Nonclassical state from two pseudoclassical states

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The quantum-mechanical superposition of two coherent states of identical mean photon number but different phases yields a state that can exhibit sub-Poissonian and oscillatory photon statistics, as well as squeezing.

I. SUB-POISSONIAN PHOTON COUNT STATISTICS AND SQUEEZING FROM COHERENT STATES

At the heart of quantum mechanics lies the superposition principle—to quote from the first chapter of Dirac's classical treatise[1]— "...any two or more states may be superposed to give a new state." Insight into the far-reaching consequences of this principle is offered by the most elementary example of superposing two coherent states[2,3] of identical mean number of photons $\langle m \rangle = \alpha^2$ but with a phase difference φ as shown in Fig. 1. The analysis of the properties of such a state $|\psi\rangle$ as a function of φ constitutes the center of interest of the present paper.

A coherent state of an electromagnetic field mode, or in the language of its mechanical analog, of a harmonic oscillator with dynamically conjugate variables x and p, minimizes the uncertainty product with identical uncertainties $(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2}$. Thus they are quantum states closest to classical states—pseudoclassical states.[2,3] In contrast, the quantum-mechanical superposition of two such coherent states forming the state $|\psi\rangle$,[4] Eq. (2.1), exhibits highly nonclassical features,[5-7] such as sub-Poissonian[8] and oscillatory photon statistics[9,10] as well as squeezing[11] of the x variable.[12]

For an appropriately large displacement, α, and $\varphi_0 = \pi$, the state $|\psi\rangle$ can be interpreted as the quantum superposition of two macroscopically distinguishable states,[13,14] that is, a Schrödinger-cat-like-state. Consequently, these states have attracted a lot of interest.[15-21] In particular, they have been shown to be extremely fragile and sensitive to dissipation:[22] The decay of their interference properties is governed by their separation in phase space.[22,23] This makes it extremely difficult to detect such states. In contrast, the nonclassical features of the state $|\psi\rangle$ discussed in the present article make their appearance when the two coherent states are not distinguishable yet, and thus the decay of the interference properties is extremely slow, as discussed in Appendix A. Various ingenious mechanisms to produce such states have been suggested.[15-21,24] Hence we in this paper confine ourselves solely to the discussion of their properties.

The paper is organized as follows: In Sec. II we analyze the photon statistics, that is, the probability W_m of finding m photons in the state $|\psi\rangle$ in its dependence on the phase difference φ. Figure 2 shows φ domains in which the photon-count probability curve gets narrower than the Poisson distribution of a single coherent state, that is, we find sub-Poissonian photon statistics—an indicator of a nonclassical state. These domains are separated from each other by zones in which W_m is broader than a Poisson distribution, that is, super-Poissonian. The resulting oscillations in the normalized variance $\sigma^2 \equiv \langle m^2 \rangle / \langle m \rangle - \langle m \rangle$—displayed in Fig. 3 and similar to those in the photon statistics of the micromaser[25]—die when the two coherent states are distinguishable. As a consequence, W_m shows rapid oscillations with the familiar Poisson envelope. The analogous effect arises in the photon statistics of a highly squeezed state.[9,10] Section III deals with the question of possible squeezing[26] in $|\psi\rangle$. A single coherent state shows identical uncertainties $(\Delta x)^2$ and $(\Delta p)^2$ in the conjugate variables x and p equal to $\frac{1}{2}$. In contrast, in the state $|\psi\rangle$ the uncertainty $(\Delta x)^2$

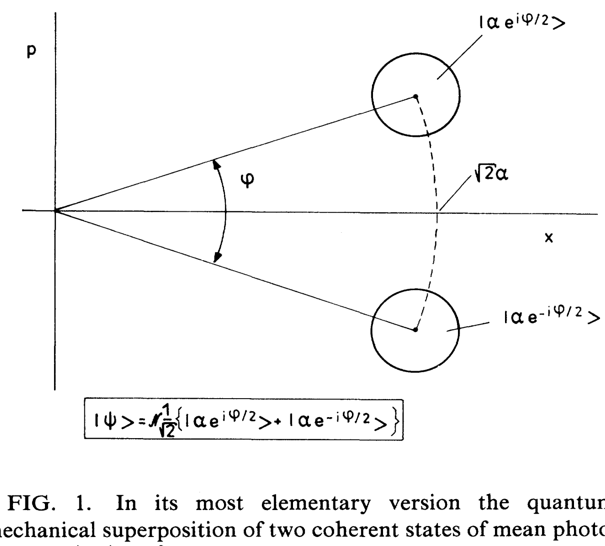

FIG. 1. In its most elementary version the quantum-mechanical superposition of two coherent states of mean photon number $\langle m \rangle = \alpha^2$ and phase difference φ can be visualized by two circles of radius unity displaced by an amount $\sqrt{2}\alpha$ from the origin and having the angle φ between them.

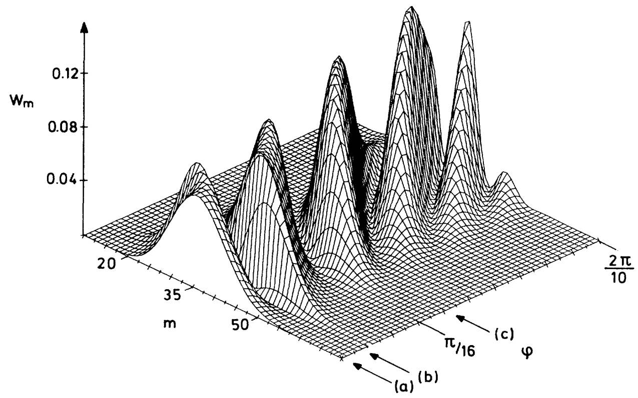

FIG. 2. The photon-count probability W_m of the quantum-mechanical superposition state $|\psi\rangle$ of Eq. (2.1) in its dependence on the relative phase difference φ. For increasing φ the Poissonian distribution for the case $\varphi=0$ (a) narrows while its maximum moves toward smaller m values (b). This curved wave front suddenly breaks off to start a new front and yields a distribution broader than a Poissonian with more than one maximum (c). (Here we have chosen $\alpha^2=36$.)

can fall below this coherent-state value provided the phase difference φ lies appropriately in the domain of sub-Poissonian photon statistics, as indicated in Fig. 4. Thus, $|\psi\rangle$ exhibits squeezing. Moreover, $(\Delta p)^2$ increases while preserving (approximately) the minimum uncertainty product $(\Delta x)^2(\Delta p)^2=\frac{1}{4}$. No squeezing in x is found in the region of super-Poissonian or oscillatory photon statistics. This example of the superposition of two coherent states exhibiting nonclassical effects such as sub-Poissonian statistics and squeezing in contrast to a single coherent state identifies once more in a striking way the so seemingly innocent principle of superposition as the main actor behind the scenes. In the same spirit is the interesting observation[27] that the quantum-mechanical superposition of two number states, such as the vacuum $|0\rangle$ and a one-photon state $|1\rangle$, shows squeezing, whereas neither the state $|0\rangle$ nor $|1\rangle$ shows any squeezing by itself. More insight into these nonclassical effects is offered by the phase-space considerations of Sec. IV within the framework of the Wigner function[28] or the concept of interference in phase space.[9,10,29] The Wigner function $P^{(W)}_{|\psi\rangle}$ for the state $|\psi\rangle$, displayed in Fig. 5, is not just the sum of the two Gaussian-bell Wigner functions of the two coherent states, but involves an interference term.[30,31] The phase difference φ can sharpen this peak and can even force it to assume negative values. In Sec. IV A we relate this Wigner-interference term to

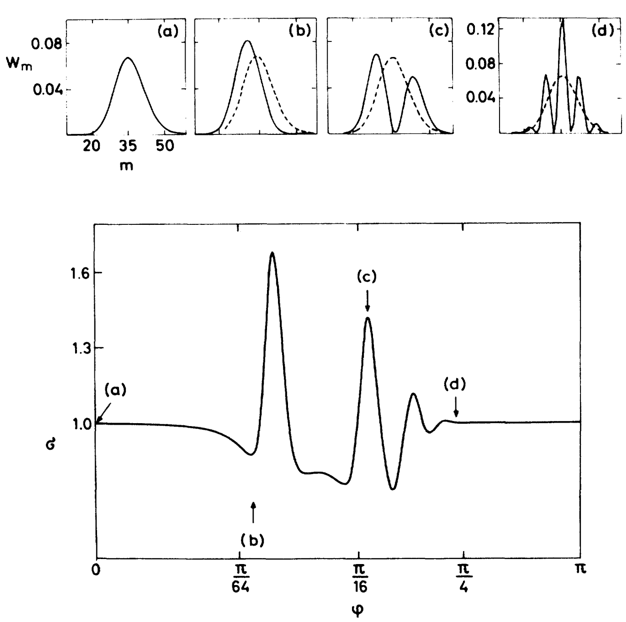

FIG. 3. The photon-count distribution W_m for the quantum-mechanical superposition of two coherent states, Eq. (2.1), is (a) Poissonian, (b) sub-Poissonian, (c) super-Poissonian, or (d) oscillatory depending on the relative phase φ between the two coherent states as expressed by the normalized variance σ, Eq. (2.9). The Poissonian distribution for $\varphi=0$ is plotted for comparison by a dashed line. To emphasize the oscillations in σ, we have chosen a logarithmic scale for φ. (Here we have chosen $\alpha^2=36$.)

