Quantum theory of propagation of nonclassical radiation in a near-resonant medium

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We develop the quantum theory of the propagation of nonclassical radiation in a nearly resonant medium. The medium is taken to have two levels, which are coupled by the radiation field. Heisenberg equations of motion are developed and compared with stochastic equations in a positive P representation. Initial conditions for absorbers and amplifiers are calculated, so that, for example, superfluorescence or self-induced transparency can be treated. The theoretical methods introduced here allow nonclassical statistics of the radiation field to be calculated such as photon antibunching or squeezing. Both radiative and collisional damping are included.

I. INTRODUCTION

With recent experimental studies[1,2] showing observable nonclassical properties of light produced by nonlinear-optical processes, it is of interest to understand the propagation of light in a near-resonant medium from a quantum-mechanical basis. More generally, the spatial propagation of intense light through near-resonant media is a fundamental problem in quantum optics, and is central to the processes of laser action, optical bistability, self-induced transparency, four-wave mixing, amplified spontaneous emission, and superfluorescence. The semiclassical treatment of these processes is well understood, but is not an adequate description if quantum fluctuations are important. We therefore intend to develop the quantum theory of propagation of nonclassical radiation in an optically pumped two-level medium that has collisional and radiative damping. The initial preparation of the medium is arbitrary, although typically it would either be in the ground state, or in an inverted state prepared via optical pumping. The states of interest decay to other states on long time scales, and are therefore treated as an isolated two-level system.

It is important to include collisional damping, since its accompanying fluctuations can mask the quantum noise of interest and change the statistical properties of the light. In fact, amplified spontaneous emission[3] (ASE) and superfluorescence[4] (SF) differ primarily in that the former is dominated by collisional dephasing. The transition from SF to ASE as the collision rate is increased has been the subject of a recent experimental study, which partially motivated the present work.[5] The application of the present formalism to this problem is given in a companion paper.[6] In the case of amplified spontaneous emission, the measured field statistics appear to be largely classical. Our methods permit the theory to treat also nonclassical radiation fields, such as were recently observed in experiments on photon antibunching[1] and quadrature squeezing.[2]

Superfluorescence, in which a macroscopic, coherent dipole spontaneously develops in an initially inverted two-level medium, provides one of the more interesting and challenging cases to study, since quantum noise and spatial propagation are both important. In the absence of a mode-selecting optical cavity, superfluorescence is inherently a three-dimensional, quantum process. Thus, "mean-field" theories cannot be used reliably. While the basic Heisenberg-picture operator equations of motion[7] were developed for the propagation problem in the 1960's, they are notoriously difficult to solve rigorously, as they are intrinsically quantum mechanical and nonlinear. In addition, the multimode case gives rise to partial differential equations in space and time.

In the case of superfluorescence with no collisional dephasing, a breakthrough[8] in solving the operator equations was achieved by realizing that to a good approximation the operator character of the variables can be ignored if the initial conditions are chosen as random c-number quantities, with statistics determined by the underlying quantum mechanics. This provides a set of differential equations that can be readily solved by standard numerical techniques, even in the nonlinear regime. This method is highly accurate in the case of many atoms, in the linear regime, where the atoms remain essentially in their excited states. The most successful theories of SF (Refs. 8 and 9) have used equations of motion that have the same form as Maxwell's wave equation for the electric field, and Bloch equations for the atomic variables, together with zero-point motion of the atomic dipoles (or, equivalently, of the electric field). The random initiation of the emission corresponds to normal (or antinormal) operator ordering.

The random initial-value method is less reliable, however, when the problem becomes nonlinear through atomic depletion. The nonlinearity then couples normally and antinormally ordered operators together. This causes ambiguities in the specification of initial conditions and Langevin terms as random c numbers, because

the system operators are intrinsically quantum mechanical. The operators cannot therefore be entirely represented by the same set of classical variables as in the linear regime, since the commutation relations would then be violated. It has been argued, rather convincingly, that in superfluorescence the violation of the commutators is not very important, since the amplified fields are nearly classical anyway. In general, however, nonclassical field states can be produced by propagation of light through resonant media.[2] By assuming *a priori* that the states are classical, one precludes the possibility of studying nonclassical states.[10]

The purpose of the present paper is to derive a set of stochastic c-number differential equations that are equivalent to the Heisenberg operator equations of motion describing light propagation in a near-resonant medium in the many-atom limit. This is accomplished through use of the positive P representation.[11] The advantage of this approach is that it provides equations that can be solved numerically while retaining the leading terms that describe the nonclassical nature of the field. The terms in the equations responsible for the nonclassical effects can be identified. When the nonclassical terms are neglected, as might be valid in superfluorescence and amplified spontaneous emission, the standard equations of previous studies are recovered. This allows a better understanding of the nature of the approximations made in earlier work. In addition, situations involving nonclassical quantum behavior can be treated. For example, there is a recent theoretical proposal that nonlinear soliton propagation in resonant media could generate nonclassical photon statistics.[12]

The model treated here is similar to the two-level superfluorescence model as extended to the three-dimensional case by Drummond and Eberly.[9] However, we also include a treatment of collisional and radiative damping as well as incoherent pumping. In these damping problems, the method uses the technique of reservoir operators introduced by Lax,[13] Haken,[15] Louisell,[15] and others. We compare different treatments of this problem, using Heisenberg equations of motion and the equivalent c-number stochastic equations. The quantum-noise sources of these two methods can be regarded as coming from different sources: in the Heisenberg case, from the damping reservoirs, and in the equivalent stochastic case, from both the damping and the nonlinearities in the Hamiltionan. We finally mention the possible applications of our methods to quantum chaos theory, as the semiclassical equations of laser theory are known to be equivalent to the Lorenz[16] equations in which chaos was discovered.

