)$ for which the trajectory $w'(t)$ will converge to $w(t)$. This generalizes the usual tests for stability (eigenvalues of Jacobians and Floquet multipliers) to Lyapunov exponents which will work for any drive $v(t)$.

Note that many of the above ideas can be carried over to maps without much trouble. For example, Eq. (2) converts to

$$v_{i+1}=f(v_i,u_i)\ ,$$
$$u_{i+1}=g(v_i,u_i)\ , \tag{6}$$
$$w_{i+1}=h(v_i,w_i)\ ,$$

and the criteria for stability of the w subsystem are the signs of the conditional Lyapunov multipliers calculated from the eigenvalues of the product

$$J=\prod_{i=1}^{\infty} D_w h(v_i,w_i)\ . \tag{7}$$

This makes maps good candidates by which to study chaotic driving, although flows usually model physical systems better (for example, the circuit considered later in this paper).

B. Rates of convergence

The above approach to the response stability can be extended to estimate the rate of convergence of a nearby response trajectory to the true trajectory. The estimate is essentially for small differences (Δw), but, in practice, appears to work well in many systems for differences on the order of the attractor size. This probably will depend strongly on the shape of the basin of attraction for the response system as well as for the general flow of the system from the starting point to the true trajectory. Both of these could cause the system to "wind around" in phase space before getting near the true trajectory. Thus we urge caution in using the convergence rates for large $|\Delta w|$ values.

To estimate actual convergence rates we want to get an estimate of the average convergence rates and their associated directions. This translates into finding the (conditional) Lyapunov exponents and their eigenvectors. We calculate these from the principal matrix solution $Z(t)$ for large t from the variational system, Eq. (5), along a trajectory which has converged to the attractor. The Lyapunov exponents are given by

$$\lambda_i=\frac{1}{t}\ln\nu_i \tag{8}$$

for large t, where ν_i are the eigenvalues of $Z(t)$, each associated with an eigenvector ζ_i.

Numerically, care must be taken in calculating λ_i and ζ_i since, as t increases, the differences between the expanding and contacting directions increase exponentially. Eckmann and Ruelle [19] suggest some stable algorithms for the calculation of λ_i, but not of ζ_i. We use the Eckmann and Ruelle *QR* decomposition technique to calculate the exponents. We get the ζ_i directly from the diagonalization of $Z(t)$, taking t as large as possible. As a check on this diagonalization we compare the eigenvalues found with those determined by the Eckmann and Ruelle technique. If the two agree well, we assume that the eigenvectors are also in good agreement with the actual eigenvectors one would obtain for large t values. We note that there now also exists a stable method for the calculation of the eigenvectors which could be used in place of our simple scheme [21].

For small $\Delta w(0)$ we approximate the convergence as

$$\Delta w(t)=e^{At}\Delta w(0)\ , \tag{9}$$

where A is generated by transforming the diagonal matrix with λ_i's on the diagonal back to the original coordinate system. In other words, we approximate $Z(t)$ with the exponentiation of a constant matrix constructed from the Lyapunov exponents and the similarity transform generated by the eigenvectors ζ_i. Below we show that this can lead to a good approximation to the convergence, including effects associated with nonorthogonal eigenvectors.

III. HOMOGENEOUS DRIVING

In this paper we focus solely on the special case in which $k=l$ and $f=h$. We call this case *homogeneous driving* because the response is the same as in that part of the drive that is not providing a drive signal. This leads to the concept of synchronization of chaotic subsystems. The more general case where $f\neq h$ we call *heterogeneous driving. A simple example of the latter is a linear* oscillator system driving a nonlinear pendulum. We will report ongoing research on this topic elsewhere [22].

Construction of a homogeneous driving system is easily visualized as starting with a system and dividing it into two subsystems (v,u). Then duplicate the subsystem which will not be used for driving (u) and call this duplicate the response (w). How to divide the drive system is determined by calculating the conditional Lyapunov exponents and choosing u as a stable subsystem. In this case the stability theorem in Sec. II guarantees that there is an open set of initial conditions containing $u(0)$ and $w(0)$ for which $w(t)$ will converge to $u(t)$ and the two subsystems will remain synchronized. We demonstrate this for three dynamical systems and for a circuit. We also show at the end of this section that the synchronization is structurally stable to small parameter variations.

TABLE I. Conditional Lyapunov exponents for various drive-response configurations for homogeneous driving systems constructed from the Lorenz system. We show the cases when there is one drive signal ($q=1$ and $k=l=2$).

