Zero-temperature directed polymers in a random potential

J. M. Kim, M. A. Moore, and A. J. Bray
Department of Theoretical Physics, The University, Manchester M13 9PL, England
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Extensive simulations of a model of directed polymers in a random potential are described. The standard deviation $\Delta E(t)$ of the lowest energy E of walks of t steps varies as t^{ω}, but there is a large spread in the values quoted in the literature for ω in d=2+1 and 3+1. In our model we are able to vary a parameter—the bending energy of the polymer—that affects the apparent value of ω, but, if allowance is made for corrections to scaling, then we find that ω is universal with the value 0.248 ± 0.004 in d=2+1 and $\omega = 0.20 \pm 0.01$ in d=3+1. The probability distribution $P(E,t)$ is found to scale as $t \to \infty$, $P(E,t)dE \to P(a)da$, where $a = (E - \langle E \rangle)/\Delta E$. The scaling function $P(a)$ is investigated and found to be universal, i.e., independent of such details as choice of distribution of the random potential, etc., and in fact identical to within our numerical accuracy to the equivalent distribution of heights in a model of ballistic aggregation.

I. INTRODUCTION

The problem of directed polymers in random potentials has been receiving much attention recently [1]. Besides being one of the simplest problems involving disorder, it is related to many other physical phenomena: the growth of an interface in the Eden model [2], ballistic aggregation [3–5], domain walls in the two-dimensional random bond Ising model [6], and a randomly stirred fluid [7] obeying Burgers's equation [8]. Despite the apparent simplicity of these directed walks, they are not understood except in two dimensions, where the exponents which characterize the walk can be calculated analytically [7, 8]. Above two dimensions, there are only numerical studies of the exponents [1, 5, 9–11]. Because the walks are directed (which means there is a special direction, the longitudinal direction, parallel to which no reverse step can be taken), it is possible to study walks of considerable length. Despite this, there is a wide variation among the values quoted for the exponents by various authors (for a review see Ref. [13]). It was this variation which prompted us to undertake yet another numerical study of directed walks in random potentials.

Our model contains an adjustable parameter γ, which is related to the bending energy of the polymer. We have studied the exponents for various values of γ and for two different types of distribution of the random potential. The apparent exponents seemed to vary continuously with γ and also depend on the choice of distribution function for the random potential. The range of variation of the exponents is similar to the spread in the quoted values of the exponents in the literature. However, we shall show that this apparent variation of the exponents is not a real effect but is a consequence of there existing large corrections to scaling. The true exponents are expected to be independent of γ and the potential.

A transfer-matrix technique [6, 12] has been employed to study our model of directed walks on hypercubic lattices of dimension d=2, 3, and 4. Let $\mathbf{x}(t)$ denote the position of the directed walk after t steps. Because of the analogy of the directed walk problem with ballistic aggregation problems, we shall usually refer to t as "time." $\mathbf{x}$ is a $(d-1)$-dimensional vector, transverse to the longitudinal direction of the directed walk. The walk is restricted by $|\mathbf{x}(t) - \mathbf{x}(t+1)| = 0$ or 1, but there is a biasing or bending factor γ against walks such that $|\mathbf{x}(t) - \mathbf{x}(t+1)| = 1$. There is a random site energy $\mu(\mathbf{x}, t)$ assigned to visiting the site $\mathbf{x}$. We shall only study directed walks at zero temperature in this paper, for which we only need to calculate the minimum energy path $E(\mathbf{x}, t)$, which can be done recursively. For example, in d=2 (which we shall write usually as d=1+1 to indicate that there is one transverse and one longitudinal direction)

$$E(x,t) = \min[E(x,t-1) + \mu(x,t-1),\ E(x+1,t-1) + \mu(x+1,t-1) + \gamma,\ E(x-1,t-1) + \mu(x-1,t-1) + \gamma], \tag{1}$$

where min takes the minimum value. In the transverse direction the lattice has periodic boundary conditions and is of length L. Using Eq. (1), the energy $E(\mathbf{x}, t)$ and the origin of the walk, $\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x}, t)$, are recorded for each $[\mathbf{x}(t), t]$.

We have employed two kinds of random site potential. One was a random potential drawn uniformly from the interval (0,1) and also a Gaussian random potential with the same variance (1/12) as the uniform random potential.