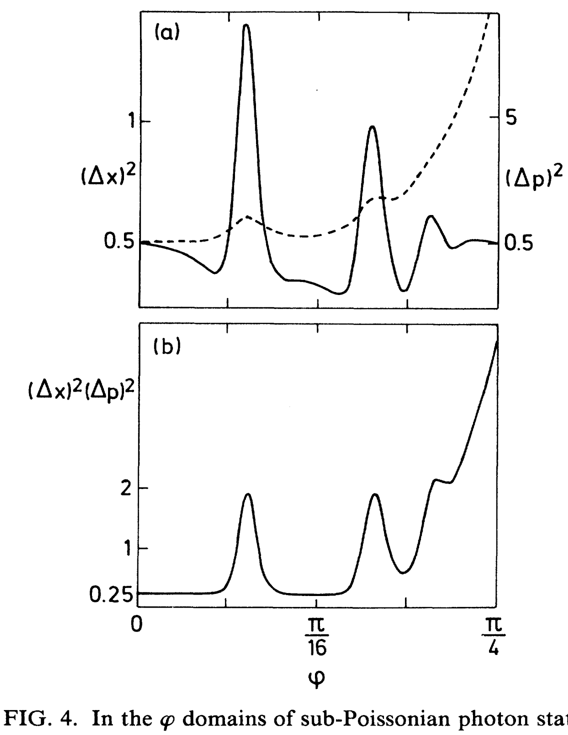

FIG. 4. In the φ domains of sub-Poissonian photon statistics (compare to Fig. 3) the x variable exhibits squeezing, that is, the variance $(\Delta x)^2$ depicted in (a) by a solid line falls below the coherent-state value of $(\Delta x)^2=0.5$. The variance $(\Delta p)^2$ shown in (a) by a dashed curve, however, never falls below 0.5, but increases such as to satisfy the uncertainty product $(\Delta x)^2(\Delta p)^2\geq\frac{1}{4}$ indicated in (b). In the first two domains of squeezing the state $|\psi\rangle$ is approximately a minimum uncertainty state whereas for $\varphi\geq\pi/8$, the x variable is still squeezed, but not in a minimum uncertainty state. (Here we have chosen $\alpha^2=36$.)

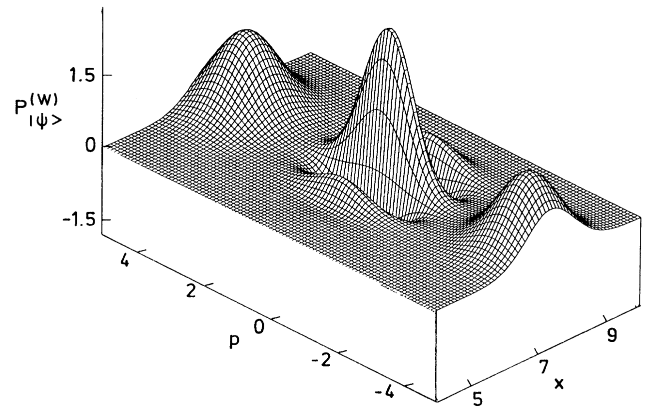

FIG. 5. The Wigner function $P^{(W)}_{|\psi\rangle}$ of the quantum-mechanical superposition of two coherent states does not consist of two Gaussian bells located in x-p oscillator phase space at $x=\sqrt{2}\alpha\cos(\varphi/2)$ and $p=\pm\sqrt{2}\alpha\sin(\varphi/2)$ corresponding to two individual coherent states $|\alpha e^{i\varphi/2}\rangle$ and $|\alpha e^{-i\varphi/2}\rangle$ but involves an interference term located on the x axis. This contribution originates from the quantum-mechanical superposition of the two coherent states and the bilinearity of the Wigner distribution, Eq. (4.1), in the wave function. This interference bell can be narrower in the x direction than the individual coherent-state Gaussian bells giving rise to squeezing in the x variable [Fig. 4(a)] or even take on negative values to create an oscillatory photon-count probability W_m, Fig. 3(d). (Here we have chosen $\alpha^2=36$ and $\varphi=\pi/3$.)

the nonclassical features of the photon-count probability W_m and the squeezing. In addition, the concept of area of overlap and interference in phase space of Sec. IV B grasps immediately the essential properties of the photon distribution W_m or the position probability curve

$$W_x=|\langle x|\psi\rangle|^2=|\psi(x)|^2 .$$

The two *interfering* areas of crossover between the two Gaussian bells[32] and the mth Planck-Bohr-Sommerfeld band or the thin phase-space highway located at x shown in Figs. 6 and 7, respectively, create the nonclassical phenomena in W_m or W_x. We conclude by summarizing our main results in Sec. V. In order to focus on the essential points, we banish all lengthy calculations to appendices.

II. SUB-POISSONIAN PHOTON STATISTICS VIA QUANTUM-MECHANICAL STATE SUPERPOSITION

In this section we discuss the photon-count probability W_m of a state[4,12]

$$|\psi\rangle=\mathcal{N}\frac{1}{\sqrt{2}}(|\alpha e^{i\varphi/2}\rangle+|\alpha e^{-i\varphi/2}\rangle) \tag{2.1}$$

built out of the quantum-mechanical superposition of two coherent states, $|\alpha\exp(\pm i\varphi/2)\rangle$, of real, positive displacement $\sqrt{2}\alpha$ in x-p oscillator phase space and real phase, $\pm\varphi/2$. Here we consider coherent states as defined[2,3] by the superposition

$$|\alpha e^{i\varphi}\rangle=\exp(-\tfrac{1}{2}\alpha^2)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}e^{im\varphi}|m\rangle \tag{2.2}$$

of number states $|m\rangle$.

As a result of the nonorthogonality of two coherent states,[2,3] $|\beta\rangle$ and $|\gamma\rangle$,

$$\langle\beta|\gamma\rangle=\exp[-\tfrac{1}{2}(|\beta|^2+|\gamma|^2)+\beta^*\gamma] , \tag{2.3}$$

the normalization constant $\mathcal{N}$ for this state takes a more complicated form,

$$\mathcal{N}^2(\varphi)=\frac{1}{[1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)]} , \tag{2.4}$$

where

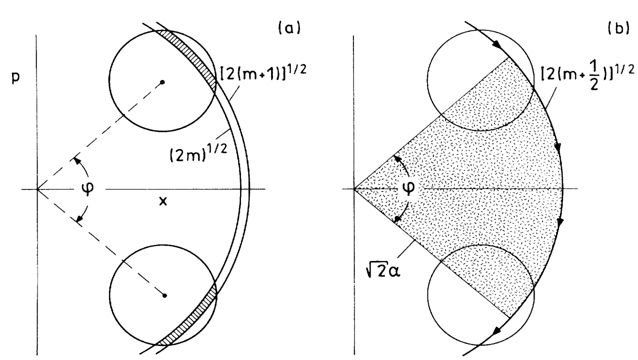

FIG. 6. Photon statistics of the superposition state $|\psi\rangle$ from interference in phase space. The mth Planck-Bohr-Sommerfeld band of the m-number state has *two* distinct areas of crossover $\mathcal{A}_m$ with the state $|\psi\rangle$ represented in phase space by two Gaussian bells and depicted in (a) in its most elementary version by circles. Interference between these zones of phase difference $2\phi_m$ given by the dotted phase-space domain of (b) gives rise to the oscillatory photon distribution W_m, shown in Fig. 3(d).

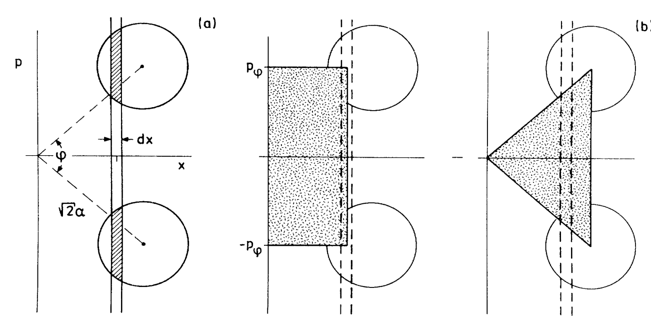

FIG. 7. Position probability of the superposition state $|\psi\rangle$ from interference in phase space. The phase-space strip parallel to the momentum axis and located at x—the representative of a position eigenstate—has *two* distinct areas of crossover $\mathcal{A}_x$ with the two Gaussian bells depicted in (a) in the most elementary way by circles. Interference between these domains with a phase difference governed by the dotted phase-space domain shown in (b) gives rise to the oscillatory position distribution, Eq. (3.5).

$$|p_\varphi| \equiv |\pm\sqrt{2}\alpha\sin(\varphi/2)| \ . \tag{2.5}$$

Here, p_φ denotes the "mean momentum" of a single of the two coherent states. In the definition of the state $|\psi\rangle$, Eq. (2.1), we have separated the normalization factor $1/\sqrt{2}$ which arises naturally in the superposition of two *orthogonal* states from the factor $\mathcal{N}$ measuring the *nonorthogonality* of the two interfering coherent states.

The probability W_m to find m photons in $|\psi\rangle$ is given by

$$W_m[|\psi\rangle] = |\langle m|\psi\rangle|^2 = \mathcal{N}^2 \tfrac{1}{2} |\langle m|\alpha e^{i\varphi/2}\rangle + \langle m|\alpha e^{-i\varphi/2}\rangle|^2 \ . \tag{2.6}$$

With the help of Eq. (2.2), that is,

$$\langle m|\beta\rangle = \frac{\beta^m}{(m!)^{1/2}} \exp(-\tfrac{1}{2}|\beta|^2) \ ,$$

we find

$$W_m[|\psi\rangle] = |\mathcal{A}_m^{1/2}\exp(i\phi_m) + \mathcal{A}_m^{1/2}\exp(-i\phi_m)|^2 \tag{2.7}$$

or

$$W_m[|\psi\rangle] = 4\mathcal{A}_m\cos^2\phi_m \ . \tag{2.8a}$$

Here we have introduced

$$\mathcal{A}_m \equiv \mathcal{N}^2 \tfrac{1}{2} \frac{\alpha^{2m}}{m!} e^{-\alpha^2} \tag{2.8b}$$

and

$$\phi_m \equiv m\varphi/2 \ . \tag{2.8c}$$

Hence the superposition of the two coherent states—two contributors of interfering probability amplitude $\langle m|\alpha e^{\pm i\varphi/2}\rangle$ as expressed by Eqs. (2.6) and (2.7)—creates the interference term $\cos^2\phi_m$, which modulates the familiar Poissonian statistics of a single coherent state.