System	Drive	Response	Lyapunov exponents
Lorenz	x	(y,z)	$(-2.5,\ -2.5)$
$\sigma=16,\ b=4,\ r=45.92$	y	(x,z)	$(-3.95,\ -16.0)$
	z	(x,y)	$(+7.89\times10^{-3},\ -17.0)$

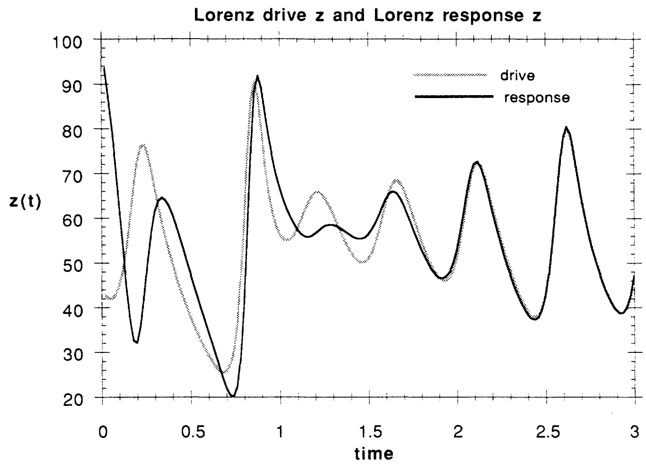

FIG. 1. Time series of the z component of the x-driven (y,z) Lorenz subsystems showing the convergence of the response $z(t)$ to the drive $z(t)$.

A. Lorenz system

We start with an example from the well-known Lorenz system:

$$\begin{aligned} \dot{x} &= \sigma(y-x) , \\ \dot{y} &= -xz+rx-y , \\ \dot{z} &= xy-bz . \end{aligned} \tag{10}$$

We choose the parameters σ, r, and b to be in the chaotic regime: $\sigma=16$, $b=4$, and $r=45.92$. We construct homogeneous driving systems by letting the response system (w) be a duplicate of a pair of the vector fields for x, y, or z, i.e., $k=l=2$. Table I shows a calculation of the conditional Lyapunov exponents for the various subsystems. The system shows two stable subsystems (x,z) driven by y,

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -xz+rx-y \\ \dot{z} &= xy-bz \end{aligned}\right\} \text{drive} , \qquad \left.\begin{aligned} \dot{x}' &= \sigma(y-x') \\ \dot{z}' &= x'y-bz' \end{aligned}\right\} \text{response} , \tag{11}$$

and (y,z) driven by x,

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -xz+rx-y \\ \dot{z} &= xy-bz \end{aligned}\right\} \text{drive} , \qquad \left.\begin{aligned} \dot{y}' &= -xz'+rx-y' \\ \dot{z}' &= xy'-bz' \end{aligned}\right\} \text{response} . \tag{12}$$

The system (x,y) is unstable, although the instability is slight as measured by the size of the positive Lyapunov exponent relative to the negative exponents. The time scale for the instability to manifest itself is orders of magnitude less than for the stable components to converge.

Figure 1 shows the time series and Fig. 2 shows the trajectories of the (y,z) subsystem for the drive and response. The Lorenz is a highly damped system and this shows up in the rapid convergence of the response to the drive. Figure 3 shows the log of the absolute value of the coordinate differences between the drive and response and a plot of the same quantities calculated from the estimate of the convergence as in Sec. II B. The convergence is generally exponential and both coordinates converge at nearly the same rate. The convergence estimate agrees with this. The convergence estimate, as we expect, does not show the detailed variations in the convergence that the actual convergence shows.

Figure 4 shows a similar situation for the (x,z) subsystem. The agreement here between the actual convergence rates and the estimated convergence shown in Fig. 5 is also very good. The maximum that appears in the Δz curve for both the real convergence and the estimated

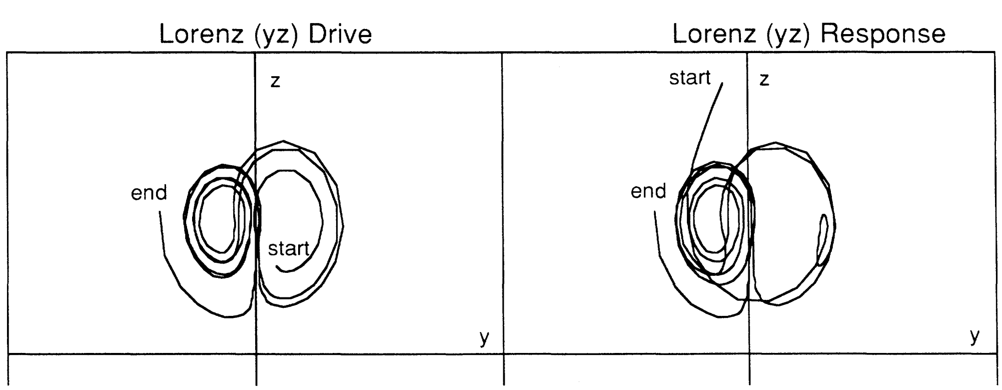

FIG. 2. Trajectories of the x-driven (y,z) Lorenz subsystems from the drive and the response.