II. HAMILTONIAN

The model treated has a group of two-level atoms, with the μth atom at location $\mathbf{x}_\mu$. The resonant frequency for the μth atom is ω_μ. The Hamiltonian[15] for the interaction of a set of two-level atoms with the radiation field can be calculated after using the usual canonical transformation to the dipole coupled form, as

$$\hat{H}=\sum_j \hat{H}_j \ ,$$

where

$$\begin{aligned}
\hat{H}_1&=\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \ , \\
H_2&=\frac{1}{2}\sum_{\mu}\hbar\omega_\mu\hat{\sigma}^{z}_{\mu} \ , \\
H_3&=\hat{H}^{a}+\hat{H}^{\sigma}+\hat{H}^{z} \ , \\
\hat{H}_4&=\hbar\sum_{\mathbf{k}}\sum_{\mu}(g\hat{a}^{\dagger}_{\mathbf{k}}\hat{\sigma}^{-}_{\mu}e^{-i\mathbf{k}\cdot\mathbf{x}_\mu}+\text{H.c.}), \\
\hat{H}_5&=\hbar\sum_{\mu}(\hat{\Gamma}^{\sigma\dagger}_{\mu}\hat{\sigma}^{-}_{\mu}+\hat{\Gamma}^{\sigma}_{\mu}\hat{\sigma}^{+}_{\mu}+\hat{\Gamma}^{z}_{\mu}\hat{\sigma}^{z}_{\mu}) \ , \\
\hat{H}_6&=\hbar\sum_{\mathbf{k}}(\hat{\Gamma}^{a\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}+\hat{\Gamma}^{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}) \ .
\end{aligned} \tag{2.1}$$

Here the rotating-wave and dipole approximations are used. The different terms in the Hamiltonian have the following physical interpretations: $\hat{H}_1$, free Hamiltonian of the paraxial modes; $\hat{H}_2$, free Hamiltonian of the atoms; $\hat{H}_3$, free Hamiltonian of the reservoirs: field modes $\hat{H}^{a}$, atomic dipoles $\hat{H}^{\sigma}$, collisions $\hat{H}^{z}$; $\hat{H}_4$, interaction of paraxial field with atoms; $\hat{H}_5$, interaction of atomic and collisional reservoirs with atoms; and $\hat{H}_6$, interaction of background reservoir with radiation field.

The frequencies $\omega_{\mathbf{k}}$ are the mode frequencies of the $\mathbf{k}$th radiation field mode with anihilation operator $\hat{a}_{\mathbf{k}}$. The sum over $\mathbf{k}$ is restricted to a single polarization and to nearly paraxial wave vectors, with $\mathbf{k}\simeq\mathbf{k}_0$, where $\mathbf{k}_0$ is directed along the z axis. Cutoffs in momentum and in volume are used, so that the allowed wave vectors are a discrete set. The paraxial modes are coupled to a background of absorbing dipoles $\hat{\Gamma}^{a}_{\mathbf{k}}$, with free Hamiltonian $\hat{H}^{a}$. This describes background absorption and reemission due to other atoms in the medium, as opposed to the resonant ones of interest. The other atoms are assumed to have nonresonant transitions, so that they are not saturated, and coherent dipole effects do not have to be treated. In general, the resonant atoms are also coupled to modes with nonparaxial wave vectors, unless there are wave guides present. The nonparaxial wave vectors are assumed to form independent radiative fluorescence reservoirs for each atom, whose operators are labeled $\hat{\Gamma}^{\sigma}_{\mu},\hat{\Gamma}^{\sigma\dagger}_{\mu}$. The free Hamiltonian of these atomic reservoirs is $\hat{H}^{\sigma}$. This approximation of independent reservoirs implies the neglect of transverse dipole-dipole coupling and radiation trapping of the nonparaxial fluorescent modes. The approximation is therefore valid only for relatively low-density optical media where local-field corrections are negligible. Any optical pumping present, if it occurs largely within the two-level manifold, is also included in these reservoirs. The operators $\hat{\Gamma}^{z}_{\mu}$ describe a coupling of the resonant atoms to a collisional phase-damping reservoir with free Hamiltonian $\hat{H}^{z}$, which would typically describe weak collisions with nonresonant atoms.

The medium may be inhomogeneously broadened around a central frequency ω_0. The atomic density can

change with position, with a number density $\rho(\mathbf{x})$. The two-level Pauli operators used to describe the μth atom are defined (at $t=0$) as

$$\hat{\sigma}_\mu^+ = |2\rangle_\mu\langle 1|_\mu ,$$
$$\hat{\sigma}_\mu^- = |1\rangle_\mu\langle 2|_\mu , \tag{2.2}$$
$$\hat{\sigma}_\mu^z = (|2\rangle_\mu\langle 2|_\mu - |1\rangle_\mu\langle 1|_\mu) .$$

The dipole coupling of each atom to the field is assumed to be identical for each atom, and approximately independent of frequency and wave vector. This, of course, is valid only for the paraxial wave vectors included in the sum over $\mathbf{k}$. For an ideal two-level atom, the coupling parameter can be written in terms of the Einstein A coefficient of the relevant transition, so that

$$g^2 = |\mathbf{d}_{12}\cdot\mathbf{e}|^2 \left[\frac{\omega_0}{2\hbar\epsilon_0 V}\right]_{\rm SI}$$
$$= |\mathbf{d}_{12}\cdot\mathbf{e}|^2 \left[\frac{2\pi\omega_0}{\hbar V}\right]_{\rm cgs} = \left[\frac{3\gamma_0\lambda_0^2 c}{8\pi V}\right] . \tag{2.3}$$

Here $\mathbf{d}_{12}=\langle 1|\hat{\mathbf{d}}|2\rangle$ is the relevant dipole matrix element, $\mathbf{e}$ is the mode polarization, γ_0 is the Einstein A coefficient or spontaneous decay rate of level $|2\rangle$ decaying to level $|1\rangle$, ω_0 is the central resonant frequency, and λ_0 the resonant free-space wavelength. In addition, V is a quantization mode volume, which will be taken as infinite in the limit of free-space propagation. The atomic wave functions have an arbitrary relative phase, so that g can always be chosen to be real.