In Fig. 2 we analyze the consequences of this contribution in more detail by depicting the photon-count probability W_m of Eq. (2.8), as a function of quantum number m and the phase difference φ. All curves here are plotted for definiteness for the same value $\alpha=6$ of the displacement parameter α. For a vanishing phase angle, that is, $\varphi=0$, the two coherent states are on top of each other, that is, $|\psi\rangle$ is a single coherent state and W_m is a Poisson distribution [Fig. 2(a)]. When we increase φ, the photon-count probability narrows, having a slightly shifted, higher maximum as shown in Fig. 2(b). This narrowing effect stands out most clearly when we compare and contrast the initial Poisson distribution of $\varphi=0$ [solid line in Fig. 3(a) and dashed lines in Figs. 3(b)–3(d)], to the photon statistics W_m for the special φ value indicated in the lower part of Fig. 3 by (b).

Mathematically, we describe this phenomenon by the normalized variance

$$\sigma^2 \equiv \langle m^2\rangle/\langle m\rangle - \langle m\rangle = 1 - 4\alpha^2\sin^2(\varphi/2)\exp[-2\alpha^2\sin^2(\varphi/2)] \times \frac{\cos(\alpha^2\sin\varphi+\varphi)+\cos^2(\varphi/2)\exp[-2\alpha^2\sin^2(\varphi/2)]}{\{1+\cos(\alpha^2\sin\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]\}\{1+\cos(\alpha^2\sin\varphi+\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]\}} \ . \tag{2.9}$$

In the last step we have evaluated the moments $\langle m\rangle \equiv \sum_{m=0}^{\infty} mW_m$ and $\langle m^2\rangle \equiv \sum_{m=0}^{\infty} m^2W_m$ of the distribution W_m, Eq. (2.8), shown in detail in Appendix B.

The lower part of Fig. 3 depicts the so-calculated variance σ as a function of the phase difference φ for fixed displacement $\alpha^2=36$. For $\varphi=0$, that is, for a coherent state with Poisson statistics, we find from Eq. (2.9), $\sigma(\varphi=0)=1$. Values $\sigma<1$ define sub-Poisson statistics whereas $\sigma>1$ indicate super-Poisson statistics. A phase difference φ such that $(\alpha\varphi)^2/2 \ll 1$ reduces Eq. (2.9) to

$$\sigma^2 \cong 1-(\alpha\varphi)^2/2 \ . \tag{2.10}$$

Consequently, there exists a range of φ values, shown in the lower part of Fig. 3, in which the photon-count probability W_m of $|\psi\rangle$ shows a substantial amount of sub-Poisson statistics. This is a remarkable result when we recall that the transition from the Poisson distribution of a coherent state to the sub-Poissonian was induced solely by the superposing of the two coherent states. This example illustrates in a striking way the power of the superposition principle.

When we increase φ further, the first "wave front" of Fig. 2 bends to the left and abruptly a second wave train breaks off, giving rise to the two peaks in W_m. As a result, the photon-count probability is broader than a Poisson distribution, that is, a super-Poissonian with $\sigma>1$ as shown in the lower part of Fig. 3. For even larger φ, this second wave front again gains height, indicating the recurrence of the narrowing of W_m to a sub-Poissonian distribution. However, this narrowing again gets abruptly interrupted by the sudden breakoff of the third wave front, at the phase value (c) of Fig. 2, again leading to super-Poissonian statistics. The transitions between $\sigma<1$ and $\sigma>1$ occur when the second contribution in Eq. (2.9) changes sign, that is, at phases $\varphi^{(\sigma)}$, satisfying the transcendental equation

$$\cos(\alpha^2\sin\varphi^{(\sigma)}+\varphi^{(\sigma)}) + \cos^2(\varphi^{(\sigma)}/2)\exp[-2\alpha^2\sin^2(\varphi^{(\sigma)}/2)] = 0 \ . \tag{2.11}$$

The oscillatory term in σ^2, Eq. (2.9), is damped by the term $\exp[-2\alpha^2\sin^2(\varphi/2)]$ resulting from the nonorthogonality of the coherent states, Eqs. (2.3). Hence, the oscillations in σ only appear for φ values when the two coherent states are not distinguishable yet, that is,

$$|p_\varphi| = \sqrt{2}\alpha \sin(\varphi/2) < 1 \ .$$

For larger φ values (in this specific example, $\varphi \geq \pi/4$), the wave fronts align themselves more and more parallel to the φ axis. This results in multipeaked photon distributions W_m, as shown in Fig. 3(d), that is, in an oscillatory photon distribution with a Poissonian envelope. This example demonstrates that $\sigma \cong 1$ does not imply Poisson statistics since the second moment of W_m is not sensitive to the oscillations in W_m. Moreover, for $\varphi \geq \pi/2$ the two coherent states are distinguishable and the oscillations in W_m merely reflect this fact analogously to the rapid variation in the photon statistics of a highly squeezed state.[9,10] This analogy stands out most clearly when we compare Eqs. (2.7) and (2.8) to the corresponding equations of Refs. 9 and 10. We conclude this section by noting that the oscillations in σ are very reminiscent of those in the corresponding quantity in the photon statistics of the one-atom maser.[25]

III. SQUEEZING VIA QUANTUM-MECHANICAL STATE SUPERPOSITION

In this section we analyze the uncertainties

$$(\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = \int_{-\infty}^{\infty} dx\, x^2 W_x - \left[\int_{-\infty}^{\infty} dx\, x W_x \right]^2 \tag{3.1a}$$

and

$$(\Delta p)^2 \equiv \langle p^2 \rangle - \langle p \rangle^2 = \int_{-\infty}^{\infty} dp\, p^2 W_p - \left[\int_{-\infty}^{\infty} dp\, p W_p \right]^2 \tag{3.1b}$$

in the dynamically conjugate variables x and p for the superposition state $|\psi\rangle$, Eq. (2.1). The corresponding position and momentum distributions W_x and W_p follow from

$$W_x = |\langle x|\psi\rangle|^2 = |\psi(x)|^2$$

and

$$W_p = |\langle p|\psi\rangle|^2 = |\psi(p)|^2 \ .$$

A single coherent state obeys

$$(\Delta x)^2_{\text{coh}} = (\Delta p)^2_{\text{coh}} = \tfrac{1}{2} \tag{3.2}$$

and therefore maintains the minimum uncertainty relation

$$(\Delta x)^2 (\Delta p)^2 = \tfrac{1}{4} \ . \tag{3.3}$$

However, a system in the state $|\psi\rangle$, Eq. (2.1), can exhibit fluctuations in x below the coherent-state limit provided the two coherent states $|\alpha e^{i\varphi/2}\rangle$ and $|\alpha e^{-i\varphi/2}\rangle$ have the appropriate phase difference φ. Thus, for certain values of φ, the state $|\psi\rangle$ is a squeezed state.[26] Moreover, in these regions the uncertainty $(\Delta p)^2$ increases such that the minimum uncertainty relation, Eq. (3.3), is (approximately) maintained. The state $|\psi\rangle$ is hence (approximately) a minimum uncertainty squeezed state.

We study these squeezing phenomena in more detail by starting from the wave function of $|\psi\rangle$ in x representation,

$$\psi(x) = \langle x|\psi\rangle = \pi^{-1/4} \mathcal{N} \frac{1}{\sqrt{2}} \exp[-\alpha^2 \sin^2(\varphi/2)] \exp[(i/2)\alpha^2 \sin\varphi] \times \{ \exp[-\tfrac{1}{2}(x - \sqrt{2}\alpha e^{i\varphi/2})^2] + \exp(-i\alpha^2 \sin\varphi) \exp[-\tfrac{1}{2}(x - \sqrt{2}\alpha e^{-i\varphi/2})^2] \} \ . \tag{3.4}$$

In the last step we have used the fact[3] that for any coherent state $|\beta\rangle$,

$$\langle x|\beta\rangle = \pi^{-1/4} \exp[\tfrac{1}{2}(\beta^2 - |\beta|^2)] \exp[-\tfrac{1}{2}(x - \sqrt{2}\beta)^2] \ .$$

We thus find from Eq. (3.4),

$$W_x \equiv |\psi(x)|^2 = 4\mathcal{A}_x \cos^2\phi_x \ , \tag{3.5a}$$

where

$$\mathcal{A}_x \equiv \tfrac{1}{2} \mathcal{N}^2 \pi^{-1/2} \exp\{ -[x - \sqrt{2}\alpha \cos(\varphi/2)]^2 \} \tag{3.5b}$$

and

$$\phi_x \equiv \sqrt{2}\alpha \sin(\varphi/2)[x - \tfrac{1}{2}\sqrt{2}\alpha \cos(\varphi/2)] \ . \tag{3.5c}$$

The probability distribution $W_x \equiv |\psi(x)|^2$ in the x variable takes a form similar to the photon distribution W_m of Eq. (2.8): The Gaussian distribution $\mathcal{A}_x[|\psi\rangle]$—analogous to $\mathcal{A}_m[|\psi\rangle]$ of Eq. (2.8b)—centered at $x_\varphi = \sqrt{2}\alpha \cos(\varphi/2)$—the x coordinate of the center of the coherent states $|\alpha e^{\pm i\varphi/2}\rangle$, shown in Fig. 1—is modulated by the oscillatory function $\cos^2\phi_x$ as a result of the quantum-mechanical superposition of the two states. The phase ϕ_x of this modulation, Eq. (3.5c), has its zero at $x = \tfrac{1}{2}x_\varphi$, that is, different from the x location of the maximum of the Gaussian, $\mathcal{A}_x$. This is in complete accordance with the photon distribution W_m, Eq. (2.8): maximum of $\mathcal{A}_m$ for $m \sim \alpha^2$ and zero of ϕ_m at $m \sim 0$. We recall the $\cos^2\phi_m$ contribution as the origin of the sub-Poissonian photon statistics of the state $|\psi\rangle$. The close similarity between Eqs. (2.8) and (3.5) hence suggests the analog for the x distribution—a distribution narrower than that of a coherent state, $(\Delta x)^2_{\rm coh}=\frac{1}{2}$, that is, squeezing.

We start the discussion of the variance $(\Delta x)^2$, Eq. (3.1a), by presenting the "mean position"

$$\langle x\rangle=\sqrt{2}\alpha\cos(\varphi/2)-\sqrt{2}\alpha\sin(\varphi/2)\exp[-2\alpha^2\sin^2(\varphi/2)]\frac{\sin(\alpha^2\sin\varphi)}{1+\cos(\alpha^2\sin\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]}$$

of a particle in the state $|\psi\rangle$, Eq. (2.1), as calculated in Appendix C. When the two coherent states are distinguishable, that is, $p_\varphi=\sqrt{2}\alpha\sin(\varphi/2)>1$, the moment $\langle x\rangle$ is identical to the x value of the center of the two coherent states, as suggested by Fig. 1. However, for small φ values or small displacements α, that is, $|p_\varphi|\ll 1$, when the Gaussian bells overlap, the mean position $\langle x\rangle$ is quite different from this geometrically determined value.