The advantage of our model is that all the exponents χ, z, and ω [see Eq. (2)] that are usually studied for directed walks can be obtained simultaneously. Thus the root-mean-square transverse displacement $\Delta x(t) = \langle[\mathbf{x}-\mathbf{x}_0(\mathbf{x},t)]^2\rangle^{1/2}$ is expected to vary with t as $t^{1/z}$ when $t^{1/z} \ll L$. The angular brackets $\langle\rangle$ mean, here and below, the average over the L^{d-1} values of the vector $\mathbf{x}$. We have furthermore averaged over many different realizations of the site energies. Each set of site energies is referred to as a run. The average energy $\langle E(t)\rangle$ and its standard deviation, $\Delta E(t)$, have also been evaluated. On a hypercubic lattice of linear dimension L, it is expected that [4]

$$\Delta E(t)=\langle(E-\langle E\rangle)^2\rangle^{1/2} \sim L^{\chi} f(t/L^z) \sim \begin{cases} t^{\omega}, & t \ll L^z \\ L^{\chi}, & t \gg L^z \end{cases} . \tag{2}$$

The exponents ω, χ, and z are connected by the scaling relations $z\omega = \chi$ and $\chi + z = 2$ [5, 14, 15] and numerical simulations [5, 9–11] are consistent with these relations. Thus, in any dimension, there is only one independent exponent that has to be determined.

In dimension two, the exponents are known [7] to be $\omega = 1/3$ and $1/z = 2/3$ from the solution of the Burgers equation, and these values have been confirmed by extensive numerical work [16]. In higher dimensions there is as yet no general agreement as to the value of the exponents [17–19]. There are two conjectures as to how the energy fluctuation exponent ω may depend on dimension, viz., $\omega = 1/(d+1)$ [10] and $\omega = 1/(2d-1)$ [9], based on the numerical simulation of growth models. Recent simulation results on directed polymers [11], ballistic aggregation [20, 21], and the hypercubic stacking model [22], are in between these values, while the direct integration of the Kardar, Parisi, and Zhang (KPZ) equation [23] [see Eq. (6) below] shows $\omega \approx 0.12$ [24, 25] to 0.25 [26] for d=2+1. This is the spread in the reported values for ω which prompted this paper.

In Sec. II, we give our results for the exponents ω, z, and χ, in dimension d=2, 3, and 4. We show that the exponent ω apparently depends on γ, but that this variation can be explained away as a consequence of large corrections to scaling. Our final results are very close to those reported earlier by one of us, viz., $\omega \approx 1/(d+1)$ [10]. Note that these exponents are the strong-coupling (or T=0) values for the exponents. In dimensions $d > 2+1$ there is a phase transition at a finite temperature T_c, between a low-temperature phase for which $\langle\mathbf{x}(t)^2\rangle_T \sim t^{2/z}$ and a high-temperature (weak-coupling) phase for which $\langle\mathbf{x}(t)^2\rangle_T \sim t$ where $\langle\rangle_T$ is the thermal average [27]. All our results pertain to the low-temperature or strong-coupling phase.

From our numerical studies, we can also calculate the full distribution function of the minimum energies E of the directed walks. For each value of $\mathbf{x}(t)$, there is a corresponding energy $E(\mathbf{x},t)$. Its distribution $P(E,t)$ is found to scale, i.e., for fixed $a \equiv (E-\langle E\rangle)/\Delta E$,

$$P(E,t)dE = P(a)da \qquad \text{for } 1 \ll t \ll L^z . \tag{3}$$

In Sec. III, $P(a)$ is determined numerically for d=1+1, 2+1, and 3+1.

We have already mentioned that the directed polymer in a random potential has strong connections with a variety of other problems. These connections are most easily seen from the continuum version of the directed polymer problem [1],

$$\frac{\partial Z(\mathbf{x},t)}{\partial t} = \frac{1}{2}\nabla^2 Z(\mathbf{x},t) + \mu(\mathbf{x},t)Z(\mathbf{x},t) , \tag{4}$$

where $Z(\mathbf{x},t)$ is the partition function of the directed polymer and $\mu(\mathbf{x},t)$ is a Gaussian random variable such that

$$\langle\mu(\mathbf{x},t)\mu(\mathbf{x}',t')\rangle = 2D\delta^{d-1}(\mathbf{x}-\mathbf{x}')\delta(t-t') . \tag{5}$$

With the variable change $h(\mathbf{x},t) = \ln Z(\mathbf{x},t)$, this becomes the KPZ equation [23] for ballistic growth,

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \frac{1}{2}\nabla^2 h + \frac{1}{2}(\nabla h)^2 + \mu(\mathbf{x},t) . \tag{6}$$

One thus expects that in the strong-coupling regime the height h at time t of the growing surface will be equivalent to the energy E ($\sim \ln Z$) in the directed walk. It is found that $\Delta h \equiv \langle(h(x,t) - \langle h\rangle)^2\rangle^{1/2} \sim t^{\omega}$ and the exponent ω in d=1+1 is the same as for the directed walk in a random potential. This is an example of universality, commonly observed in critical phenomena, where quite different physical systems have the same critical exponents. Another feature of universality is that probability distributions, appropriately scaled, should also be universal. We shall show in Sec. III that the scaling function $P(a)$ for our model of a directed walk is numerically identical to the analogous function describing the height distribution in a model [10] studied in ballistic aggregation.