The spatial and frequency regions of interest are divided up into a lattice of cells. There are $N_{\mathbf{n}}$ atoms in the $\mathbf{n}$th frequency and volume cell, where $\mathbf{n}=(\mathbf{j},n)$. These atoms are all located in spatial cell labeled $(\mathbf{j})$ and have an inhomogeneously broadened frequency indexed by (n). In each of the frequency and volume cells, collective operators are defined, relative to the central wave vector $\mathbf{k}_0$. Here $\mathbf{k}_0$ is in the z direction, and corresponds to the principal propagation direction of radiation in the medium. The medium is typically pencil shaped for the paraxial approximation to be applicable.

The set of all atoms (μ) in the $\mathbf{n}$th cell is denoted $s(\mathbf{n})$. The central frequency of the nth cell is ω_n and there can be several frequencies ω_n with their corresponding collective operators $\hat{J}_{\mathbf{n}}$ in a given spatial location labeled $\mathbf{j}$. The different spatial locations are denoted $\mathbf{x}(\mathbf{j})$, so that

$$\mathbf{x}(\mathbf{j}) = \mathbf{x}_0 + (j_1\Delta_1, j_2\Delta_2, j_3\Delta_3) . \tag{2.4}$$

Here we suppose that in the ith dimension, there are $(2M_i+1)$ lattice points with a lattice spacing of Δ_i, with $j_i=(-M_i,\ldots,0,\ldots,M_i)$. The spatial volume of the cell at location $\mathbf{j}$ is $\Delta V=\Delta_1\Delta_2\Delta_3$, containing a total of $N_{\mathbf{j}}=\sum_n N_{\mathbf{n}}$ atoms, on summing over all the frequencies. Clearly, the ratio $N_{\mathbf{j}}/\Delta V$ equals the number density $\rho(\mathbf{x}(\mathbf{j}))$ at the given location. The combined frequency and spatial density $\rho(\mathbf{x},\omega)$ is given by the ratio $N_{\mathbf{n}}/(\Delta V\Delta\omega_n)$, for an nth frequency bandwidth of $\Delta\omega_n$.

We note that a two-level atom is formally equivalent to a spin-$\frac{1}{2}$ particle. Accordingly, the collective operators $\hat{J}_{\mathbf{n}}^{\pm},\hat{J}_{\mathbf{n}}^z$, are defined as having the usual angular momentum commutation relations appropriate for a system with maximum spin of $(N/2)$. Thus

$$\hat{J}_{\mathbf{n}}^+ = \sum_{\mu\in s(\mathbf{n})} \hat{\sigma}_\mu^+ e^{i\mathbf{k}_0\cdot\mathbf{x}_\mu} = (\hat{J}_{\mathbf{n}}^-)^\dagger ,$$
$$\hat{J}_{\mathbf{n}}^z = \tfrac{1}{2}\sum_{\mu\in s(\mathbf{n})} \sigma_\mu^z . \tag{2.5}$$

Local-field operators $\hat{\alpha}_{\mathbf{j}}$ are now defined as the discrete Fourier transforms of the set of mode operators $\hat{a}_{\mathbf{k}}$ on the wave-vector lattice $\mathbf{m}$ reciprocal to $\mathbf{j}$. The wave vectors included in the summation over $\mathbf{k}$ are, therefore, $\mathbf{k}(\mathbf{m})$, which is given by

$$\mathbf{k}(\mathbf{m}) = \mathbf{k}_0 + \Delta\mathbf{k}(\mathbf{m}) ,$$

with

$$\Delta\mathbf{k}(\mathbf{m}) = (m_1\Delta k_1, m_2\Delta k_2, m_3\Delta k_3) \tag{2.6}$$

and

$$m_i = -M_i,\ldots,0,\ldots,+M_i .$$

Here $\Delta k_i = 2\pi/[\Delta_i(2M_i+1)]$ is the lattice spacing in the ith direction of wave-number space. Using this wave-number lattice, $\hat{\alpha}_{\mathbf{j}}$ will be defined as

$$\hat{\alpha}_{\mathbf{j}} = M^{-1/2}\sum_{\mathbf{m}} \hat{a}_{\mathbf{k}(\mathbf{m})}\exp[i\Delta\mathbf{k}(\mathbf{m})\cdot\mathbf{x}(\mathbf{j})] , \tag{2.7}$$

where

$$M = \prod_{i=1}^{3}(2M_i+1) .$$

This implies that the mode operator $\hat{\alpha}_{\mathbf{j}}$ describes a localized field excitation of a slowly varying field envelope, at the location $\mathbf{x}(\mathbf{j})$ on the spatial lattice indexed by (j_1,j_2,j_3). The definition of $\hat{\alpha}_{\mathbf{j}}$ guarantees that these operators have boson commutation relations, i.e.,

$$[\hat{\alpha}_{\mathbf{k}},\hat{\alpha}_{\mathbf{j}}^\dagger] = \delta_{\mathbf{k},\mathbf{j}}^{(3)} . \tag{2.8}$$

In order to treat the time evolution of the quantum system, it is often simplest to use an interaction picture in which the states evolve in time only through their interaction terms. This is achieved by writing the Hamiltonian as $\hat{H}=\hat{H}_0+\hat{H}_I$. For ease of calculation, the free Hamiltonian $\hat{H}_0$ is chosen to have only one resonant frequency ω_0 for the system operators, which therefore all have frequencies of multiples of ω_0 in the interaction picture. With this choice, there is an extensive cancellation of explicit time dependences in the resulting equations, provided the rotating-wave approximation is employed. The free Hamiltonian in the interaction picture is defined as

$$\hat{H}_0 = \hbar\omega_0\left[\sum_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \tfrac{1}{2}\sum_\mu\hat{\sigma}_\mu^z\right] + \hat{H}_3$$
$$= \hbar\omega_0\left[\sum_{\mathbf{j}}\hat{\alpha}_{\mathbf{j}}^\dagger\hat{\alpha}_{\mathbf{j}} + \sum_{\mathbf{n}}\hat{J}_{\mathbf{n}}^z\right] + \hat{H}_3 . \tag{2.9}$$