We now turn to the variance $(\Delta x)^2$, Eq. (3.1a). When we substitute Eq. (3.5) into Eq. (3.1a) and perform the integrations, we find after minor algebra shown in Appendix C,

$$(\Delta x)^2=\frac{1}{2}\left[1-4\alpha^2\sin^2(\varphi/2)\exp[-2\alpha^2\sin^2(\varphi/2)]\frac{\cos(\alpha^2\sin\varphi)+\exp[-2\alpha^2\sin^2(\varphi/2)]}{\{1+\cos(\alpha^2\sin\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]\}^2}\right], \tag{3.6}$$

a result quite similar to the variance σ, Eq. (2.9), of the photon distribution W_m.

The solid line in Fig. 4(a) depicts the uncertainty $(\Delta x)^2$, Eq. (3.6), as a function of the phase difference φ for a fixed displacement $\alpha^2=36$. We note that $(\Delta x)^2$ repeatedly falls below the coherent-state value of 0.5, Eq. (3.2), thus indicating squeezing in the x variable. Moreover, φ domains of no squeezing follow domains of squeezing analogous to the oscillations in the variance σ of the photon distribution W_m. The phases, $\varphi^{(x)}$ separating the domains of squeezing and no squeezing are given by the zeros of the second contribution of Eq. (3.6), that is,

$$\cos(\alpha^2\sin\varphi^{(x)})+\exp[-2\alpha^2\sin^2(\varphi^{(x)}/2)]=0 . \tag{3.7}$$

This equation is quite similar to the corresponding equation (2.11), determining the phase angles $\varphi^{(\sigma)}$, separating zones of sub- and super-Poissonian statistics. We recall that due to the exponential decay term $\exp[-2\alpha^2\sin^2(\varphi/2)]$ in Eqs. (2.9) and (3.6), these nonclassical effects only appear for phase angles $\varphi\le 2\arcsin(\sqrt{2}\alpha)^{-1}$. Hence for $\alpha^2\gg 1$, the approximations

$$\cos(\alpha^2\sin\varphi+\varphi)\cong\cos(\alpha^2\sin\varphi)$$

and

$$\cos^2(\varphi/2)\cong 1$$

make (apart from the prefactor $\frac{1}{2}$) the two expressions for σ^2 and $(\Delta x)^2$, Eqs. (2.9) and (3.6), and, hence, Eqs. (2.11) and (3.7) identical. Therefore, φ regions exhibiting sub-Poissonian photon-count probability also show squeezing in the x variable, as shown in Figs. 3 and 4(a) for the case of $\alpha^2=36$. For small values of α, however, these two regions do not coincide.

We now evaluate the uncertainty $(\Delta p)^2$, Eq. (3.1b), in the conjugate variable p by starting from the momentum distribution

$$W_p=|\psi(p)|^2=\tfrac{1}{2}\mathcal{N}^2((\pi^{-1/2}\exp\{-[p-\sqrt{2}\alpha\sin(\varphi/2)]^2\}+\pi^{-1/2}\exp\{-[p+\sqrt{2}\alpha\sin(\varphi/2)]^2\}$$
$$+2\pi^{-1/2}\cos(\alpha^2\sin\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]\exp(-p^2)) \tag{3.8}$$

calculated in Appendix D. After minor algebra outlined in Appendix E, we find

$$\langle p\rangle=0$$

and

$$(\Delta p)^2=\frac{1}{2}\left[1+4\alpha^2\sin^2(\varphi/2)\frac{1}{1+\cos(\alpha^2\sin\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]}\right]. \tag{3.9}$$

Hence a particle in the state $|\psi\rangle$, Eq. (2.1), has zero mean momentum, as suggested by the symmetry of Fig. 1. The second contribution of $(\Delta p)^2$, Eq. (3.9), is positive for all φ values. Hence no squeezing in the p variable is possible, as shown in Fig. 4(a) by the dashed line.

More insight into these quantities is offered by considering Eqs. (3.6) and (3.9) in the limit of small phase angles φ, that is, for $(\alpha\varphi)^2/2\ll 1$, or large angles, that is, for $(\alpha\varphi)^2/2\gg 1$.

In the case of $(\alpha\varphi)^2/2\ll 1$, Eq. (3.9) reduces to

$$(\Delta p)^2\cong\tfrac{1}{2}[1+(\alpha\varphi)^2/2] , \tag{3.10a}$$

and thus yields an uncertainty larger than the corresponding $\frac{1}{2}$ value of a coherent state, Eq. (3.2).

On the other hand, in the same limit the uncertainty in

x, Eq. (3.6), simplifies to

$$(\Delta x)^2 \cong \tfrac{1}{2}[1-(\alpha\varphi)^2/2] , \qquad (3.10b)$$

confirming the squeezing in the x variable displayed in Fig. 4. We note the analogy to the corresponding expression for the nonclassical variance σ, Eq. (2.10). Moreover, from Eq. (3.10) we find

$$(\Delta x)^2(\Delta p)^2 \cong \tfrac{1}{4} ,$$

that is, for this particular choice of parameters the state $|\psi\rangle$ is approximately a minimum uncertainty squeezed state, Eq. (3.3), as shown in Fig. 4(b). Here we have depicted the product $(\Delta x)^2(\Delta p)^2$ given by Eqs. (3.6) and (3.9).

In the φ zone of oscillatory photon distribution, that is, when $(\alpha\varphi)^2/2 \gg 1$, Eq. (3.6) reduces to

$$(\Delta x)^2 \cong \tfrac{1}{2} ,$$

that is, to the uncertainty of a coherent state, Eq. (3.2), whereas Eq. (3.9) simplifies to

$$(\Delta p)^2 \cong \tfrac{1}{2}[1+4\alpha^2 \sin^2(\varphi/2)] ,$$

that is, the sum of the p width $(=\frac{1}{2})$ of the two coherent states and their relative displacement in the p direction,

$$2p_\varphi^2 = 2[\sqrt{2}\alpha \sin(\varphi/2)]^2 .$$

The uncertainty product then reads

$$(\Delta x)^2(\Delta p)^2 \cong \tfrac{1}{4}[1+4\alpha^2 \sin^2(\varphi/2)] .$$

Thus, the state $|\psi\rangle$ does not remain a minimum uncertainty state, as shown in Fig. 4(b).

We conclude this section by emphasizing again that whereas a single coherent states does not exhibit squeezing, a state built out of the superposition of two coherent states can exhibit a considerable amount of squeezing.[11,12] This example illustrates in a striking way the power of the superposition principle in quantum mechanics promoting two pseudoclassical states to a single, highly nonclassical state.[5–7]

IV. CONSIDERATIONS IN PHASE SPACE

More insight into the power of the superposition principle and its consequences for nonclassical features springs from the Wigner distribution[30] $P^{(W)}_{|\psi\rangle}$ of the state $|\psi\rangle$ on the one hand and the area-of-overlap approach[29] on the other.

A. Wigner function approach

One possible representation of the state $|\psi\rangle$, Eq. (2.1), in x-p oscillator phase space consists of the Wigner function[28]

$$P^{(W)}_{|\psi\rangle}(x,p) = \pi^{-1}\int_{-\infty}^{\infty} dy\, \exp(2ipy)\psi^*(x+y)\psi(x-y) , \qquad (4.1)$$

where $\psi(x)=\langle x|\psi\rangle$ is the x representation of the state $|\psi\rangle$, Eq. (3.4).

When we substitute Eq. (3.4) into Eq. (4.1) and perform the integration, we arrive after minor algebra shown in Appendix D at

$$P^{(W)}_{|\psi\rangle} = \tfrac{1}{2}\mathcal{N}^2\left(P^{(W)}_{|\alpha\exp(i\varphi/2)\rangle} + P^{(W)}_{|\alpha\exp(-i\varphi/2)\rangle} + \mathcal{P}_{\text{int}}\right) , \qquad (4.2a)$$

where

$$P^{(W)}_{|\alpha\exp(\pm\varphi/2)\rangle} \equiv \pi^{-1}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2\} \times \exp\{-[p \mp \sqrt{2}\alpha \sin(\varphi/2)]^2\} \qquad (4.2b)$$

denotes the Wigner function of a single coherent state[32] of displacement α and phase $\pm\varphi/2$. The interference term

$$\mathcal{P}_{\text{int}}(x,p) \equiv \frac{2}{\pi}\cos\{2\sqrt{2}\alpha\sin(\varphi/2)[x-\tfrac{1}{2}\sqrt{2}\alpha\cos(\varphi/2)]\} \times \exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2 - p^2\} \qquad (4.2c)$$

arises as a consequence of the bilinearity of the Wigner distribution, Eq. (4.1), in the wave function. Therefore, $P^{(W)}_{|\psi\rangle}$ is not the sum of the two Wigner functions, $P^{(W)}_{|\alpha e^{\pm i\varphi/2}\rangle}$, Eq. (4.2b), of the two coherent states but involves the Gaussian bell $\mathcal{P}_{\text{int}}$, Eq. (4.2c), located at the positive x axis at $x=\sqrt{2}\alpha\cos(\varphi/2)$ and modulated by the oscillatory function of phase $2\phi_x$, Eq. (3.5c). We note from Eq. (4.2) that the local widths of the three peaks in the variable p are identical and equal to unity. Moreover, they are independent of φ. The same holds true for the width in the x direction of $P^{(W)}_{|\alpha e^{\pm i\varphi/2}\rangle}$. In contrast, the x width of the interference term, $\mathcal{P}_{\text{int}}$, Eq. (4.2c) due to the $\cos(2\phi_x)$ modulation, strongly depends on φ. For appropriate values of φ, it gets narrower than the coherent-state Gaussian bell, Eq. (4.2b), that is, it causes the squeezing discussed in Sec. III. In a quite different φ region, however, domains in phase space exist in which $\mathcal{P}_{\text{int}}$ takes on negative as well as positive values—ditches in phase space, as illustrated in Fig. 5. These Wigner wave crests and troughs are the origin of these nonclassical features of the superposition state $|\psi\rangle$.