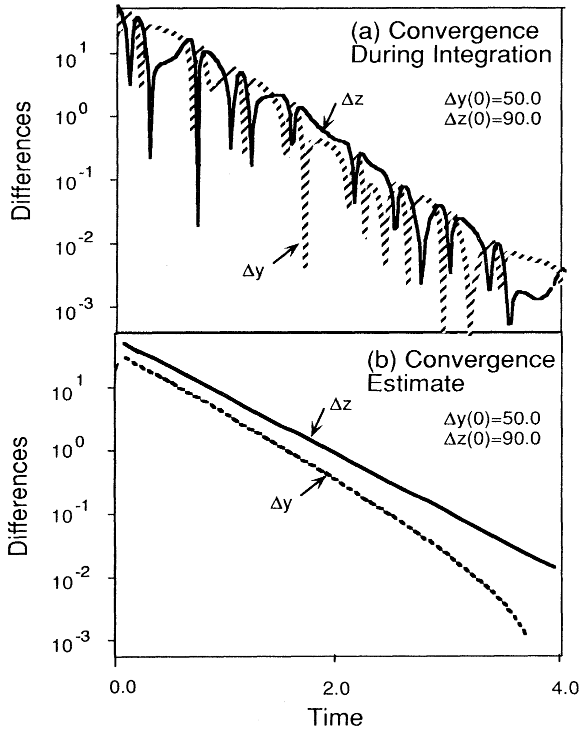

FIG. 3. Absolute values of the differences between the y and z components of the drive and response (y,z) Lorenz subsystem calculated (a) during numerical integration of the system and (b) from Eq. (9).

convergence is real. This results from the simultaneous existence of two things. One is the nonorthogonality of the eigenvectors and the other is the existence of two very different (in magnitude) conditional Lyapunov exponents. Figure 6 shows this schematically. If the exponent along the vector μ_1 is more negative than that along μ_2, then the component of the difference Δw along μ_1 will decrease rapidly. In a short time this will essentially leave only the component along μ_2. The dots in Fig. 6 show the evolution of Δw. When projected back into the Cartesian coordinates of the dynamical system, this leads to an increasing z component early on in the evolution of Δw.

B. Rössler system

The Rössler system shows that stability can change dramatically with parameters, even though the behavior of the basic system (chaotic in this case) does not appear to go through any bifurcations at the same points. The Rössler system is given by

$$\dot{x} = -(y+z) ,$$
$$\dot{y} = x + ay , \tag{13}$$
$$\dot{z} = b + z(x-c) .$$

Because the (x,y) subsystem is linear it is easy to show that for all positive a parameter values this subsystem is unstable for a z Rössler drive $(\lambda_1 \approx \lambda_2 \approx a/2)$. Likewise, for x driving (y,z) there will always be one positive conditional Lyapunov exponent equal to a and so this subsystem is unstable. The situation for y driving (x,z) is not obvious. We study the typical case $a = b = 0.2$ and c spans part of the range 3.0–11.0. The c parameter is the same in both drive and response throughout.

Figures 7 and 8 show the time series and the trajectories of the drive system and the response plus the drive variable (y) system for $c = 4.7$ (the Rössler behavior is actually period 5 here). Although the response initially converges to the x-y plane, the trajectories never do converge fully. Figures 7 and 9 show that they are actually diverging after the initial apparent convergence. The estimated convergence misses the initial dip in the Δx difference and the oscillations in the Δz component caused by the hopping of the Rössler trajectory to a point near the center of the unstable focus, but it does pick up the overall divergences properly.

Figure 10 shows the value of the largest conditional Lyapunov exponent for the response (x,z) subsystem.

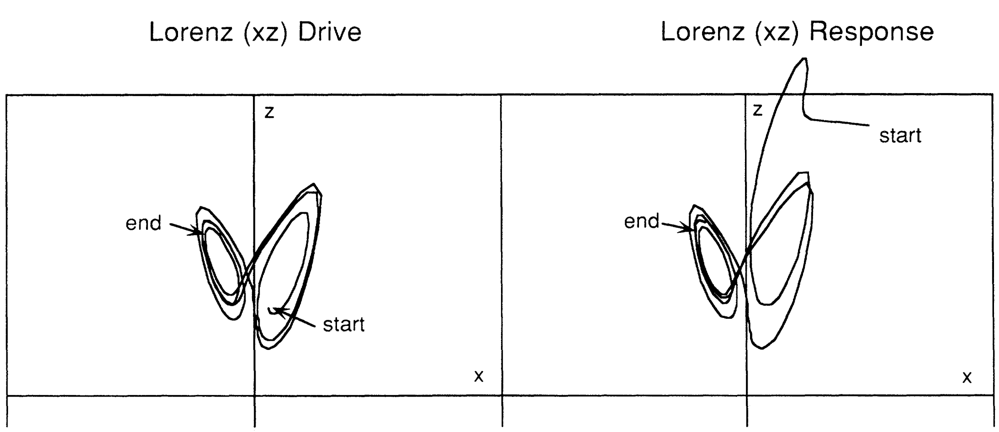

FIG. 4. Trajectories of the y-driven (x,z) Lorenz subsystems from the drive and the response.