The plan of the paper is as follows. In Sec. II, we present the results of the transfer-matrix studies of our model and give values of the exponents ω, z, and χ. The apparent dependence of the exponents on the bending energy γ is removed by adding corrections to the scaling limit. Studies of energy distributions are given in Sec. III. A brief discussion of some outstanding points is given in Sec. IV.

II. THE SCALING EXPONENTS

The root-mean-square deviation $\Delta E(t)$ of the energies has been studied to determine ω, the exponent governing the rate of growth of the energy fluctuation. ΔE grows with time and it is expected that $\Delta E \sim t^{\omega}$ for $1 \ll t \ll L^z$. The $\ln(\Delta E) \sim \ln t$ curves are shown in Fig. 1 for the largest systems used (L=40 000 for d=1+1, 1000 for d=2+1, and 100 for d=3+1) for a Gaussian random potential, the bending energy γ=0.5, and averaged over 100 runs. From a least-squares fit to the data we get

$$\omega = \begin{cases} 0.332 \pm 0.003 , & d = 1+1 \\ 0.248 \pm 0.004 , & d = 2+1 \\ 0.20 \pm 0.01 , & d = 3+1 . \end{cases} \tag{7}$$

Notice that these values are consistent with the ω =

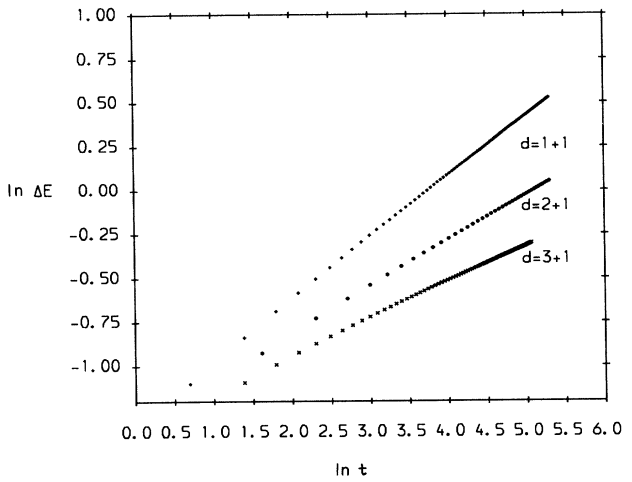

FIG. 1. Energy fluctuation ΔE as a function of length t for the largest systems with Gaussian $\mu(\mathbf{x},t)$ and $\gamma = 0.5$.

$1/(d+1)$ conjecture. (N.B. in all the figures the statistical uncertainties are smaller than the symbols indicating the points.)

In Fig. 2 we have shown $\ln \Delta E(t)$ vs $\ln t$ plots for the uniform distribution of site energies as well as the Gaussian distribution for various values of γ in dimension d=2+1. Notice that, except for the case of the uniform distribution with $\gamma = 0$, the curves are remarkably straight, but with quite different slopes, i.e., values of ω. These values of ω vary from 0.2 to 0.25. One would naturally expect that as $t \rightarrow \infty$ all the slopes would become equal, but there are few obvious signs of this actually happening in Fig. 2. It is this apparent constant value of the slope for each value of γ which has probably led to the spread in the values quoted for ω in the literature. (We are supposing that different models are equivalent to ours but with a particular value for γ and choice of distribution function of site energy.) However, on closer inspection the problem goes away. Suppose that as $t \rightarrow \infty$

$$\Delta E(t)^2 = At^{2\omega} + B \ , \tag{8}$$

where A and B are constants which would be expected to depend on γ and the type of random potential. In other words, we are postulating that the leading correction to the asymptotic term ($At^{2\omega}$) is a constant B. We know of no *a priori* reason for supposing that the largest correction to scaling takes this form for directed polymer problems, but it seems to fit the data well. A similar correction has also been found to work well in describing the intrinsic width of the surface in the Eden model [9]. In Fig. 3 we have plotted the running slope $2\omega_{\text{eff}}$, defined by $\partial \ln(\Delta E)^2/\partial \ln t$ against $t^{-2\omega}$, which if Eq. (8) is valid, should approach the true exponent ω linearly as $t \rightarrow \infty$. For a wide range of γ, Fig. 3 indicates that Eq. (8) describes the data reasonably well, and that there is a single universal value of $\omega \approx 0.25$. Furthermore, Fig. 3 suggests that there exists a value of γ for which the leading correction to scaling, i.e., B, vanishes. For this value of γ (≈ 1 for a uniform distribution of site energies), the asymptotic behavior sets in for relatively short walks.

An alternative way to calculate ω is to study the quantity

$$G(\mathbf{x},t) = \langle [E(\mathbf{x},t) - E(\mathbf{0},t)]^2 \rangle \ . \tag{9}$$

At $t \ll L^z$, $G(\mathbf{x},t)$ increases like $|\mathbf{x}|^{2\chi}$ and saturates when $|\mathbf{x}| \simeq \Delta x(t)$. To obtain ω, $G(L/2,t)$ was calculated, which scales as $t^{2\omega}$. Since points separated by $L/2$ and small t are not correlated, this method yields potentially accurate values for ω. For d=2+1 and γ=0.5, we found $\omega \approx 0.247$ for a Gaussian random potential.