We now focus on the interference term $\mathcal{P}_{\text{int}}$ giving rise to the photon-count probability W_m, Eq. (2.6). In the Wigner function approach the probability W_m to find m photons in the state $|\psi\rangle$ is given by the phase-space integral[28]

$$W_m = 2\pi\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, P^{(W)}_{|\psi\rangle}(x,p) P^{(W)}_m(x,p) , \qquad (4.3)$$

where $P^{(W)}_{|\psi\rangle}$ is the Wigner function of the state $|\psi\rangle$, Eq. (4.2), and

$$P^{(W)}_m(x,p) = \frac{(-1)^m}{\pi}\exp[-(x^2+p^2)]L_m(2(x^2+p^2)) \qquad (4.4)$$

denotes the Wigner function of the mth number state. Here L_m is the mth Laguerre polynomial.[33]

When we substitute Eqs. (4.2a) into Eq. (4.3), we arrive at

$$W_m = \tfrac{1}{2}\mathcal{N}^2\{W_m[|\alpha\exp(i\varphi/2)\rangle] + W_m[|\alpha\exp(-i\varphi/2)\rangle] + W_m[\text{int}]\}$$
$$\equiv \tfrac{1}{2}\mathcal{N}^2\left[2\pi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\, P^{(W)}_{|\alpha\exp(i\varphi/2)\rangle}P^{(W)}_m + 2\pi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\, P^{(W)}_{|\alpha\exp(-i\varphi/2)\rangle}P^{(W)}_m + 2\pi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\, \mathcal{P}_{\text{int}}P^{(W)}_m\right] . \tag{4.5}$$

The photon-count probability, W_m is thus given by (1) the overlap in phase space between the two coherent states $|\alpha\exp(\pm i\varphi/2)\rangle$ represented by the corresponding Wigner functions $P^{(W)}_{|\alpha\exp(\pm i\varphi/2)\rangle}$, Eq. (4.2b), and the mth number state Wigner function $P^{(W)}_m$, Eq. (4.4), which yields the standard Poisson result[32]

$$W_m[|\alpha\exp(\pm i\varphi/2)\rangle] = \frac{\alpha^{2m}}{m!}e^{-\alpha^2} \tag{4.6a}$$

and (2) by the overlap between the interference term $\mathcal{P}_{\text{int}}$, Eq. (4.2c), with the mth number state function $P^{(W)}_m$ providing

$$W_m[\text{int}] = 2\frac{\alpha^{2m}}{m!}e^{-\alpha^2}\cos(m\varphi) . \tag{4.6b}$$

The details of the integration may be found in Appendix F.

When we substitute Eq. (4.6) into Eq. (4.5), we arrive at Eq. (2.8). We can hence trace back the origin of the nonclassical features of $|\psi\rangle$, that is, of the $\cos^2\phi_m$ contribution, to the interference contribution $\mathcal{P}_{\text{int}}$ in the Wigner function phase-space representation of $|\psi\rangle$.

B. Area of overlap and interference in phase space

In this section we bring out the striking features of the photon-count probability, $W_m \equiv |\langle m|\psi\rangle|^2$, Eq. (2.8), and the position distribution $W_x \equiv |\langle x|\psi\rangle|^2$, Eq. (3.5), of the state $|\psi\rangle$, Eq. (2.1), using the two central ingredients of the concept of *interference in phase space*:[29] (1) In the semiclassical limit the quantum-mechanical scalar product between two states is governed by the area of overlap between the two states represented in phase space. (2) In the case of two or more distinct zones of crossover, the corresponding contributions have to be added with a phase difference given by the area caught between the center lines of the state. We focus the present discussion on the case $p_\varphi = \sqrt{2}\alpha\sin(\varphi/2) \gg 1$, that is, on a case when the two coherent states do not have considerable overlap, that is, when they are distinguishable.

1. Photon-count probability

In the semiclassical limit the mth number state $|m\rangle$ can be represented[9,10,29,32] in phase space as a circular Planck-Bohr-Sommerfeld band with its inner edge given by $r^{(\text{in})}_m = (2m)^{1/2}$ and its outer edge by

$$r^{(\text{out})}_m = [2(m+1)]^{1/2} .$$

A single coherent state, $|\alpha e^{i\varphi/2}\rangle$, we depict[32] by a Gaussian bell, Eq. (4.2b), located at $x = \sqrt{2}\alpha\cos(\varphi/2)$ and $p = \sqrt{2}\alpha\sin(\varphi/2)$. The probability[32]

$$W_m[|\alpha\exp(i\varphi/2)\rangle] \equiv |\langle m|\alpha e^{i\varphi/2}\rangle|^2 = \frac{\alpha^{2m}}{m!}\exp(-\alpha^2) \cong (2\pi)^{-1/2}\alpha^{-1} \times\exp\left[-\left[\frac{m+\frac{1}{2}-\alpha^2}{\sqrt{2}\alpha}\right]^2\right] \tag{4.7a}$$

of finding m photons in a coherent state of large displacement $\alpha \gg 1$ is governed by the area of the overlap[32]

$$A_m[|\alpha\exp(i\varphi/2)\rangle] \equiv \int dx\int dp\, P^{(W)}_{|\alpha\exp(i\varphi/2)\rangle}(x,p) \cong (2\pi)^{-1/2}\alpha^{-1}\exp\left[-\left[\frac{m+\frac{1}{2}-\alpha^2}{\sqrt{2}\alpha}\right]^2\right] = W_m , \tag{4.7b}$$

where $\int dx\int dp$ is over the mth band, between this Gaussian bell and the mth Planck-Bohr-Sommerfeld band.

We depict the superposition state, Eq. (2.1), in its elementary way by *two* Gaussian bells, Eq. (4.2b), located at $x = \sqrt{2}\alpha\cos(\varphi/2)$ and $p = \pm\sqrt{2}\alpha\sin(\varphi/2)$, that is,

$$P_{|\psi\rangle}(x,p) \equiv \tfrac{1}{2}[P^{(W)}_{|\alpha\exp(i\varphi/2)\rangle}(x,p) + P^{(W)}_{|\alpha\exp(-i\varphi/2)\rangle}(x,p)] . \tag{4.8}$$

Here we have introduced the factor $\frac{1}{2}$ to ensure the normalization

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\, P_{|\psi\rangle}(x,p) = 1 .$$

Hence the mth Planck-Bohr-Sommerfeld band enjoys two distinct, symmetrically located zones of overlap with the state representation $P_{|\psi\rangle}$, Eq. (4.8), as shown in Fig. 6(a). Each domain has an area

$$A_m = \tfrac{1}{2}\int dx\int dp\, P_{|\psi\rangle}(x,p) = \tfrac{1}{2}(2\pi)^{-1/2}\alpha^{-1}\exp\left[-\left[\frac{m+\frac{1}{2}-\alpha^2}{\sqrt{2}\alpha}\right]^2\right] , \tag{4.9}$$

where $\int dx\int dp$ is over the mth band, following from Eq. (4.8) with the help of Eq. (4.7). Each contribution represents a complex-valued probability amplitude of absolute value $A_m^{1/2}$ which interferes with its counterpart, that is,

$$W_m[|\psi\rangle]=|A_m^{1/2}\exp(i\tilde{\phi}_m)+A_m^{1/2}\exp(-i\tilde{\phi}_m)|^2 . \tag{4.10a}$$

The phase difference $2\tilde{\phi}_m[|\psi\rangle]$ is the domain embraced by the center line of the mth band and that of the two coherent states, as indicated in Fig. 6(b). The area of this phase-space segment of angle φ reads

$$2\tilde{\phi}_m[|\psi\rangle]=\tfrac{1}{2}r_{m+1/2}^2\varphi=\tfrac{1}{2}2(m+\tfrac{1}{2})\varphi=(m+\tfrac{1}{2})\varphi . \tag{4.10b}$$

This result, Eq. (4.10), is very reminiscent of the corresponding exact expressions, Eq. (2.7) and Eq. (2.8). This stands out most clearly when we recall the Gaussian approximation (4.7a) of the Poisson distribution (2.8b) as well as the property $\mathcal{N}\cong 1$ for $p_\varphi \gg 1$, which yields

$$A_m \cong \mathcal{A}_m .$$

The phase $\tilde{\phi}_m$, Eq. (4.10b), however, is different from the corresponding exact expression (2.8c). We note that the factor $\frac{1}{2}$ resulting from the zero-point energy of the harmonic oscillator is missing.[34]

2. *Position probability*

We now turn to the position distribution W_x, Eq. (3.5). This distribution results from the scalar product between an x eigenstate $|x\rangle$ and the state $|\psi\rangle$. The state $|x\rangle$ represented in x-p oscillator phase space as an infinitely long thin strip parallel to the p axis located at x cuts out of the distribution $P_{|\psi\rangle}$, Eqs. (4.8) and (4.2b), *two* distinct symmetrically located domains shown in Fig. 7(a). The area of each overlap reads

$$\begin{aligned}A_x[|\psi\rangle]&=\tfrac{1}{2}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dp\;\delta(x-x')P_{|\psi\rangle}(x,p)\\&=\tfrac{1}{2}\pi^{-1/2}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2\}=\mathcal{A}_x .\end{aligned}$$

In the last step we have made use of Eq. (3.5b) and $\mathcal{N}\cong 1$. These two contributing areas interfere,

$$W_x[|\psi\rangle]=|\mathcal{A}_x^{1/2}\exp(i\phi_x)+\mathcal{A}_x^{1/2}\exp(-i\phi_x)|^2 . \tag{4.11}$$

The phase difference $2\phi_x$, Eq. (3.5c), between the two amplitudes,

$$\begin{aligned}2\phi_x=2[&\sqrt{2}\alpha\sin(\varphi/2)x\\&-\tfrac{1}{2}\sqrt{2}\alpha\cos(\varphi/2)\sqrt{2}\alpha\sin(\varphi/2)] ,\end{aligned}$$

also allows a simple geometrical interpretation in phase space: It is the difference between the rectangular phase-space area, $\sqrt{2}\alpha\sin(\varphi/2)x$, that is, the phase xp of the position eigenstate $\exp(ixp)$ and the phase difference of the two coherent states expressed by the area

$$\tfrac{1}{2}\sqrt{2}\alpha\cos(\varphi/2)\sqrt{2}\alpha\sin(\varphi/2) ,$$

of the phase-space triangle shown in Fig. 7(b).