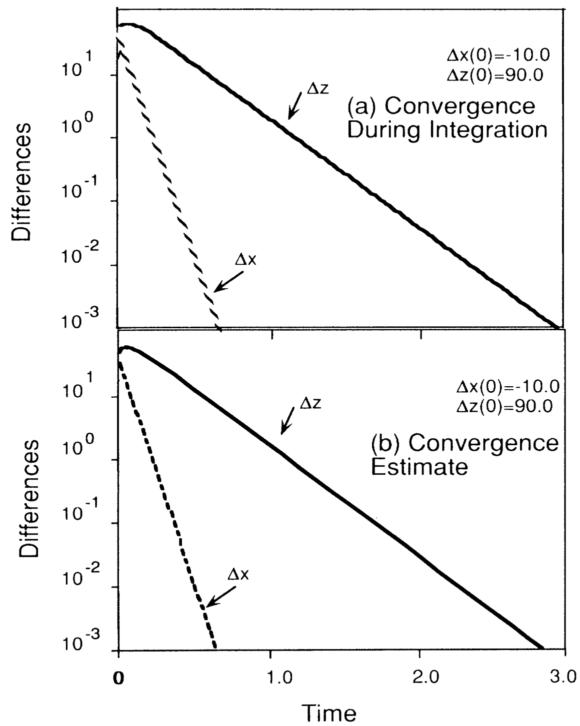

FIG. 5. Absolute values of the differences between the y and z components of the drive and response (x,z) Lorenz subsystem calculated (a) during numerical integration of the system and (b) from Eq. (9).

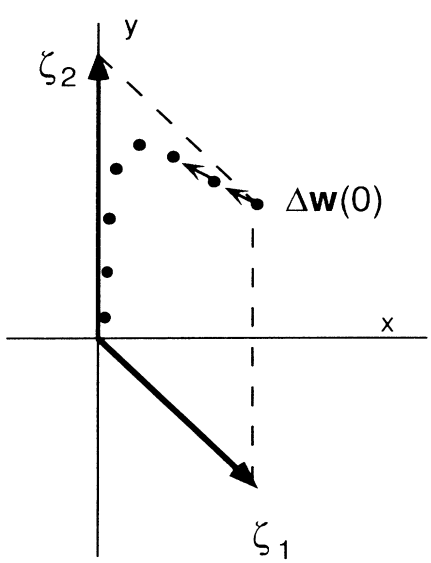

FIG. 6. The dots mark the "trajectory" of the difference $\Delta w(t)$ between the response variables and their counterparts in the drive system. The eigenvalue associated with ζ_1 is assumed to be much less (more negative) than that associated with ζ_2, causing the y component of $\Delta w(t)$ to actually increase initially.

The general behavior of the driving Rössler system is also labeled. There are three regions of instability for the (x,z) system. These appear to occur near the parameter values for various chaotic regions between periodic windows. In both stable and unstable regimes dips appear in the conditional exponent near many of the periodic windows and even in chaotic regimes which are close to multiple-period behavior (see the regions near $c=4.3$ and 8.9). The chaos appears to be unbroken by such periodic windows above $c=9.0$ and the (x,z) system remains stable from this value to $c=20.0$, the highest value we tested. There may be some connection between the windows and the loss of (x,z) stability, but this is not clear.

C. Hysteretic circuit

We examine the case of a set of synchronizing chaotic circuits. More details on the circuits appear in Refs. [9] and [10]. The circuit behaves like a system with two unstable foci, with a hysteretic component that depends on one of the salient voltages and causes those voltages to spiral out from alternating foci. The model for this system is

$$\begin{aligned}\dot{x}_1&=x_2+\gamma x_1+c x_3\,,\\ \dot{x}_2&=-\omega x_1-\delta_2 x_2\,,\\ \epsilon\dot{x}_3&=(1-x_3^2)(Sx_1-D+x_3)-\delta_3 x_3\,.\end{aligned}\tag{14}$$

The hysteresis comes from the x_3 equation. The x_3 voltage flips between two values causing the center of oscillation(the unstable focus) in the (x_1,x_2) subsystem to shift back and forth along the x_1 axis.

The actual two-dimensional duplicate (response) subsystem consists of a circuit in which a combination of x_1, x_2, and x_3 voltages appear. Hence we make a transformation to new variables to isolate the drive and response parts of the dynamical equations. Let

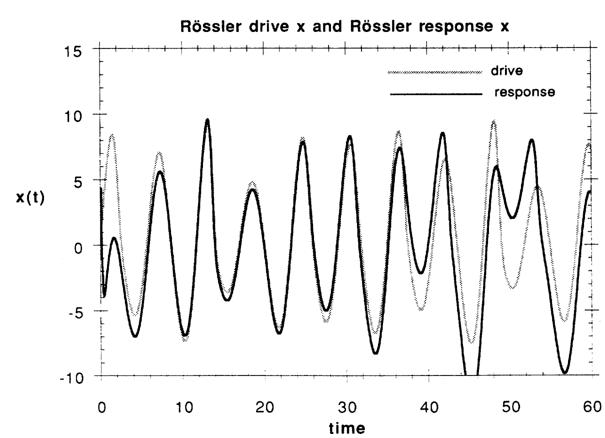

FIG. 7. Time series of the x component of the y-driven (x,z) Rössler subsystems showing the lack of asymptotic convergence of the response $x(t)$ to the drive $x(t)$.