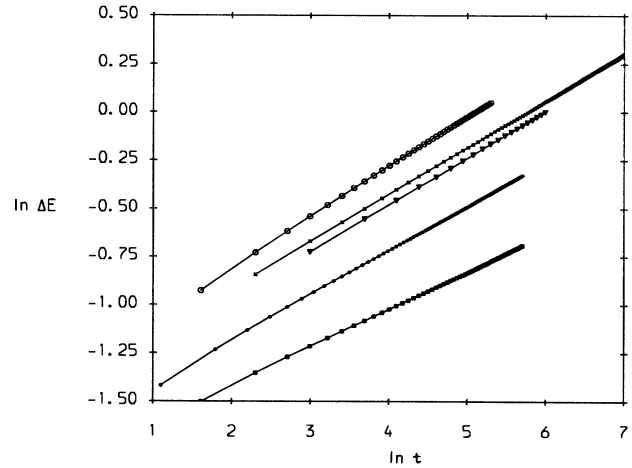

FIG. 2. Energy fluctuation ΔE as a function of length t for uniform $\mu(\mathbf{x},t)$ with γ=0, 0.25, and 0.5, and Gaussian $\mu(\mathbf{x},t)$ with $\gamma = 0$ and 0.5 in d=2+1. Curves are from top to bottom: Gaussian $\mu(\mathbf{x},t)$ with $\gamma = 0.5(\omega = 0.25)$, uniform $\mu(\mathbf{x},t)$ with $\gamma = 0.5(\omega = 0.24)$, Gaussian $\mu(\mathbf{x},t)$ with $\gamma = 0(\omega = 0.25)$, uniform $\mu(\mathbf{x},t)$ with $\gamma = 0.25(\omega = 0.23)$, and uniform $\mu(\mathbf{x},t)$ with $\gamma = 0(\omega = 0.2)$. The bottom one shows slightly increasing slopes for large t.

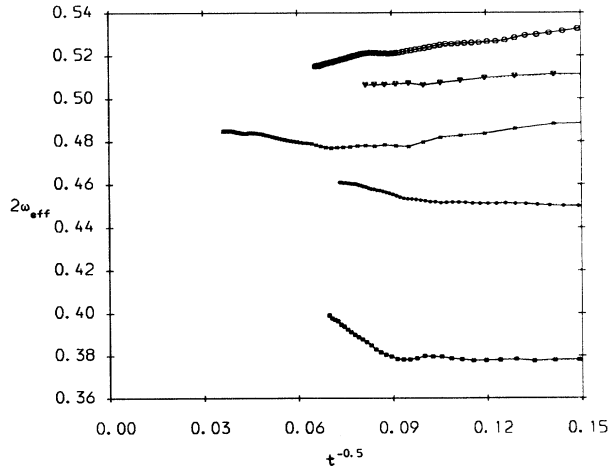

FIG. 3. The effective exponent $2\omega_{\text{eff}}$ as a function of $t^{-0.5}$ for uniform $\mu(\mathbf{x},t)$ with different γ in d=2+1. Curves are from top to bottom: $\gamma = 3$, 1, 0.5, 0.25, and 0.

Although according to the scaling relations there is only one independent exponent, we have determined χ independently in d=2+1 in order to check the scaling relations. The exponent χ is determined by $\Delta E(t,L) \sim L^{\chi}$ for $t \gg L^z$. For these large values of t, $\Delta E(t,L)$ is not self-averaging and has to be averaged over many different configurations (runs). From $\Delta E(t \gg L^z, L)$ with L=10, 20, 40, 60, 80, 100, and 120 as in Fig. 4 we get

$$\chi = 0.40 \pm 0.02 \ , \quad d = 2+1 \ . \tag{10}$$

We did not attempt to determine χ directly in higher dimensions because of the attendant cost in computing time.

To determine the exponent $1/z$, we calculate $\Delta x(t)$. In d=1+1, we get $1/z = 0.664 \pm 0.003$ in good agreement with the exact result 2/3. Our results in higher dimensions are given in Fig. 5 with

$$1/z = \begin{cases} 0.62 \pm 0.01 \ , & d = 2+1 \\ 0.59 \pm 0.01 \ , & d = 3+1 \ . \end{cases} \tag{11}$$

Since the log-log plot in Fig. 5 curves slightly upward as t increases in d=3+1, the above value might be a lower bound for $1/z$. If we apply the same correction to scaling as exemplified in Eq. (8) to the above data (with $1/z$ instead of ω), we get $1/z = 0.622 \pm 0.006$ in d=2+1 and $1/z = 0.595 \pm 0.007$ in d=3+1. This $1/z$ exponent is relatively insensitive to the choice of random site distribution function and the bending energy γ. In d=2+1, the independently measured exponents $\chi = 0.40$, $\omega = 0.25$, and $1/z = 0.62$ agree well with $\chi = z\omega$ and $\chi + z = 2$.