In the interaction picture the Hamiltonian $\hat{H}_0$ determines the operator time evolution, and the state evolution is

determined by the interaction Hamiltonian $\hat{H}_I$. In terms of the local operators $\hat{\alpha}_j, \hat{J}_n$, the original free Hamiltonians can now be written as

$$\hat{H}_1+\hat{H}_2+\hat{H}_3=\hat{H}_0+\hbar\left[\sum_j\sum_{j'}\Delta\omega(\mathbf{j},\mathbf{j}')\hat{\alpha}_j^\dagger\hat{\alpha}_{j'}+\sum_n\Delta\omega_n\hat{J}_n^z\right], \tag{2.10}$$

where

$$\Delta\omega_n=\omega_n-\omega_0 ,$$
$$\Delta\omega(\mathbf{j},\mathbf{j}')=\omega(\mathbf{j},\mathbf{j}')-\omega_0\delta_{\mathbf{j},\mathbf{j}'}^{(3)} ,$$
$$\omega(\mathbf{j},\mathbf{j}')=\frac{1}{M}\sum_{\mathbf{m}}\omega_{\mathbf{k}(\mathbf{m})}\exp\{i\Delta\mathbf{k}(\mathbf{m})\cdot[\mathbf{x}(\mathbf{j})-\mathbf{x}(\mathbf{j}')]\} .$$

Here the difference in detunings between atoms in a given frequency cell is ignored, and a discrete number of modes is assumed. Both of these approximations are increasingly reliable in the limit of large quantization volume and small cell volume in frequency and space. In addition to this, it is necessary to reexpress the direct interaction term $\hat{H}_4$ using the new operators. This requires a similar type of approximation, with small phase differences between neighboring atoms of order $\Delta\mathbf{k}\cdot\Delta\mathbf{x}$ being ignored. A new coupling constant $g'=gM^{1/2}$ must now be used for the collective operators, where M is the number of lattice points.

With this small phase-difference approximation, and leaving the reservoir interactions unchanged, the total Hamiltonian is now divided up into a nominally free and an interacting part, as follows

$$\hat{H}=\hat{H}_0+\hat{H}_I ,$$

where

$$\hat{H}_I=\hat{H}'_I+\hat{H}_5+\hat{H}_6$$

and

$$\hat{H}'_I=\hbar\left[\sum_j\sum_{j'}\Delta\omega(\mathbf{j},\mathbf{j}')\hat{\alpha}_j^\dagger\hat{\alpha}_{j'}+\sum_n\Delta\omega_n\hat{J}_n^z+g'\left[\sum_j\hat{\alpha}_j^\dagger\sum_n\hat{J}_n^-+\text{H.c.}\right]\right] . \tag{2.11}$$

We note here that time-invariant operators in the Schrödinger picture are denoted as $\hat{A}$. Corresponding time-dependent operators in the Heisenberg picture will be denoted as $\hat{A}(t)$ in all cases. The interaction picture operators of the system variables have a time dependence of

$$\begin{aligned}&\hat{\alpha}_{[I]}^\dagger(t)=\hat{\alpha}^\dagger\exp(i\omega_0 t) ,\\ &\hat{\alpha}_{[I]}(t)=\hat{\alpha}\exp(-i\omega_0 t) ,\\ &\hat{J}_{[I]}^+(t)=\hat{J}^+\exp(i\omega_0 t) ,\\ &\hat{J}_{[I]}^-(t)=\hat{J}^-\exp(-i\omega_0 t) ,\\ &\hat{J}_{[I]}^z(t)=\hat{J}^z .\end{aligned} \tag{2.12}$$

As system observables will usually appear in combinations which cancel these rapid time dependences, it will be generally appropriate to work with Schrödinger-picture operators, rather than interaction-picture operators. This standard procedure will be used in this paper, except when calculations take place explicitly in the Heisenberg picture.

III. HEISENBERG-LANGEVIN EQUATIONS OF MOTION

Of interest is either the temporal evolution of the density operator in the interaction picture or the evolution of the system operators in the Heisenberg picture. Here we will review the latter case, which was first applied to the propagation problem by Graham and Haken.[7] We will take a somewhat different approach to obtaining the Heisenberg equations of motion, based on the Hamiltonian $\hat{H}$ in Eq. (2.1). The derivation will be useful for establishing notation and for later comparison with the c-number stochastic equations.

In general terms, a system operator $\hat{A}_l(t)$ in the Heisenberg picture obeys the Heisenberg equation of motion

$$\frac{d}{dt}\hat{A}_l(t)=(i\hbar)^{-1}[\hat{A}_l(t),\hat{H}(t)] . \tag{3.1}$$

It was shown by Lax[13] and by Haken and Weidlich[14] that, within the Markov approximation for the coupling between the system and the reservoirs, this equation can be well approximated as

$$\frac{d}{dt}\hat{A}_l(t)=(i\hbar)^{-1}[\hat{A}_l(t),\hat{H}_s(t)]+\hat{D}_l(t)+\hat{F}_l(t) , \tag{3.2}$$

where

$$\hat{H}_s(t)=\hat{H}_1(t)+\hat{H}_2(t)+\hat{H}_4(t)$$

represents the Hamiltonian evolution of the system in the absence of the reservoirs and $\hat{D}_l(t)$ and $\hat{F}_l(t)$ represent the influence of the reservoirs. It is assumed that the average over reservoir variables of the Langevin operators $\hat{F}_l(t)$ is zero, i.e., $\langle\hat{F}_l(t)\rangle_R=0$. This means that $\langle\hat{D}_l(t)\rangle_R$, which is still a system operator, is equal to the mean "drift" or damping of the system operator under the action of the reservoirs. The Markovian approximation means that the Langevin operators are assumed to be δ correlated:

$$\langle\hat{F}_l(t)\hat{F}_m(t')\rangle_R=2\langle\hat{D}_{lm}(t)\rangle_R\delta(t-t') , \tag{3.3}$$

where the diffusion "coefficients" $\langle\hat{D}_{lm}(t)\rangle_R$, which are also system operators, will be given below. This approximation requires that the correlation time of $\hat{F}_l(t)$ is short compared to the damping times, but may be long compared to the natural oscillation times of the system variables. Lax[13] and Haken[14] showed that the diffusion coefficients are given by the time-dependent Einstein relation

$$2\langle \hat{D}_{lm}(t)\rangle_R = \left[\frac{d}{dt}\langle \hat{A}_l(t)\hat{A}_m(t)\rangle_R\right]_{\rm NH} - \langle \hat{D}_l(t)\hat{A}_m(t)\rangle_R - \langle \hat{A}_l(t)\hat{D}_m(t)\rangle_R \ . \tag{3.4}$$

The bracket labeled NH, around the first term, means that only the non-Hamiltonian terms $[\hat{D}_l(t)+\hat{F}_l(t)]$ are to be included in evaluating the derivative. This occurs because the Hamiltonian evolution of the system does not contribute to damping or diffusion.

Explicitly, Eq. (3.2) is written for the system (field and atoms) operators in the Heisenberg picture as

$$\begin{aligned}
\frac{d}{dt}\hat{a}_{\mathbf{k}}(t) &= -i\omega_{\mathbf{k}}\hat{a}_{\mathbf{k}}(t) - i\sum_{\mu} g\hat{\sigma}_{\mu}^{-}(t)e^{-i\mathbf{k}\cdot\mathbf{x}_\mu} \\
&\quad -\tfrac{1}{2}c\kappa\hat{a}_{\mathbf{k}}(t)+\hat{F}_{\mathbf{k}}^{a}(t) \ , \\
\frac{d}{dt}\hat{\sigma}_{\mu}^{-}(t) &= -i\omega_{\mu}\hat{\sigma}_{\mu}^{-}(t) + ig\hat{\sigma}_{\mu}^{z}(t)\sum_{\mathbf{k}}\hat{a}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}_\mu} \\
&\quad -\gamma_{\perp}\hat{\sigma}_{\mu}^{-}(t)+\hat{F}_{\mu}^{\sigma}(t) \ , \\
\frac{d}{dt}\hat{\sigma}_{\mu}^{z}(t) &= 2\sum_{\mathbf{k}}(ig\hat{a}_{\mathbf{k}}^{\dagger}(t)\hat{\sigma}_{\mu}^{-}(t)e^{-i\mathbf{k}\cdot\mathbf{x}_\mu}+\text{H.c.}) \\
&\quad -\gamma_{\parallel}[\hat{\sigma}_{\mu}^{z}(t)-\sigma^{\rm SS}]+\hat{F}_{\mu}^{z}(t) \ .
\end{aligned} \tag{3.5}$$

The equations for $\hat{a}_{\mathbf{k}}^{\dagger}(t)$ and $\hat{\sigma}_{\mu}^{+}(t)$ are given by the hermitian conjugate equations. In each of these equations the first two terms describe free evolution, the third term is the drift part $\hat{D}_l(t)$, and the fourth term is the Langevin operator. In Eq. (3.5), $c\kappa$ is the damping rate of the mode intensity caused by the background absorbers, where c is the speed of light. In the atomic equations the longitudinal and transverse damping rates are given by

$$\begin{aligned}
\gamma_{\parallel} &= W_{12}+W_{21} \ , \\
\gamma_{\perp} &= \gamma_P+\tfrac{1}{2}\gamma_{\parallel} \ ,
\end{aligned} \tag{3.6}$$

where W_{21} is the relaxation rate from the excited to the ground state, while W_{12} is an incoherent pumping rate from the ground to excited atomic state, and γ_P is the pure dephasing rate caused by quasielastic atomic collisions (in the impact approximation). The steady-state atomic inversion, denoted by $\sigma^{\rm SS}$ in Eq. (3.5), is given by

$$\sigma^{\rm SS}=\frac{W_{12}-W_{21}}{W_{12}+W_{21}} \tag{3.7}$$

and is equal to -1 in the absence of incoherent pumping. The population within the two-level system is assumed to be conserved: $|1\rangle\langle 1|+|2\rangle\langle 2|=\hat{1}$. The system operators obey their Schrödinger-picture commutation relations at any given time. Thus, the ordering of $\hat{a}$ and $\hat{\sigma}$ operators in Eqs. (3.5) is arbitrary since $[\hat{a}(t),\hat{\sigma}(t)]=0$.

From Eq. (3.4) it can be shown that the Langevin operator for the field-mode amplitude obeys the well-known relations[13–15]

$$\begin{aligned}
\langle \hat{F}_{\mathbf{k}}^{a\dagger}(t)\hat{F}_{\mathbf{k}'}^{a}(t')\rangle_R &= c\kappa\bar{n}\delta(t-t')\delta_{\mathbf{k}\mathbf{k}'}^{(3)} \ , \\
\langle \hat{F}_{\mathbf{k}}^{a}(t)\hat{F}_{\mathbf{k}'}^{a\dagger}(t')\rangle_R &= c\kappa(\bar{n}+1)\delta(t-t')\delta_{\mathbf{k}\mathbf{k}'}^{(3)} \ ,
\end{aligned} \tag{3.8}$$

where

$$\bar{n}=[\exp(\hbar\omega_0/k_BT_a)-1]^{-1} \ .$$

Here $\bar{n}$ is the mean equilibrium photon number in each mode of interest (in the absence of coupling to the active atoms), with T_a being the temperature of the background absorber reservoir.