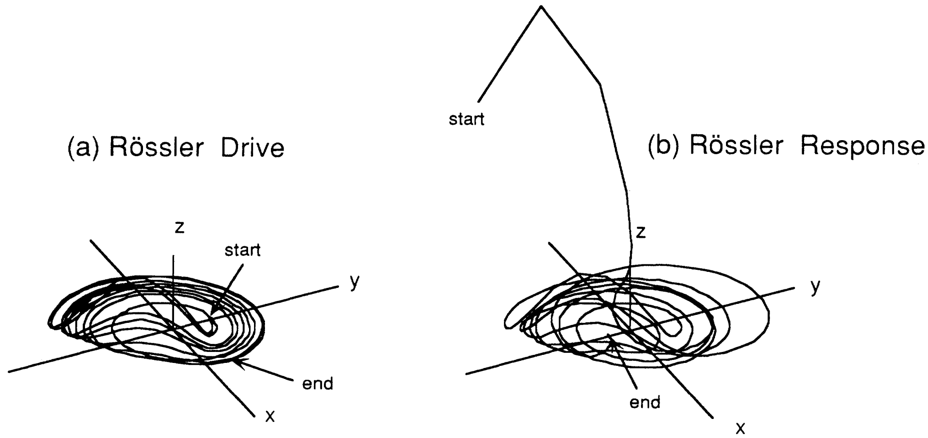

FIG. 8. Trajectories of (a) the Rössler (drive) system and (b) the Rössler response system including the y drive, i.e, y is the same in (a) and (b).

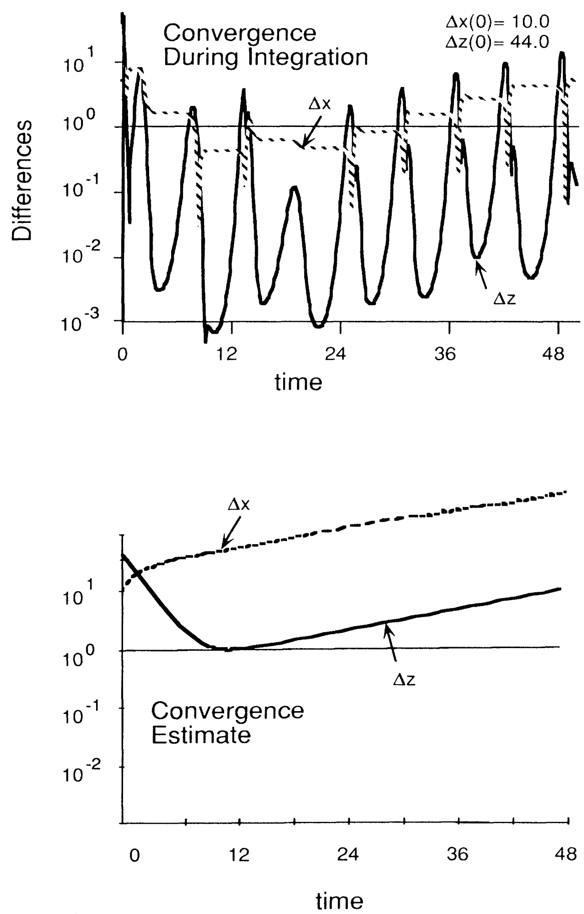

FIG. 9. Absolute values of the differences between x and z components of the drive and response (x,z) Rössler subsystem calculated (a) during numerical integration of the system and (b) from Eq. (9).

$$x_4=\frac{1}{\alpha}(x_3+\beta x_1)\ . \tag{15}$$

Then in terms of x_1, x_2, and x_4, the equations of motion are

$$\begin{aligned}
\dot{x}_1&=x_2+\gamma x_1+c(\alpha x_4-\beta x_1)\ ,\\
\dot{x}_2&=-\omega x_1+\delta_2 x_2\ ,\\
\dot{x}_4&=\frac{1}{\alpha}\{[1-(\alpha x_4-\beta x_1)^2]Sx_1-D+\alpha x_4-\beta x_1)/\epsilon\\
&\quad-\delta_3(\alpha x_4-\beta x_1)/\epsilon\\
&\quad-\beta x_2-\beta\gamma x_1-c\beta(\alpha x_4-\beta x_1)\}\ ,
\end{aligned} \tag{16}$$

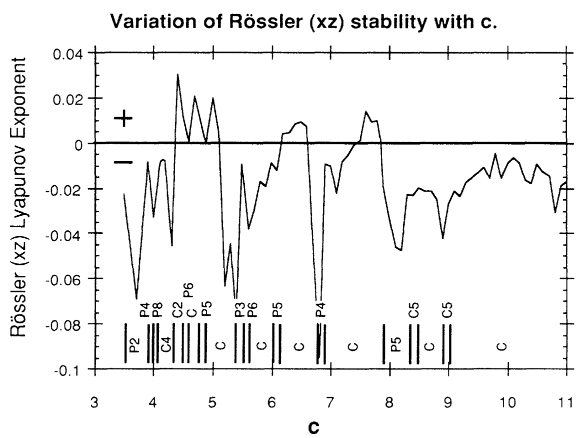

FIG. 10. The largest conditional Lyapunov exponent of the y-driven (x,z) Rössler subsystem as a function of the parameter c. Across the bottom are labels signifying the behavior of the Rössler (3D) drive system: $P2$ represents period 2, $P3$ represents period 3, $P4$ represents period 4, $P5$ represents period 5, $P8$ represents period 8, $C2$ represents mildly chaotic with period-2-like trajectory, $C4$ represents mildly chaotic with period-4-like trajectory, and C represents chaotic behavior.