As a final check for the values of the exponents, we generated a random potential following the same distribution as $P(a)$ (given in Sec. III below). With variance 1/12 and $\gamma = 0.5$, it yields $\omega \approx 0.252$ and $1/z \approx 0.627$ in d=2+1, showing effectively the same values as found

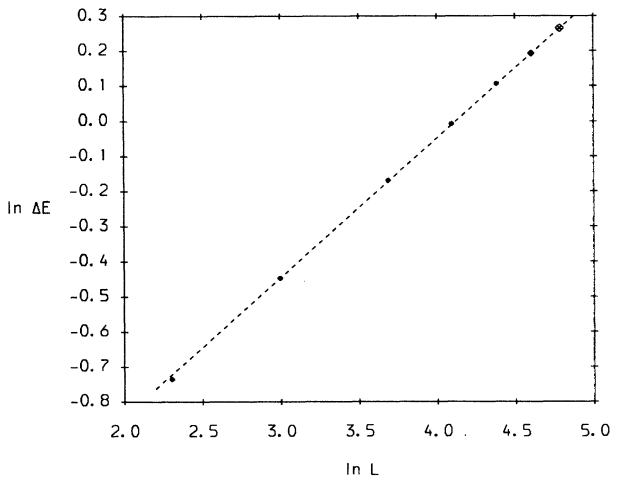

FIG. 4. Energy fluctuation in the saturated region as a function of system size L with uniform $\mu(\mathbf{x},t)$ and $\gamma = 0.5$ in d=2+1. The broken line is a guide line for $\chi = 0.4$. The number of independent runs are 10 000, 5000, 3000, 1500, 1200, 1000, and 800 for L=10, 20, 40, 60, 80, 100, and 120, respectively.

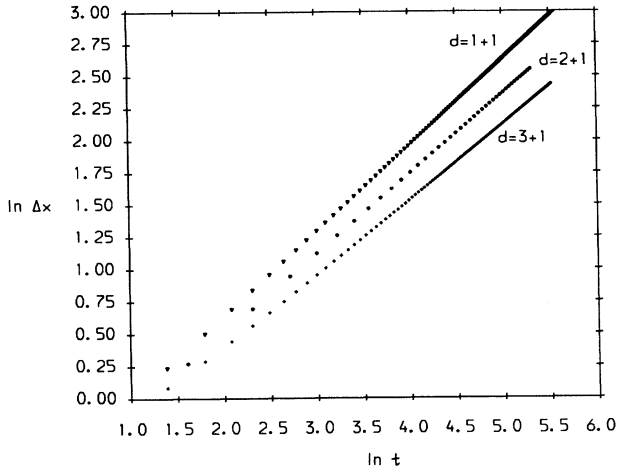

FIG. 5. Transverse fluctuation Δx as a function of length t with Gaussian $\mu(\mathbf{x},t)$ and $\gamma = 0.5$ (100 runs). In d=1+1, 0.5 is added to the $\ln \Delta x$ to avoid overlap with d=2+1.

before and indicating that the random potentials of different shape [Gaussian, uniform and $P(a)$ distributions] give the same exponents.

$\langle E(t) \rangle$ is obviously proportional to the polymer length t as $t \to \infty$, but what is the correction to this leading term? In the nonlinear growth model, such as the Eden model, ballistic growth, and the restricted-solid-on-solid growth model, it has been shown that the height average $\langle h \rangle$ is proportional to t with a correction of $0(t^{\omega})$ by a scaling argument [28]. With the assumed relation between $\langle E(t) \rangle$ and $\langle h(t) \rangle$, one expects that

$$V(t) \equiv \frac{d\langle E(t) \rangle}{dt} = C_1 + C_2 t^{-(1-\omega)} \tag{12}$$

for $t \ll L^z$ where C_1 and C_2 are constants. To test this relation, we have calculated $V(t)$ in d=2+1. Figure 6 shows a good agreement with the Eq. (12) with $\omega = 0.25$.