We conclude this section by noting that the concept of interference in phase space readily explains the similarity between the photon number distribution W_m, Eqs. (2.8) and (4.10), and the position distribution W_x, Eqs. (3.5) and (4.11): In the limit of large average numbers of photons and small angles φ, the two coherent states are in the immediate neighborhood of the x axis. Moreover, the Planck-Bohr-Sommerfeld bands for large m values are infinitely thin and, close to the x axis, their edges are straight lines.[32] This makes an m band almost indistinguishable from the phase-space representation of a position state.

V. SUMMARY

The striking consequences of the superposition principle of quantum mechanics—a single coherent state, a quasiclassical state, the quantum-mechanical superposition of two coherent states of identical average number of photons but well-defined phase difference, a highly nonclassical state that exhibits sub-Poissonian and oscillatory photon statistics—are the central results of the present article. The phase difference between the two states, φ, determines in a sensitive way the statistics of this superposition state: φ domains characterized by sub-Poissonian photon-count probability interchange repeatedly with ones of super-Poisson statistics. The resulting oscillations in the normalized variance of the photon-count probability curve make their appearance when the two coherent states have still considerable overlap, that is, when they are not distinguishable yet. Hence in this φ region the nonclassical features are not sensitive to dissipation. When the two states are well separated, the photon distribution displays an oscillatory behavior with a Poissonian envelope. Consequently, the normalized variance of the photon-count probability which is insensitive to these oscillations approaches the Poissonian value of unity. This superposition state also displays another interesting nonclassical feature of the radiation field, namely, squeezing. There exist φ domains in which the uncertainty in one of the conjugate variables, in the present example of the x variable, falls below the corresponding coherent-state value. These domains are separated from each other by domains of nonsqueezing, that is, fluctuations larger than that of a single coherent state. In the limit of large average number of photons the critical phase angles at which transitions from nonsqueezing to squeezing and vice versa occur are identical to the ones where the photon-count probability shifts from super- to sub-Poissonian statistics. Hence, here sub-Poissonian statistics is always accompanied by squeezing. However, when the state $|\psi\rangle$ contains only a few photons, these phase angles differ. The strong correlation between sub-Poissonian statistics and squeezing stands out most clearly when we view this superposition state from phase space: The phase-space representatives of a number state of large photon number (a thin Planck-Bohr-Sommerfeld band of large radius) and a position eigenstate (a thin phase-space strip) are almost identical in the neighborhood of the x axis where the two superposing coherent states exist. Hence the resulting areas of overlaps between the highway and the superposition state $|\psi\rangle$ on one the hand and the band and the $|\psi\rangle$ state on the other give similar results for the absolute value of interfering probability amplitudes. Their phase differences however, differ in the two cases. Another insight into these nonclassical factors created by the seemingly innocent principle of quantum mechanics springs from the Wigner function. This distribution also contains, apart from the two Gaussian bells corresponding to the two coherent states, a contribution which for the state $|\psi\rangle$ exists on the x axis of x-p oscillator phase space and can assume negative as well as positive values. It is from these ditches and troughs that the nonclassical features discussed in this paper originate.

APPENDIX A: DECAY OF INTERFERENCE FEATURES

In this appendix we discuss the influence of dissipation on the photon statistics

$$W_m = |\mathcal{A}_m^{1/2} e^{im\varphi/2} + \mathcal{A}_m^{1/2} e^{-im\varphi/2}|^2 \tag{A1}$$

of the superposition state $|\psi\rangle$, Eq. (2.1). A harmonic oscillator weakly coupled to a zero-temperature heat bath serves as a model.[3,22,23] The density operator ρ of the oscillator then obeys the master equation

$$\frac{d\rho}{dt} = \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) , \tag{A2}$$

where a and $a^\dagger$ denote the annihilation and creation operators for the energy eigenstates $|n\rangle$, and γ is the decay constant. The time dependence of the photon statistics, $W_m(t) \equiv \langle m|\rho(t)|m\rangle$, resulting from Eq. (A2) follows from the differential-recurrence relation

$$\dot{W}_m = \gamma(m+1)W_{m+1} - \gamma m W_m . \tag{A3}$$

We can easily verify that[22,23]

$$W_m(t) = e^{-m\gamma t} \sum_{j=0}^{\infty} W_{m+j}(t=0) \begin{bmatrix} m+j \\ m \end{bmatrix} (1-e^{-\gamma t})^j \tag{A4}$$

is a solution of Eq. (A3). When we substitute the photon distribution, Eq. (A1), into Eq. (A4) and use the relation

$$\mathcal{A}_{m+j} \begin{bmatrix} m+j \\ m \end{bmatrix} = \mathcal{N}^2 \frac{1}{2} \frac{\alpha^{2m}}{m!} e^{-\alpha^2} \frac{\alpha^{2j}}{j!} ,$$

following from the definition of $\mathcal{A}_m$, Eq. (2.8b), we arrive after minor algebra at

$$\begin{aligned} W_m(t) = \mathcal{N}^2 \frac{(\alpha^2 e^{-\gamma t})^m}{m!} \\ &\times (\exp(-\alpha^2 e^{-\gamma t}) \\ &\quad + \tfrac{1}{2}\exp\{im\varphi + \alpha^2[(1-e^{-\gamma t})e^{i\varphi} - 1]\} \\ &\quad + \tfrac{1}{2}\exp\{-im\varphi + \alpha^2[(1-e^{-\gamma t})e^{-i\varphi} - 1]\}) \end{aligned}$$

or

$$\begin{aligned} 2W_m(t) = \mathcal{N}^2 \frac{(\alpha e^{-\gamma t/2})^{2m}}{m!} \exp[-(\alpha e^{-\gamma t/2})^2] \\ &\times \{1 + \cos[m\varphi + \alpha^2(1-e^{-\gamma t})\sin\varphi] \\ &\quad \times \exp[-2\alpha^2 \sin^2(\varphi/2)(1-e^{-\gamma t})]\} . \end{aligned}$$

This is the central result of this appendix. Three important features of the decay of the photon statistics stand out most clearly:

(1) Due to the coupling of the oscillator to the heat bath, the amplitude common to the two coherent states $|\alpha \exp(i\varphi/2)\rangle$ and $|\alpha \exp(-i\varphi/2)\rangle$ building the state $|\psi\rangle$ decays as $\alpha e^{-\gamma t/2}$, giving rise to the Poisson envelope of decaying mean photon number

$$\langle m\rangle(t) = \alpha^2 e^{-\gamma t} .$$

(2) The interference term experiences a time-dependent phase shift $\alpha^2(1-e^{-\gamma t})\sin\varphi$, which converges towards the time-independent value $\alpha^2 \sin\varphi$.

(3) For short times, that is, $\gamma t \ll 1$, this interference contribution decays exponentially with the decay constant $2\alpha^2 \sin^2(\varphi/2)\gamma$ and hence with the separation of the two coherent states. Since many of the nonclassical features of this state discussed in the article, such as sub-Poissonian photon statistics or squeezing arise when the two coherent states are not distinguishable yet, that is, when $|\alpha \sin(\varphi/2)| \ll 1$, the decay of these nonclassical phenomena is slow. This is in strong contrast to the ordinary investigations[22,23] of superposition states, which focus on the limit $|\alpha \sin(\varphi/2)| \gg 1$.

APPENDIX B: NORMALIZED VARIANCE σ

In this appendix we calculate the normalized variance

$$\sigma^2 \equiv \langle m^2\rangle / \langle m\rangle - \langle m\rangle , \tag{B1}$$

where the moments

$$\langle m\rangle = \sum_{m=0}^{\infty} m W_m \tag{B2}$$

and

$$\langle m^2\rangle = \sum_{m=0}^{\infty} m^2 W_m \tag{B3}$$

follow from the photon-count probability W_m, Eq. (2.8), in the form

$$W_m = \mathcal{N}^2 \left[\frac{(\alpha^2)^m}{m!} + \frac{1}{2} \left[\frac{(\alpha^2 e^{i\varphi})^m}{m!} + \frac{(\alpha^2 e^{-i\varphi})^m}{m!} \right] \right] \exp(-\alpha^2) .$$

The relation

$$\sum_{m=0}^{\infty} \frac{m}{m!} \lambda^m = \lambda \sum_{m=1}^{\infty} \frac{\lambda^{m-1}}{(m-1)!} = \lambda e^{\lambda}$$

yields for the first moment, Eq. (B2),

$$\langle m\rangle=\mathcal{N}^2\{\alpha^2+\tfrac{1}{2}\alpha^2 e^{i\varphi}\exp[-\alpha^2(1-e^{i\varphi})]+\tfrac{1}{2}\alpha^2 e^{-i\varphi}\exp[-\alpha^2(1-e^{-i\varphi})]\}$$
$$=\alpha^2\mathcal{N}^2[1+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)] \tag{B4a}$$

or

$$\langle m\rangle=\alpha^2\frac{1+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)}{1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)} . \tag{B4b}$$

Here we have recalled the "mean momentum," Eq. (2.5), of a single Gaussian bell,

$$|p_\varphi|\equiv\sqrt{2}\alpha\sin(\varphi/2) ,$$

and have made use of the normalization, Eq. (2.4).