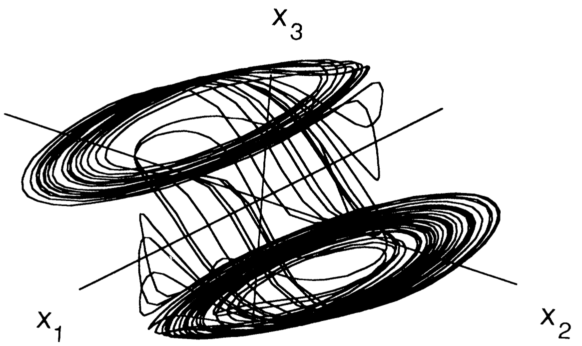

FIG. 11. The attractor for the hysteresis circuit as calculated from the model, Eq. (14).

and the x_4 voltage drives the (x_1,x_2) subsystem.

The parameters chosen to model [23] the behavior of the circuit are $\alpha=6.67$, $\beta=7.87$, $\gamma=0.2$, $c=2.2$, $\delta_2=0.001$, $\delta_3=0.001$, $\epsilon=0.3$, $\omega=10$, $S=1.667$, and $D=0.0$. The attractor for this system model is shown in Fig. 11 in terms of x_1, x_2, and x_3. The unstable foci behavior combined with the hysteresis is easily seen. We visualize the actual attractor from the operating circuit on an oscilloscope (see Fig. 12) by measuring a combination of circuit voltages which mimic the two-dimensional (2D) projection of the 3D attractor. The model duplicates the topology of the actual circuit behavior rather well.

Equations (16) make a rather straightforward model using x_4 to drive a (x_1,x_2) subsystem. The (x_1,x_2) subsystem is linear and, in this case, the Lyapunov exponents can be calculated from the Jacobian of the (x_1,x_2) subsystem. These are -16.587 and $-0.603\ \text{ms}^{-1}$. The eigenvectors are $(-0.855, -0.518)$ and $(-0.07, -1.21)$, respectively. A measurement of the x_2 voltage during

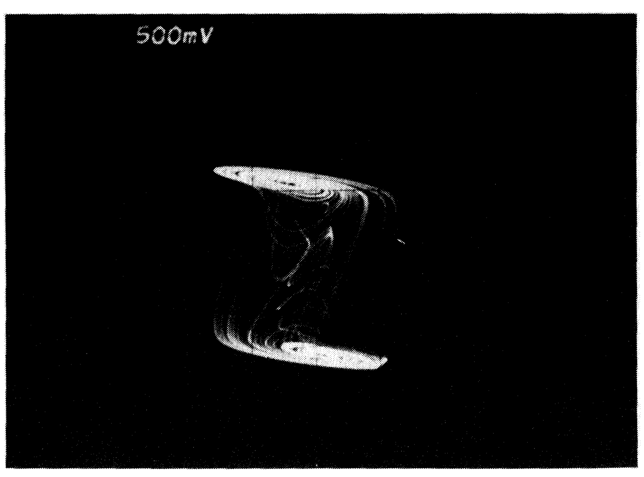

FIG. 12. The attractor for the hysteresis circuit as seen on the oscilloscope.

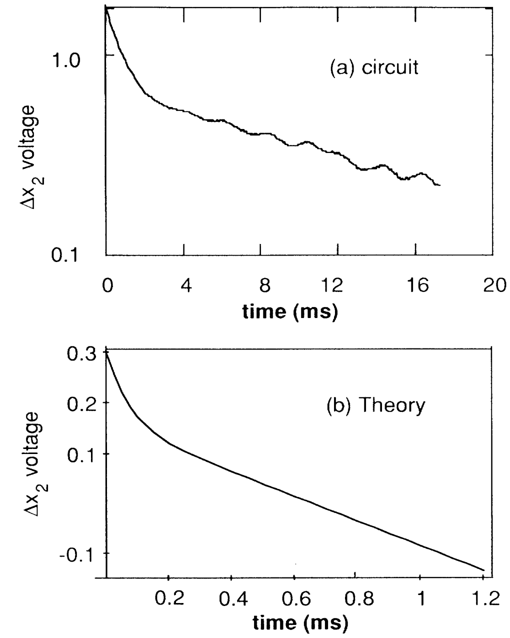

FIG. 13. Convergence of the x_2 voltage from the response to its counterpart in the drive circuit as a function of time: (a) as measured in the actual circuit and (b) from theory, Eq. (9) applied to the trajectory from Eq. (14).

the convergence to synchronization of the (x_1,x_2) subsystem would generally show a rapid convergence corresponding to the most negative exponent (-16.587) and then a slower convergence corresponding to the less negative exponent -0.603). Figure 13 shows the convergence as measured from the difference in x_2 voltages in the drive and response circuits and as calculated from the convergence estimation formula. The model is adjusted on the time scale to mimic the circuit's attractor, so except for this time-scaling factor the two agree on the form of the convergence to synchronization.