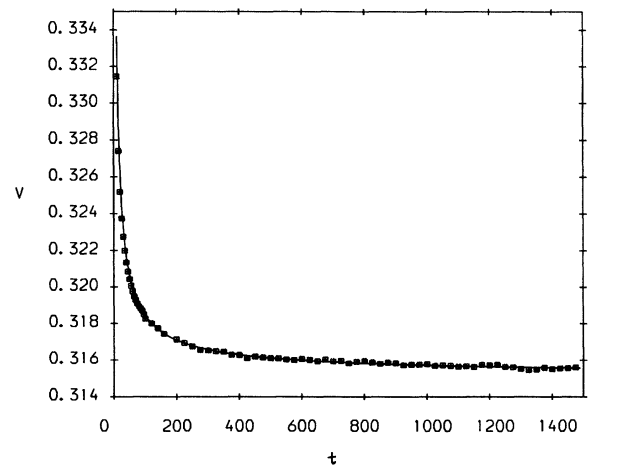

FIG. 6. $V(t) \equiv d\langle E(t) \rangle / dt$ as a function of t for uniform $\mu(\mathbf{x},t)$ with $\gamma = 0.5$ in d=2+1. The line is the best fit $V(t) = C_1 + C_2 t^{-3/4}$ ($C_1 = 0.315\,13$ and $C_2 = 0.104\,35$).

III. DISTRIBUTION FUNCTIONS FOR THE ENERGY

We have calculated the probability distribution $P(a,t)da$ at time t so that the scaled energy $a = (E - \langle E \rangle)/\Delta E$ is between a and $a + da$. Figure 7 shows a very good data collapse for ten different times implying that $P(a,t)$ rapidly tends to a scaling limit $P(a)$ independent of t. The curves $P(a)$ are asymmetric, i.e., $P(a) \neq P(-a)$. This asymmetry appears to increase slowly with dimension. The presence of asymmetry is not surprising given that in the mapping to ballistic aggregation [see Eq. (6)], $h \sim \ln Z \sim E$, but the nonlinear term in Eq. (6) breaks the symmetry between positive and negative values of h.

Since $P(a,t)$ seems to be independent of time for $t \ll L^z$, as shown by Fig. 7, the expectation value of higher moments ($m \geq 2$) should have the following form:

$$M_m \equiv \langle (E - \langle E \rangle)^m \rangle \sim (\Delta E)^m \sim t^{m\omega} . \tag{13}$$

Likewise, the nth cumulant should scale as $C_n(t) \simeq t^{n\omega}$ ($n \geq 2$). In d=1+1, we calculate the cumulants and moments up to fourth order. It is convenient to study the normalized nth cumulant

$$\hat{C}_n \equiv C_n(t)/[C_2(t)]^{n/2} \tag{14}$$

which should be t independent for large t. Figure 8 shows $\hat{C}_3$ and Fig. 9 shows $\hat{C}_4(t)$ for d=1+1. One would expect that the $t \to \infty$ limit of $\hat{C}_3(t)$ and $\hat{C}_4(t)$ would be universal, i.e., the same for all values of the bending energy γ and choice of site energy distribution function. However, this universality is not very obvious from Fig. 8, where $\hat{C}_3(t)$ for the uniform distribution of random site energies approaches the asymptotic value much more slowly than for the case of a Gaussian distribution. This is perhaps due to the fact that the asymptotic distribution of energies $P(E,t)$ is closer to a Gaussian than a uniform distribution. Again there is a certain value of γ at which $\hat{C}_3$ arrives at the asymptotic value more quickly. In fact, a measurement of $\hat{C}_3$ represents a sensitive test of whether data are in the scaling regime. We estimate that $\hat{C}_3 = -0.29 \pm 0.02$ in d=1+1 for uniform $\mu(x,t)$ with $\gamma \geq 1$ and for Gaussian $\mu(x,t)$. Since the asymmetry of $P(a)$ grows with d, $-\hat{C}_3$ becomes larger in higher dimensions.

We note that the standard replica calculation [29] seems to imply that $C_4 \equiv 0$ for d=1+1. Figure 9 shows that this is plainly not the case. The limitations of the replica method [29] are well known [30] and stem from the inappropriate interchange of the limit $t \to \infty$ and the replica index $n \to 0$.

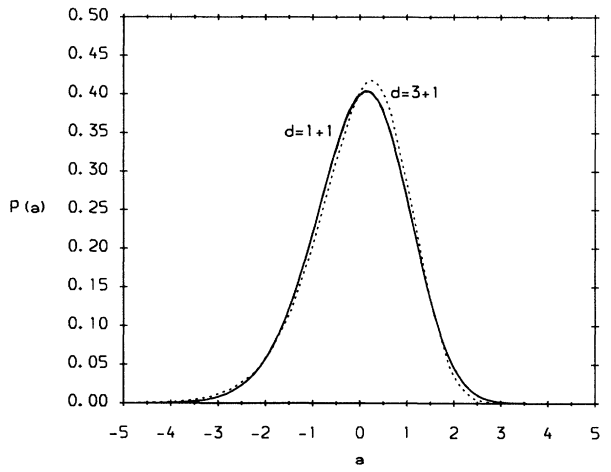

FIG. 7. The probability distribution $P(a,t)$ of minimum energy E as a function of $a = (E - \langle E \rangle)/\Delta E$ with Gaussian $\mu(\mathbf{x},t)$ and $\gamma = 0.5$ in d=1+1 and d=3+1. We collect many millions of $E(\mathbf{x},t)$ at ten different times (t=30,60,..., and 300 in d=1+1) and they collapse to one curve very well. There is a trend for the asymmetry of $P(a)$ to grow with dimension. The data for d=2+1 lie between the two curves shown, but have been omitted to avoid overcrowding.