From Eq. (3.4) it is also easy to show that the Langevin operators for the atoms obey the relations

$$\begin{aligned}
&\langle \hat{F}_{\mu}^{z}(t)\hat{F}_{\mathbf{k}}^{a}(t')\rangle = \langle \hat{F}_{\mu}^{\sigma}(t)\hat{F}_{\mathbf{k}}^{a}(t')\rangle = 0 \ , \\
&\langle \hat{F}_{\mu}^{\sigma\dagger}(t)\hat{F}_{\nu}^{\sigma}(t')\rangle_R \\
&\qquad = [\gamma_P(1+\langle \hat{\sigma}_{\mu}^{z}(t)\rangle_R)+W_{12}]\delta(t-t')\delta_{\mu\nu} \ , \\
&\langle \hat{F}_{\mu}^{\sigma}(t)\hat{F}_{\nu}^{\sigma\dagger}(t')\rangle_R \\
&\qquad = [\gamma_P(1-\langle \hat{\sigma}_{\mu}^{z}(t)\rangle_R)+W_{21}]\delta(t-t')\delta_{\mu\nu} \ , \\
&\langle \hat{F}_{\mu}^{\sigma\dagger}(t)\hat{F}_{\nu}^{\sigma\dagger}(t')\rangle_R = \langle \hat{F}_{\mu}^{\sigma}(t)\hat{F}_{\nu}^{\sigma}(t')\rangle_R = 0 \ , \\
&\langle \hat{F}_{\mu}^{z}(t)\hat{F}_{\nu}^{z}(t')\rangle_R = 2\gamma_{\parallel}[1-\sigma^{\rm SS}\langle \hat{\sigma}_{\mu}^{z}(t)\rangle_R]\delta(t-t')\delta_{\mu\nu} \ ,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\langle \hat{F}_{\mu}^{\sigma\dagger}(t)\hat{F}_{\nu}^{z}(t')\rangle_R = -2W_{12}\langle \hat{\sigma}_{\mu}^{+}(t)\rangle_R\delta(t-t')\delta_{\mu\nu} \ , \\
&\langle \hat{F}_{\mu}^{\sigma}(t)F_{\nu}^{z}(t')\rangle_R = -2W_{21}\langle \hat{\sigma}_{\mu}^{-}(t)\rangle_R\delta(t-t')\delta_{\mu\nu} \ , \\
&\langle \hat{F}_{\mu}^{z}(t)\hat{F}_{\nu}^{\sigma\dagger}(t')\rangle_R = -2W_{21}\langle \hat{\sigma}_{\mu}^{+}(t)\rangle_R\delta(t-t')\delta_{\mu\nu} \ , \\
&\langle \hat{F}_{\mu}^{z}(t)\hat{F}_{\nu}^{\sigma-}(t')\rangle_R = -2W_{12}\langle \hat{\sigma}_{\mu}^{-}(t)\rangle_R\delta(t-t')\delta_{\mu\nu} \ .
\end{aligned}$$

To derive these results, we used Eqs. (3.5) and the relations

$$\begin{aligned}
\hat{\sigma}_{\mu}^{\pm}(t)\hat{\sigma}_{\mu}^{\mp}(t) &= \tfrac{1}{2}[1\pm\hat{\sigma}_{\mu}^{z}(t)] \ , \\
\hat{\sigma}_{\mu}^{z}(t)\hat{\sigma}_{\mu}^{z}(t) &= 1 \ .
\end{aligned}$$

Using Eqs. (3.5), we can now derive equations of motion for the collective atomic operators $\hat{J}_{\mathbf{n}}^{\pm}(t),\hat{J}_{\mathbf{n}}^{z}(t)$ defined in Eq. (2.5), and the spatially localized field operators $\hat{\alpha}_{\mathbf{j}}(t)$ defined in Eq. (2.7). These are

$$\begin{aligned}
\frac{d}{dt}\hat{\alpha}_{\mathbf{j}}(t) &= -\tfrac{1}{2}c\kappa\hat{\alpha}_{\mathbf{j}}(t) - i\sum_{\mathbf{j}'}\omega(\mathbf{j},\mathbf{j}')\hat{\alpha}_{\mathbf{j}'}(t) \\
&\quad -i\sum_{n} g'\hat{J}_{\mathbf{n}}^{-}(t)+\hat{F}_{\mathbf{j}}^{\alpha}(t) \ , \\
\frac{d}{dt}\hat{J}_{\mathbf{n}}^{-}(t) &= -(\gamma_{\perp}+i\omega_n)\hat{J}_{\mathbf{n}}^{-}(t) \\
&\quad +2ig'\hat{\alpha}_{\mathbf{j}}(t)\hat{J}_{\mathbf{n}}^{z}(t)+\hat{F}_{\mathbf{n}}^{J}(t) \ , \\
\frac{d}{dt}\hat{J}_{\mathbf{n}}^{z}(t) &= -\gamma_{\parallel}[\hat{J}_{\mathbf{n}}^{z}(t)-\tfrac{1}{2}\sigma^{\rm SS}(t)N_{\mathbf{n}}] \\
&\quad +[ig'\hat{\alpha}_{\mathbf{j}}^{\dagger}(t)\hat{J}_{\mathbf{n}}^{-}(t)+\text{H.c.}]+\hat{F}_{\mathbf{n}}^{z}(t) \ ,
\end{aligned} \tag{3.10}$$

where $g'=M^{1/2}g$, and the Langevin operators have the nonvanishing correlations

$$\langle \hat{F}^{\alpha\dagger}_{\mathbf{j}}(t)\hat{F}^{\alpha}_{\mathbf{j}'}(t')\rangle_R = c\kappa\bar{n}\,\delta(t-t')\delta^{(3)}_{\mathbf{jj}'} ,$$