D. The effect of parameter differences

A natural question to ask is what is the effect on stability and synchronization when the response (w) has parameters slightly different from its counterpart (u) in the drive system? In other words, how would the convergence $w(t)\rightarrow u(t)$ be affected? This is a question that would come up in any practical applications of chaotic driving.

In this case we can begin as above and examine the equation of motion for $\Delta w(t)=w(t)-u(t)$. Letting μ stand for the parameters in the h vector field and recalling that w and u have the same vector fields,

$$\begin{aligned}\Delta\dot{w} &= h(v,w,\mu')-h(v,u,\mu)\\ &= h(v,w,\mu')-h(v,u,\mu')+h(v,u,\mu')-h(v,u,\mu)\\ &= D_w h\,\Delta w+o(v,w)+h(v,u,\mu')-h(v,u,\mu)\ ,\end{aligned} \tag{17}$$

where $D_w h$ and $o(v,w)$ are as before. Note that the above theorem for uniform asymptotic stability does not apply. Because of the extra terms $h(v,u,\mu')$ and $h(v,u,\mu)$, criteria (i) of the theorem is not satisfied.

We can get an estimate of Δw by assuming that $\Delta w(0) \ll 1$. That is, the systems have nearly identical initial conditions and we can ignore $o(v,w)$ for times less than some t_1. Then we can approximate

$$\Delta\dot{w} = D_w h\, \Delta w + h(v,u,\mu') - h(v,u,\mu) \ . \tag{18}$$

Using the principal matrix solution $Z(t)$, the solution to Eq. (9) is

$$\Delta w(t) = Z(t)\Delta w(0) + \int_0^t Z(t-\tau)B(\tau)d\tau \ , \tag{19}$$

where $B(t) = h(v,u,\mu') - h(v,u,\mu)$. We can now show that $\Delta w(t)$ may be bounded. In general, there exist positive constants [18] c_1 and c_2 such that

$$\|Z(t)\| \le c_1 e^{-c_2 t} \ . \tag{20}$$

In fact, $-c_2$ should be the largest negative conditional Lyapunov exponent. If $B(t)$ is bounded by a constant b_1 (as it will be for most cases), then we have

$$|\Delta w| \le c_1 e^{-c_2 t} + \frac{c_1 b_1}{c_2}(1 - e^{-c_2 t}) \ . \tag{21}$$

If our assumption about $o(v,w)$ holds for long enough times, then

$$|\Delta w| \le \frac{c_1 b_1}{c_2} \ . \tag{22}$$

So, if the differences in parameters are not large (which determines b_1), Δw will remain small. This will hold only if $o(v,w)$ is "small enough" which will generally require Δw to remain small (which is to be shown). Hence, Eq. (22) is not the result of a theorem, but a heuristic argument that for small enough parameter differences the actual response trajectory will remain near its drive counterpart. This means that the systems will remain nearly synchronized. We find this to be the case in our numerical studies and in actual circuits for as long a time as we let the systems evolve [9,10]. This in fact shows that even in heterogeneous driving situations a change in parameters of the response will not greatly affect its behavior.

IV. CONCLUSIONS

The concepts of drive decomposition and homogeneous and heterogeneous driving generalize many standard cases of driven systems. These generalizations are straightforward and constructive in nature. They provide a scheme for building up driven systems, especially homogeneously driven systems. The concept of conditional Lyapunov exponents also generalizes the usual stability condition for driven systems. It provides a test for the response system that guarantees predictability (when the exponents are negative) of certain events and dynamical behavior (e.g., synchronization) even when the overall motion is chaotic. As the model of the circuit shows, transformations (diffeomorphisms) can be used to create new versions of the drive-response system which can possibly be tailored to specific needs.

Because of the existence of chaotic signals in physiological systems [24–27], it is tempting to speculate about the relationship to stable chaotic driving and/or synchronization. For one thing, our results show that the existence of such signals does not automatically mean that all subsystems will behave in chaotic or random fashion. Synchronization itself is structurally stable in chaotic driving [9] and using chaotic signals may be preferable to periodic signals in certain cases where increased robustness is advantageous. Certainly, we now know that neural response to stimuli need not be dynamically simple (periodic, for example) for the same predictable, albeit chaotic state to be reached, provided that the driven neurons form a stable response. In order for nervous response to be useful it may only be necessary that the same pattern emerge for similar stimuli, but not that the pattern be in any way regular.