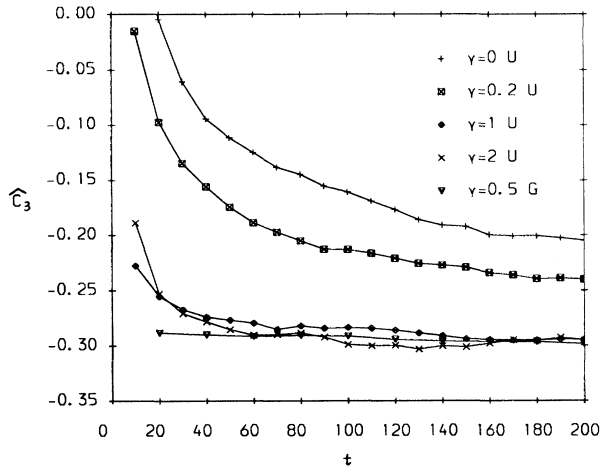

FIG. 8. The normalized third cumulant $\hat{C}_3$ as a function of length t for uniform (U) $\mu(\mathbf{x},t)$ with γ=0, 0.2, 1, and 2; and Gaussian (G) $\mu(\mathbf{x},t)$ with $\gamma = 0.5$ in d=1+1.

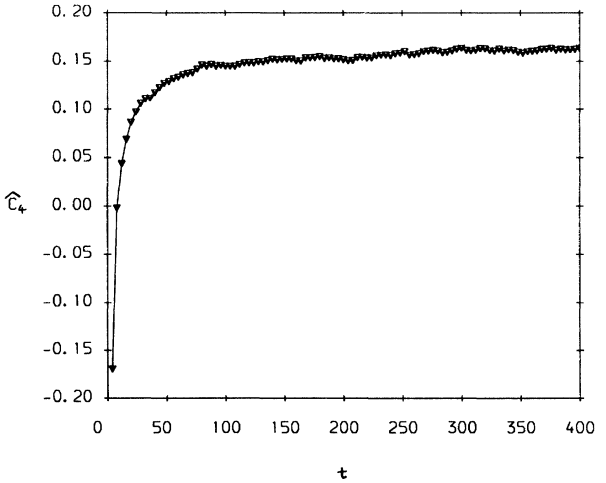

FIG. 9. The normalized fourth cumulant $\hat{C}_4$ as a function of length t for uniform $\mu(\mathbf{x},t)$ with $\gamma = 1$ in d=1+1.

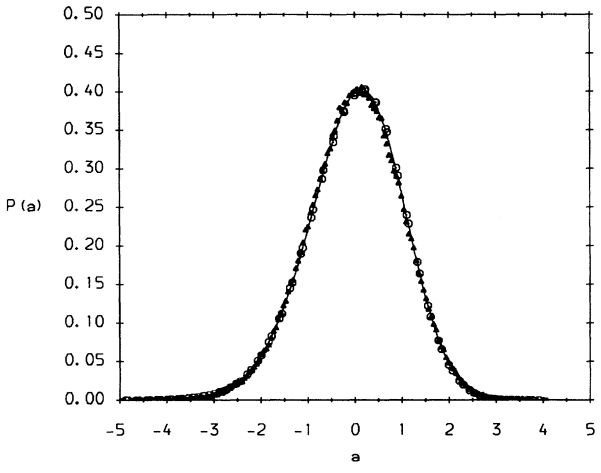

FIG. 10. The probability distribution $P(a,t)$ of height h as a function of $a = (h - \langle h \rangle)/\Delta h$ in a restricted-solid-on-solid growth model for d=1+1 (t=300, L=40 000 and 20 runs; circle) and the probability distribution $P_1(a,t)$ of E_1 as a function of $a = (E_1 - \langle E_1 \rangle)/\Delta E_1$ in our model (t=200 and 400 000 runs; triangle). The continuous line is for the $P(a)$ of Fig. 7 for d=1+1.

Zhang has suggested [31], using a replica argument that when a is large and negative $P(a) \sim e^{-c|a|^\zeta}$ with $\zeta = 1/(1-\omega)$. We have tried to fit the data with

$$P(a) \simeq \begin{cases} e^{-c_-|a|^{\zeta_-}}, & a < -1 \\ e^{-c_+|a|^{\zeta_+}}, & a > 1 \end{cases} \tag{15}$$

for $1 < |a| < 5$ where c_- and c_+ are constant. We found that $\zeta_- = 1.6 \pm 0.2$ and $\zeta_+ = 2.4 \pm 0.2$ in d=1+1, $\zeta_- = 1.5 \pm 0.2$ and $\zeta_+ = 2.6 \pm 0.2$ in d=2+1, and $\zeta_- = 1.45 \pm 0.2$ and $\zeta_+ = 2.7 \pm 0.2$ in d=3+1. Thus the numerical values of ζ_- are close to Zhang's prediction but a little larger.