$$\langle \hat{F}^{\alpha}_{\mathbf{j}}(t)\hat{F}^{\alpha\dagger}_{\mathbf{j}'}(t')\rangle_R = c\kappa(\bar{n}+1)\delta(t-t')\delta^{(3)}_{\mathbf{jj}'} ,$$

$$\langle \hat{F}^{J\dagger}_{\mathbf{n}}(t)\hat{F}^{J}_{\mathbf{n}'}(t')\rangle_R = [2\gamma_P\langle \hat{J}^{z}_{\mathbf{n}}\rangle_R + N_{\mathbf{n}}(\gamma_P + W_{12})]\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} ,$$

$$\langle \hat{F}^{J}_{\mathbf{n}}(t)\hat{F}^{J\dagger}_{\mathbf{n}'}(t')\rangle_R = [-2\gamma_P\langle \hat{J}^{z}_{\mathbf{n}}\rangle_R + N_{\mathbf{n}}(\gamma_P + W_{21})]\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} ,$$

$$\langle \hat{F}^{z}_{\mathbf{n}}(t)\hat{F}^{z}_{\mathbf{n}'}(t')\rangle_R = \left[\frac{\gamma_\parallel}{2}\right](N_{\mathbf{n}} - 2\sigma^{\mathrm{SS}}\langle \hat{J}^{z}_{\mathbf{n}}\rangle_R)\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} , \quad (3.11)$$

$$\langle \hat{F}^{J\dagger}_{\mathbf{n}}(t)\hat{F}^{z}_{\mathbf{n}'}(t')\rangle_R = -W_{12}\langle \hat{J}^{+}_{\mathbf{n}}(t)\rangle_R\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} ,$$

$$\langle \hat{F}^{J}_{\mathbf{n}}(t)\hat{F}^{z}_{\mathbf{n}'}(t')\rangle_R = -W_{21}\langle \hat{J}^{-}_{\mathbf{n}}(t)\rangle_R\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} ,$$

$$\langle \hat{F}^{z}_{\mathbf{n}}(t)\hat{F}^{J\dagger}_{\mathbf{n}'}(t')\rangle_R = -W_{21}\langle \hat{J}^{+}_{\mathbf{n}}(t)\rangle_R\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} ,$$

$$\langle \hat{F}^{z}_{\mathbf{n}}(t)\hat{F}^{J}_{\mathbf{n}'}(t')\rangle_R = -W_{12}\langle \hat{J}^{-}_{\mathbf{n}}(t)\rangle_R\delta(t-t')\delta^{(4)}_{\mathbf{n},\mathbf{n}'} .$$

The collective Langevin operators are made up of sums of the individual atom Langevin operators, e.g.,

$$\hat{F}^{z}_{\mathbf{n}}(t) = \sum_{\mu\in s(\mathbf{n})}\hat{F}^{z}_{\mu}(t) . \quad (3.12)$$

There are assumed to be a large number of atoms within each cell, so the collective Langevin operators approximately obey the Gaussian decorrelation property which relates higher-order correlation functions to the two-time correlation functions given in Eq. (3.11).[8,15] Thus the full statistics of the quantum noise terms are specified.

Since our goal is to derive equations of motion that are useful for propagation problems, we will take the limit that the cell size becomes small, while still containing many atoms, and thereby obtain equations for continuum-field variables. In this limit the terms involving $\omega(\mathbf{j},\mathbf{j}')$ must generate the paraxial wave equation, and hence are equivalent to derivatives of the field variables,

$$i\sum_{\mathbf{j}'}\omega(\mathbf{j},\mathbf{j}')\hat{\alpha}_{\mathbf{j}'}(t) \simeq i\omega_0\hat{\alpha}(t,\mathbf{x}) + c\frac{\partial}{\partial z}\hat{\alpha}(t,\mathbf{x}) - \frac{ic^2}{2\omega_0}\nabla_\perp\hat{\alpha}(t,\mathbf{x}), \quad (3.13)$$

where $\mathbf{x}$ corresponds to $\mathbf{x}(\mathbf{j})$ and $\nabla_\perp$ is the transverse Laplacian $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$.

We now wish to define an appropriate set of continuum operators. We choose to use a normalization and units[17] that correspond to the Bloch vector and the Rabi frequency. That is, the continuum atomic operators will correspond to the coordinates of a Bloch[18] vector of unit length, while the field operators will correspond to the Rabi[19] frequency. In this convention, the atomic operators are dimensionless, while the field operators have dimensions of s^{-1}. The continuum variables are defined in a rotating frame by

$$\hat{R}^{\pm}(t,\mathbf{x}(\mathbf{j}),\omega_n) = [2\hat{J}^{\pm}_{\mathbf{n}}(t)e^{\mp i\omega_0 t}]/N_{\mathbf{n}} ,$$
$$\hat{R}^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n) = 2\hat{J}^{z}_{\mathbf{n}}(t)/N_{\mathbf{n}} , \quad (3.14)$$

while the continuum electric-field variable or Rabi frequency is

$$\hat{\Omega}(t,\mathbf{x},(\mathbf{j})) = 2ig'\hat{\alpha}_{\mathbf{j}}(t)e^{i\omega_0 t} = \left[\frac{2|\mathbf{d}_{12}\cdot\mathbf{e}|}{\hbar}\right]\hat{E}^{(+)}(t,\mathbf{x}(\mathbf{j}))e^{i(\omega_0 t - k_0 z_{\mathbf{j}})} ,$$

where, in SI units,

$$\hat{E}^{(+)}(t,\mathbf{x}(\mathbf{j})) = i\left[\frac{\hbar\omega_0 M}{2\epsilon_0 V}\right]^{1/2}\hat{\alpha}_{\mathbf{j}}(t)e^{ik_0 z_{\mathbf{j}}} .$$

This definition is used widely in semiclassical treatments of the two-level atom.[17] We prefer to use a Rabi frequency having dimensions of s^{-1} in the general theory presented here. Recent work in superfluorescence has used dimensionless Rabi-frequency variables.[8] Our notation can be adapted to this convention with the introduction of appropriate scaled time variables. However, our equations are also useful in other applications as well as superfluorescence, and for this reason, the choice of dimensionless time variables is left open at this stage.