The relation

$$\sum_{m=0}^{\infty}\frac{m^2}{m!}\lambda^m=\lambda^2\sum_{m=2}^{\infty}\frac{(m-1)}{(m-1)!}\lambda^{m-2}+\lambda\sum_{m=1}^{\infty}\frac{\lambda^{m-1}}{(m-1)!}=(\lambda^2+\lambda)e^\lambda$$

allows to calculate the second moment, Eq. (B3),

$$\langle m^2\rangle=\mathcal{N}^2\{\alpha^2(\alpha^2+1)+\tfrac{1}{2}\alpha^2(\alpha^2e^{2i\varphi}+e^{i\varphi})\exp[-\alpha^2(1-e^{i\varphi})]+\tfrac{1}{2}\alpha^2(\alpha^2e^{-2i\varphi}+e^{-i\varphi})\exp[-\alpha^2(1-e^{-i\varphi})]\}$$
$$=\alpha^2\mathcal{N}^2\{1+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)+\alpha^2[1+\cos(\alpha^2\sin\varphi+2\varphi)\exp(-p_\varphi^2)]\} . \tag{B5}$$

When we divide Eq. (B5) by Eq. (B4a) and subtract Eq. (B4b), we find for σ^2, Eq. (B1),

$$\sigma^2=1-\alpha^2\left[\frac{1+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)}{1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)}-\frac{1+\cos(\alpha^2\sin\varphi+2\varphi)\exp(-p_\varphi^2)}{1+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)}\right] .$$

We combine the two terms in large parentheses by making use of the identities

$$2\cos(\alpha^2\sin\varphi+\varphi)-[\cos(\alpha^2\sin\varphi+2\varphi)+\cos(\alpha^2\sin\varphi)]=2(1-\cos\varphi)\cos(\alpha^2\sin\varphi+\varphi)=4\sin^2(\varphi/2)\cos(\alpha^2\sin\varphi+\varphi)$$

and

$$\cos^2(\alpha^2\sin\varphi+\varphi)-\cos(\alpha^2\sin\varphi)\cos(\alpha^2\sin\varphi+2\varphi)=\tfrac{1}{2}\{1+\cos[2(\alpha^2\sin\varphi+\varphi)]\}-\tfrac{1}{2}\{\cos(2\varphi)+\cos[2(\alpha^2\sin\varphi+\varphi)]\}$$
$$=\sin^2\varphi=4\sin^2(\varphi/2)\cos^2(\varphi/2)$$

to arrive at

$$\sigma^2=1-2p_\varphi^2\exp(-p_\varphi^2)\frac{\cos(\alpha^2\sin\varphi+\varphi)+\cos^2(\varphi/2)\exp(-p_\varphi^2)}{[1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)][1+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)]} .$$

APPENDIX C: UNCERTAINTY IN x

In this appendix we calculate the uncertainty

$$(\Delta x)^2\equiv\langle x^2\rangle-\langle x\rangle^2 \tag{C1}$$

for the state $|\psi\rangle$, Eq. (2.1).

We start from the probability distribution $|\psi(x)|^2$ in the form

$$W_x=|\psi(x)|^2=\pi^{-1/2}\mathcal{N}^2(\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2\}+\tfrac{1}{2}\exp(-p_\varphi^2)\times\{\exp(i\alpha^2\sin\varphi)\exp[-(x-\sqrt{2}\alpha e^{i\varphi/2})^2]+\exp(-i\alpha^2\sin\varphi)\exp[-(x-\sqrt{2}\alpha e^{-i\varphi/2})^2]\}) . \tag{C2}$$

When we recall the formula[35]

$$\pi^{-1/2}\int_{-\infty}^{\infty}dx\,x^n\exp[-(x-z)^2]=\begin{cases}1 & \text{for } n=0\\ z & \text{for } n=1\\ z^2+\frac{1}{2} & \text{for } n=2 ,\end{cases} \tag{C3}$$

we can easily verify the normalization

$$\int_{-\infty}^{\infty}dx\,W_x=\mathcal{N}^2\{1+\exp[-2\alpha^2\sin^2(\varphi/2)]\cos(\alpha^2\sin\varphi)\}=1 .$$

With the help of Eqs. (C2) and (C3), we find

$$\langle x\rangle=\int_{-\infty}^{\infty}dx\,xW_x=\mathcal{N}^2[\sqrt{2}\alpha\cos(\varphi/2)+\sqrt{2}\alpha\cos(\alpha^2\sin\varphi+\varphi/2)\exp(-p_\varphi^2)] .$$

When we make use of the relation

$$\cos(\alpha^2\sin\varphi+\varphi/2)=\cos(\varphi/2)\cos(\alpha^2\sin\varphi)-\sin(\varphi/2)\sin(\alpha^2\sin\varphi)\ ,$$

we arrive at

$$\langle x\rangle=\sqrt{2}\alpha\cos(\varphi/2)-\sqrt{2}\alpha\sin(\varphi/2)\exp(-p_\varphi^2)\frac{\sin(\alpha^2\sin\varphi)}{1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)}\ . \tag{C4}$$

The second moment $\langle x^2\rangle$ follows from Eqs. (C2) and (C3),

$$\begin{aligned}\langle x^2\rangle=\int_{-\infty}^{\infty}dx\,x^2W_x&=\mathcal{N}^2\{[2\alpha^2\cos^2(\varphi/2)+\tfrac{1}{2}]+\tfrac{1}{2}\exp(-p_\varphi^2)\\&\qquad\times[\exp(i\alpha^2\sin\varphi)(2\alpha^2e^{i\varphi}+\tfrac{1}{2})+\exp(-i\alpha^2\sin\varphi)(2\alpha^2e^{-i\varphi}+\tfrac{1}{2})]\}\\&=\tfrac{1}{2}\mathcal{N}^2[1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)]+2\alpha^2\mathcal{N}^2[\cos^2(\varphi/2)+\cos(\alpha^2\sin\varphi+\varphi)\exp(-p_\varphi^2)]\ .\end{aligned}$$

When we use the relation

$$\cos(\alpha^2\sin\varphi+\varphi)=\cos^2(\varphi/2)\cos(\alpha^2\sin\varphi)-\sin^2(\varphi/2)\cos(\alpha^2\sin\varphi)-\sin\varphi\sin(\alpha^2\sin\varphi)$$

and the normalization, Eq. (2.4), we arrive at

$$\begin{aligned}\langle x^2\rangle&=2\alpha^2\cos^2(\varphi/2)-2\alpha^2\sin\varphi\exp(-p_\varphi^2)\frac{\sin(\alpha^2\sin\varphi)}{1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)}\\&\quad+\tfrac{1}{2}\left[1-4\alpha^2\sin^2(\varphi/2)\exp(-p_\varphi^2)\frac{\cos(\alpha^2\sin\varphi)}{1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)}\right]\ .\end{aligned}\tag{C5}$$

We substitute Eqs. (C4) and (C5) into the expression, Eq. (C1), for the variance $(\Delta x)^2$, to find

$$(\Delta x)^2=\tfrac{1}{2}\left[1-2p_\varphi^2\exp(-p_\varphi^2)\frac{\cos(\alpha^2\sin\varphi)+\exp(-p_\varphi^2)}{[1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)]^2}\right]\ .$$

APPENDIX D: WIGNER FUNCTION OF STATE $|\psi\rangle$

In this appendix we calculate the Wigner function $P_{|\psi\rangle}^{(W)}(x,p)$ of the state $|\psi\rangle$, Eq. (2.1), from the definition[28]

$$P_{|\psi\rangle}^{(W)}(x,p)=\pi^{-1}\int_{-\infty}^{\infty}dy\,\exp(2ipy)\psi^*(x+y)\psi(x-y)$$

and Eq. (3.4), that is,

$$\begin{aligned}P_{|\psi\rangle}^{(W)}(x,p)&=(2\pi)^{-1}\mathcal{N}^2\exp(-p_\varphi^2)\pi^{-1/2}\int_{-\infty}^{\infty}dy\,\exp(2ipy)\\&\quad\times\{\exp[-\tfrac{1}{2}(x+y-\sqrt{2}\alpha e^{-i\varphi/2})^2-\tfrac{1}{2}(x-y-\sqrt{2}\alpha e^{i\varphi/2})^2]\\&\qquad+\exp(-i\alpha^2\sin\varphi)\exp[-\tfrac{1}{2}(x+y-\sqrt{2}\alpha e^{-i\varphi/2})^2-\tfrac{1}{2}(x-y-\sqrt{2}\alpha e^{-i\varphi/2})^2]+\text{c.c.}\}\ .\end{aligned}$$

When we recall the identity

$$\begin{aligned}2ipy-\tfrac{1}{2}(x+y-z_1)^2-\tfrac{1}{2}(x-y-z_2)^2&=\left[\frac{1}{2i}(z_1-z_2)\right]^2-[x-\tfrac{1}{2}(z_1+z_2)]^2\\&\quad-\left[p+\frac{1}{2i}(z_1-z_2)\right]^2-[y-\tfrac{1}{2}(z_1-z_2+2ip)]^2\ ,\end{aligned}$$

together with Eq. (C3), we can perform the integration, that is,

$$\begin{aligned}P_{|\psi\rangle}^{(W)}(x,p)&=(2\pi)^{-1}\mathcal{N}^2\exp(-p_\varphi^2)\\&\quad\times(\exp(p_\varphi^2)\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2-[p-\sqrt{2}\alpha\sin(\varphi/2)]^2\}\\&\qquad+\exp(p_\varphi^2)\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2-[p+\sqrt{2}\alpha\sin(\varphi/2)]^2\}\\&\qquad+e^{-p^2}\{\exp(-i\alpha^2\sin\varphi)\exp[-(x-\sqrt{2}\alpha e^{-i\varphi/2})^2]+\exp(i\alpha^2\sin\varphi)\exp[-(x-\sqrt{2}\alpha e^{i\varphi/2})^2]\})\end{aligned}$$

or

$$P^{(W)}_{|\psi\rangle}(x,p)=\tfrac{1}{2}\mathcal{N}^2(\pi^{-1}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2-[p-\sqrt{2}\alpha\sin(\varphi/2)]^2\}$$
$$+\pi^{-1}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2-[p+\sqrt{2}\alpha\sin(\varphi/2)]^2\}$$
$$+2\pi^{-1}\cos\{\alpha^2\sin\varphi+2\sqrt{2}\alpha\sin(\varphi/2)[x-\sqrt{2}\alpha\cos(\varphi/2)]\}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2-p^2\})\ . \quad \text{(D1)}$$

We conclude this appendix by evaluating the marginal, that is, the momentum distribution

$$W_p\equiv|\psi(p)|^2=\int_{-\infty}^{\infty}dx\,P^{(W)}_{|\psi\rangle}(x,p)$$
$$=\tfrac{1}{2}\mathcal{N}^2(\pi^{-1/2}\exp\{-[p-\sqrt{2}\alpha\sin(\varphi/2)]^2\}+\pi^{-1/2}\exp\{-[p+\sqrt{2}\alpha\sin(\varphi/2)]^2\}$$
$$+2\pi^{-1/2}\cos(\alpha^2\sin\varphi)\exp[-2\alpha^2\sin^2(\varphi/2)]\exp(-p^2))\ .$$

Here we have made use of the integral[35]

$$\pi^{-1/2}\int_{-\infty}^{\infty}dy\,\cos(zy)e^{-y^2}=\exp(-z^2/4)\ . \quad \text{(D2)}$$

When we integrate Eq. (D1) over p, we find the position distribution

$$W_x\equiv|\psi(x)|^2=\int_{-\infty}^{\infty}dp\,P^{(W)}_{|\psi\rangle}(x,p)=\mathcal{N}^2(\pi^{-1/2}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2\}$$
$$+\pi^{-1/2}\cos\{\alpha^2\sin\varphi+2\sqrt{2}\alpha\sin(\varphi/2)[x-\sqrt{2}\alpha\cos(\varphi/2)]\}\exp\{-[x-\sqrt{2}\alpha\cos(\varphi/2)]^2\})\ ,$$

in agreement with Eq. (3.5).