We believe that the function $P(a)$ is a universal function. We have studied the distribution of heights in the restricted solid-on-solid model [10] in d=1+1. Defining $a = (h - \langle h \rangle)/\Delta h$, one sees from Fig. 10 that the scaling form of the height distribution $P(a)$ is identical to within numerical accuracy to $P(a)$ of Fig. 7 in our model of directed polymers. Even though the height is a discrete variable, the value of the normalized third cumulant in the model is around -0.29, which is the same as that of the directed polymers. This result strongly implies the universality of the function $P(a)$.

We have also calculated the minimum energy $E_1(\mathbf{x} = \mathbf{0}, \mathbf{x}_0 = \mathbf{0})$ for polymers with no final transverse displacement. The distribution function $P_1(a)$ is shown for d=1+1 in Fig. 10, where $a = (E_1 - \langle E_1 \rangle)/\Delta E_1$. To within the limits of our numerical accuracy the scaling form of P_1 seems to be identical to that of $P(a)$. We are unable to give any argument as to why the scaling form of these distributions should be the same.

IV. DISCUSSION

In Secs. II and III, we showed that, if allowance is made for corrections to scaling, then the exponents ω, χ, z and the normalized cumulants are universal, depending only on the dimension d. However, while few would doubt the universality of the exponents, the fact that we could provide only numerical arguments indicates what is the major problem in the whole area of strong-coupling disordered systems, namely the lack of a renormalization-group treatment for such systems. There have recently been attempts to remedy this situation [32], but it is clear that there still remains a long way to go. Moreover, a renormalization-group procedure for strong-coupling fixed points would have application to a number of important fields, e.g., spin-glass, random-field systems, as well as to problems connected with ballistic aggregation and directed polymers.

[1] M. Kardar and Y. C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[2] M. Eden, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probabilities*, edited by F. Neyman (University of California Press, Berkeley, 1961), p. 223.
[3] M. J. Vold, J. Colloid Sci. **14**, 168 (1959).
[4] F. Family and T. Vicsek, J. Phys. A **18**, L75 (1985).
[5] P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A **34**, 5091 (1986).
[6] D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985).
[7] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[8] J. M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Boston, 1974).
[9] D. E. Wolf and J. Kertész, Europhys. Lett. **4**, 651 (1987).
[10] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).
[11] W. Renz (unpublished).
[12] M. Kardar, Phys. Rev. Lett. **55**, 2923 (1985).
[13] J. Krug and H. Spohn, in *Solids Far From Equilibrium: Growth, Morphology and Defects*, edited by C. Godreche (Cambridge University Press, Cambridge, England, 1990).
[14] J. Krug, Phys. Rev. A **36**, 5465 (1987).
[15] E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A **39**, 3053 (1989).
[16] J. G. Zabolitzky and D. Stauffer, Phys. Rev. A **34**, 1523 (1986); Phys. Rev. Lett. **57**, 1809 (1986).
[17] A. J. McKane and M. A. Moore, Phys. Rev. Lett. **60**, 527 (1988).
[18] T. Halpin-Healy, Phys. Rev. Lett. **62**, 442 (1989).
[19] B. Derrida and R. B. Griffiths, Europhys. Lett **8**, 111 (1989).
[20] H. Yan, D. Kessler, and L. M. Sander, Phys. Rev. Lett. **64**, 926 (1990).
[21] Y. P. Pellegrini and R. Jullien, Phys. Rev. Lett. **64**, 1745

(1990).
[22] B. M. Forrest and L. H. Tang, Phys. Rev. Lett. **64**, 1405 (1990).
[23] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[24] A. Chakrabarti and R. Toral, Phys. Rev. B **40**, 11 419 (1989).
[25] H. Guo, B. Grossmann, and M. Grant, Phys. Rev. Lett. **64**, 1262 (1990).
[26] J. G. Amar and F. Family, Phys. Rev. A **41**, 3399 (1990).
[27] B. Derrida and O. Golinelli, Phys. Rev. A **41**, 4160 (1990).
[28] J. Krug and P. Meakin, J. Phys. A **23**, L987 (1990).
[29] M. Kardar, Nucl. Phys. B **290**, 582 (1987).
[30] J. P. Bouchaud and H. Orland, J. Stat. Phys. **61**, 877 (1990).
[31] Y. C. Zhang, Europhys. Lett. **9**, 113 (1989).
[32] M. Mezard and G. Parisi, J. Phys. A **23**, L1229 (1990); T. Nattermann and H. Leschhorn, Europhys. Lett. **14**, 603 (1991).