At any rate, the use of chaotic signals to drive systems is in its infancy, unlike periodic drives, and leaves many open questions. In this paper we have focused mostly on stability, but many topologically related issues need to be explored.

Since derivatives of vector fields (Jacobians) do not transform in a simple (covariant) way under diffeomorphisms [28,29], one would not expect the type and number of stable subsystems to be topological invariants. The question remains as to what, if any, are the topological invariants of these systems and concepts? It would seem that there should be a relation to stable, center, and unstable manifolds [1,2,30] of a system, but other than a possible local relationship, nothing is obvious.

From the example of the Rössler system, a chaotic system need not have a stable subsystem at all. There appears to be in this case, a weak relation to the behavior of the full Rössler system (either chaotic or periodic), but a rigorous link (if it exists) has not yet been made. The converse question is also interesting: Does a chaotic system necessarily have to have at least one unstable subsystem? Perhaps, but the example of the Lorenz system in which only the (x,y) subsystem is marginally unstable suggests that this might not be the case.

Higher-dimensional chaotic systems with more than one positive Lyapunov exponent call into question whether it is still possible in this case to construct synchronous (stable) subsystems of dimension $n-1$ (i.e., with only one drive variable).

Despite these unanswered questions, we feel that the idea of driving stable subsystems with chaotic signals opens up the possibility of using chaos; that is, purposefully designing drive systems to act chaotically, then using the properties unique to chaos to obtain novel and useful behavior.

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APPENDIX

A more mathematically elaborate method can be used to show the division of a system into drive and response subsystems. Although it is more intricate than that used in Sec. II, it does show that there is a rigorous decomposition method for any dynamical system.

Let $\dot{x}=X(x)$ be an n-dimensional dynamical system. The vector field is the mapping $X:R^n \rightarrow R^n$. We call the vector field *drive decomposable* if it can be divided into drive and response subsystems. That is, there is a dimension $q<n$ and corresponding functions $Q:R^q \rightarrow R^q$ (for the drive system) and $P:R^n \rightarrow R^{n-q}$ (for the response system), so that with suitable reordering of X indices, $Q=(X_1,\ldots,X_q)(x_1,\ldots,x_1)$ and $P=(X_{q+1},\ldots,X_n)(x_1,\ldots,x_n)$. That is, the drive (Q) does not depend on all n variables. This is the mathematical form of the statement in the above paragraph. The decomposed equations of motion become

$$\left.\begin{aligned}\dot{x}_1&=Q_1(x_1,\ldots,x_q)\\&\vdots\\\dot{x}_q&=Q_q(x_1,\ldots,x_q)\end{aligned}\right\}\text{ drive subsystem ,}$$

$$\left.\begin{aligned}\dot{x}_{q+1}&=P_1(x_1,\ldots,x_n)\\&\vdots\\\dot{x}_n&=P_{n-q}(x_1,\ldots,x_n)\end{aligned}\right\}\text{ response subsystem .}$$

In many cases the response is driven by some subset of the drive system's variables. We make the following definitions to further refine the above drive decomposition and put the drive-response system in a final form appropriate for many cases to be presented later. Again, with suitable reorderings, let the response system depend only on drive variables $(x_1,\ldots,x_m)$ for some $m \leq q$. Let $v=(x_1,\ldots,x_m)$. Let u equal the remaining drive variables which are equal to $(x_{m+1},\ldots,x_{m+k})$, where $k=q-m$. The response variables are equal to $w=(x_{m+k+1},\ldots,x_{m+k+l})$, where $l=n-q$. Along with these variables we define the vector fields (subsets of X's components) $f:R^q \rightarrow R^m$, $g:R^q \rightarrow R^k$, and $h:R^{m+l} \rightarrow R^l$, so that the dynamical system becomes Eq. (1) in Sec. II. In other words, we have written h in place of P (explicitly showing only v and w arguments) and defined $f=(Q_1,\ldots,Q_m)$ and $g+(Q_{m+1},\ldots,Q_{m+k})$.

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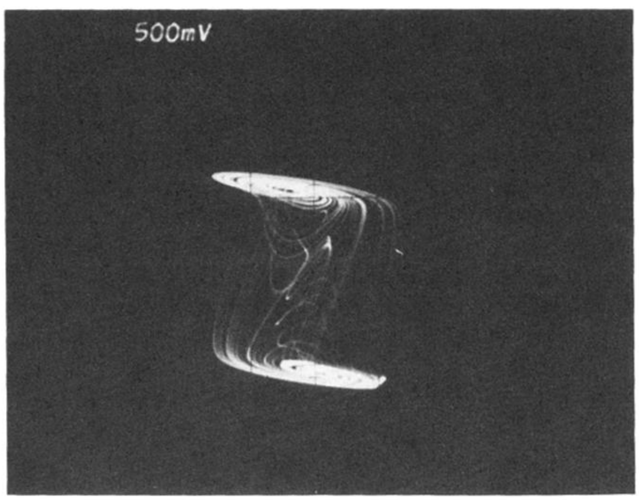

FIG. 12. The attractor for the hysteresis circuit as seen on the oscilloscope.