APPENDIX E: UNCERTAINTY IN p

We now calculate the uncertainty $(\Delta p)^2\equiv\langle p^2\rangle-\langle p\rangle^2$ for the state $|\psi\rangle$, Eq. (2.1), starting from the momentum distribution, Eq. (3.8),

$$W_p\equiv|\psi(p)|^2=\tfrac{1}{2}\mathcal{N}^2(\pi^{-1/2}\exp\{-[p-\sqrt{2}\alpha\sin(\varphi/2)]^2\}+\pi^{-1/2}\exp\{-[p+\sqrt{2}\alpha\sin(\varphi/2)]^2\}$$
$$+2\pi^{-1/2}\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)\exp(-p^2))\ . \quad \text{(E1)}$$

When we make use of Eqs. (C3) and (2.4), we can convince ourselves that W_p is properly normalized, that is,

$$\int_{-\infty}^{\infty}dp\,W_p=\tfrac{1}{2}\mathcal{N}^2[1+1+2\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)]=1\ .$$

Moreover, we recognize from Eq. (E1) the symmetry relation

$$W_{-p}=W_p$$

and, hence,

$$\langle p\rangle=\int_{-\infty}^{\infty}dp\,pW_p=0\ . \quad \text{(E2)}$$

The second moment $\langle p^2\rangle$ follows from Eq. (C3),

$$\langle p^2\rangle=\int_{-\infty}^{\infty}dp\,p^2W_p=\tfrac{1}{2}\mathcal{N}^2\{2[2\alpha^2\sin^2(\varphi/2)+\tfrac{1}{2}]+2\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)\tfrac{1}{2}\}\ .$$

We thus find, with the help of Eq. (E2),

$$(\Delta p)^2=\tfrac{1}{2}+\frac{2\alpha^2\sin^2(\varphi/2)}{1+\cos(\alpha^2\sin\varphi)\exp(-p_\varphi^2)}\ .$$

APPENDIX F: PHOTON DISTRIBUTION W_m VIA WIGNER FUNCTION

In this appendix we evaluate the photon-count probability W_m of the state $|\psi\rangle$ via the Wigner function technique. A rotation of the coordinate system,

$$\bar{x}=\cos(\varphi/2)x\pm\sin(\varphi/2)p\ ,$$

$$\bar{p}=\mp\sin(\varphi/2)x+\cos(\varphi/2)p\ ,$$

allows to express the coherent-state contributions

$$W_m[|\alpha\exp(\pm i\varphi/2)\rangle]=2\pi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,P^{(W)}_{|\alpha\exp(\pm i\varphi/2)\rangle}P_m^{(W)}$$
$$=2\frac{(-1)^m}{\pi}\exp(-2\alpha^2)\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,\exp\{2\sqrt{2}\alpha[\cos(\varphi/2)x\pm\sin(\varphi/2)p]\}$$
$$\times\exp[-2(x^2+p^2)]L_m(2(x^2+p^2))$$

to the photon distribution W_m of the state $|\psi\rangle$ as

$$W_m[|\alpha\exp(\pm i\varphi/2)\rangle]=\exp(-2\alpha^2)I_m(\varphi=0)\ , \tag{F1}$$

that is, in terms of the integral

$$I_m(\varphi)\equiv2\frac{(-1)^m}{\pi}\int_{-\infty}^{\infty}d\bar{x}\int_{-\infty}^{\infty}d\bar{p}\,\exp(2\sqrt{2}\alpha e^{i\varphi/2}\bar{x})\exp[-2(\bar{x}^2+\bar{p}^2)]L_m(2(\bar{x}^2+\bar{p}^2))\ . \tag{F2}$$

The interference contribution

$$W_m[\text{int}]\equiv2\pi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,\mathcal{P}_{\text{int}}P_m^{(W)}$$
$$=\exp[-i\alpha^2\sin\varphi-2\alpha^2\cos^2(\varphi/2)]$$
$$\times2\frac{(-1)^m}{\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,\exp(2\sqrt{2}\alpha e^{i\varphi/2}x)\exp[-2(x^2+p^2)]L_m(2(x^2+p^2))+\text{c.c.}$$

can also be expressed in terms of I_m, Eq. (F2),

$$W_m[\text{int}]\equiv\exp(-\alpha^2-\alpha^2e^{i\varphi})I_m(\varphi)+\text{c.c.} \tag{F3}$$

We now evaluate the integral I_m, Eq. (F2), following a technique which we have already applied in Ref. 32 for the case of a single coherent state.

When we introduce the new variables ρ and φ' via $x=(\rho/2)^{1/2}\cos\varphi'$ and $p=(\rho/2)^{1/2}\sin\varphi'$, we find

$$I_m(\varphi)=(-1)^m\int_0^{\infty}d\rho\,e^{-\rho}L_m(\rho)\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi'\exp(2\alpha e^{i\varphi/2}\rho^{1/2}\cos\varphi')$$
$$=(-1)^m\int_0^{\infty}d\rho\,e^{-\rho}L_m(\rho)J_0(2[\rho(-\alpha^2e^{i\varphi})]^{1/2})\ ,$$

where we have performed[35] the integration over φ' with the help of the Bessel function J_0 of zeroth order. The generating function of the Laguerre polynomial,[33]

$$J_0(2[\rho(-\alpha^2e^{i\varphi})]^{1/2})=\exp(\alpha^2e^{i\varphi})\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha^2e^{i\varphi})^k}{k!}L_k(\rho)\ ,$$

reduces the above integral to

$$I_m(\varphi)=\exp(\alpha^2e^{i\varphi})\sum_{k=0}^{\infty}\frac{(-1)^{k+m}(\alpha^2e^{i\varphi})^k}{k!}\int_0^{\infty}d\rho\,e^{-\rho}L_m(\rho)L_k(\rho)$$

or

$$I_m(\varphi)=(\alpha^2e^{i\varphi})^m(m!)^{-1}\exp(\alpha^2e^{i\varphi})\ ,$$

where in the last step we have used the orthonormalization property of the Laguerre polynomials.[33]

When we substitute this result into Eqs. (F1) and (F3), we arrive at

$$W_m[|\alpha\exp(\pm i\varphi/2)\rangle]=\frac{\alpha^{2m}}{m!}e^{-\alpha^2}$$

and

$$W_m[\text{int}]=2\frac{\alpha^{2m}}{m!}e^{-\alpha^2}\cos(m\varphi)\ .$$

[1]P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Clarendon, Oxford, 1984), p. 12.

[2]R. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).

[3]See, for example, W. H. Louisell, *Quantum Statistical Properties of Radiation*, (Wiley, New York, 1973); or M. Sargent III, M. O. Scully, and W. E. Lamb Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

[4]We emphasize that the state $|\psi\rangle=\mathcal{N}(1/\sqrt{2})(|\alpha e^{i\varphi/2}\rangle$

$+|\alpha e^{-i\varphi/2}\rangle$) defined in Eq. (2.1) is quite different from the state $|\beta\rangle \equiv |\alpha e^{i\varphi/2}+\alpha e^{-i\varphi/2}\rangle$, which is a coherent state.

[5]According to M. Hillery [Phys. Lett. **111A**, 409 (1985)], any pure state which is not a coherent state is a nonclassical state, that is, its P representation is more singular than a δ function or can assume negative values. The state $|\psi\rangle$, Eq. (2.1), is a pure state and not a coherent state. Hence, it must exhibit nonclassical features. Another pure state, a logarithmic state of the radiation field, exhibiting nonclassical features, has been investigated in R. Simon and M. Venkata Satyanarayana, J. Mod. Opt. **35**, 719 (1988).

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[8]For an excellent review on sub-Poissonian statistics and antibunching see, for example, M. C. Teich and B. A. E. Saleh, *Photon Bunching and Antibunching,* Vol. XXVI of *Progress in Optics,* edited by E. Wolf (Elsevier, Amsterdam, 1988), and M. C. Teich and B. E. A. Saleh, Quantum Opt. **1**, 153 (1989); Phys. Today **43** (6), 26 (1990).

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[11]Squeezing has also been found in the state $|\psi\rangle$, Eq. (2.1), for a fixed phase difference $\varphi=\pi$ and a small average number of photons. See, for example, J. Janszky and A. V. Vinogradov, Phys. Rev. Lett. **64**, 2771 (1990). Note that their result is a special case of Eq. (3.6) for $\varphi=\pi$: $(\Delta x)^2_{\varphi=\pi}=\frac{1}{2}[1-4\alpha^2 e^{-2\alpha^2}(1+e^{-2\alpha^2})^{-1}]$.

[12]A superposition state consisting of two coherent states of identical phase but different average photon numbers shows squeezing in the phase variable, as discussed by W. Schleich, J. P. Dowling, R. J. Horowicz, and S. Varro, in *New Frontiers in Quantum Optics and Quantum Electrodynamics,* edited by A. Barut (Plenum, New York, 1990).

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