Using Eq. (3.9), the new variables are easily shown to obey the equations of motion

$$\left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} - \frac{i}{2k_0}\nabla_\perp\right]\hat{\Omega}(t,\mathbf{x}) = -\tfrac{1}{2}\kappa\Omega(t,\mathbf{x}) + G\int\rho(\mathbf{x},\omega)\hat{R}^{-}(t,\mathbf{x},\omega)d\omega + \hat{F}^{\Omega}(t,\mathbf{x}) ,$$

$$\frac{\partial}{\partial t}\hat{R}^{-}(t,\mathbf{x},\omega) = -(\gamma_\perp + i\Delta\omega)\hat{R}^{-}(t,\mathbf{x},\omega) + \hat{\Omega}(t,\mathbf{x})\hat{R}^{z}(t,\mathbf{x},\omega) + \hat{F}^{R}(t,\mathbf{x},\omega) , \quad (3.15)$$

$$\frac{\partial}{\partial t}\hat{R}^{z}(t,\mathbf{x},\omega) = -\gamma_\parallel[\hat{R}^{z}(t,\mathbf{x},\omega) - \sigma^{\mathrm{SS}}] - \tfrac{1}{2}[\hat{\Omega}(t,\mathbf{x})\hat{R}^{+}(t,\mathbf{x},\omega) + \mathrm{H.c.}] + \hat{F}^{z}(t,\mathbf{x},\omega) .$$

Here we define

$$G = (Vg^2/c) = \omega_0|\mathbf{d}_{12}\cdot\mathbf{e}|^2/(2\epsilon_0\hbar c) .$$

The sum over atomic frequencies has been replaced by an integral with detunings defined by $\Delta\omega$, and the Langevin operators are defined by

$$\hat{F}^{\Omega}(t,\mathbf{x}(\mathbf{j})) = 2(g'/c)\hat{F}^{\alpha}_{\mathbf{j}}(t) ,$$
$$\hat{F}^{R}(t,\mathbf{x}(\mathbf{j}),\omega_n) = 2\hat{F}^{J}_{\mathbf{n}}(t)/N_{\mathbf{n}} , \quad (3.16)$$
$$\hat{F}^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n) = 2\hat{F}^{z}_{\mathbf{n}}(t)/N_{\mathbf{n}} .$$

Equations (3.15) are identical with the usual semiclassical propagation equations[17] for the slowly varying envelope fields, except for the presence of the Langevin operators. These operators have the nonvanishing correlation functions

$$\langle \hat{F}^{\Omega\dagger}(t,\mathbf{x}(\mathbf{j}))\hat{F}^{\Omega}(t',\mathbf{x}(\mathbf{j}'))\rangle_R = 4G\delta^{(3)}_{\mathbf{jj}'}\kappa\bar{n}\delta(t-t')/\Delta V ,$$

$$\langle \hat{F}^{\Omega}(t,\mathbf{x}(\mathbf{j}))\hat{F}^{\Omega\dagger}(t',\mathbf{x}(\mathbf{j}'))\rangle_R = 4G\delta^{(3)}_{\mathbf{jj}'}\kappa(\bar{n}+1)\delta(t-t')/\Delta V ,$$

$$\langle \hat{F}^{R\dagger}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{R}(t',\mathbf{x}(\mathbf{j}'),\omega_{\mathrm{n}})\rangle_R = 4\delta^{(4)}_{\mathbf{jj}'}[\gamma_P(1+\langle R^Z\rangle_R)+W_{12}]\delta(t-t')/N_{\mathrm{n}} ,$$

$$\langle \hat{F}^{R}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{R\dagger}(t',\mathbf{x}(\mathbf{j}'),\omega_{n'})\rangle_R = 4\delta^{(4)}_{\mathrm{nn}'}[\gamma_P(1-\langle R^z\rangle_R)+W_{21}]\delta(t-t')/N_{\mathrm{n}},$$

$$\hat{F}^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{z}(t',\mathbf{x}(\mathbf{j}'),\omega_{n'})\rangle_R = 2\delta^{(4)}_{\mathrm{nn}'}[\gamma_{\parallel}(1-\sigma^{\mathrm{SS}}\langle \hat{R}^{z}\rangle_R)]\delta(t-t')/N_{\mathrm{n}} , \quad (3.17)$$

$$\langle \hat{F}^{R\dagger}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{z}(t',\mathbf{x}(\mathbf{j}',\omega_{n'})\rangle_R = -2\delta^{(4)}_{\mathrm{nn}'}W_{12}\langle \hat{R}^{+}\rangle_R\delta(t-t')/N_{\mathrm{n}} ,$$

$$\langle \hat{F}^{R}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{z}(t',\mathbf{x}(\mathbf{j}'),\omega_{n'})\rangle_R = -2\delta^{(4)}_{\mathrm{nn}'}W_{21}\langle \hat{R}^{-}\rangle_R\delta(t-t')/N_{\mathrm{n}} ,$$

$$\langle \hat{F}^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{R\dagger}(t',\mathbf{x}(\mathbf{j}'),\omega_{n'})\rangle_R = -2\delta^{(4)}_{\mathrm{nn}'}W_{21}\langle \hat{R}^{+}\rangle_R\delta(t-t')/N_{\mathrm{n}} ,$$

$$\langle \hat{F}^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n)\hat{F}^{R}(t',\mathbf{x}(\mathbf{j}'),\omega_{n'}\rangle_R = -2\delta^{(4)}_{\mathrm{nn}'}W_{12}\langle \hat{R}^{-}\rangle_R\delta(t-t')/N_{\mathrm{n}} ,$$

where

$$N_{\mathrm{n}}=\Delta V\Delta\omega\rho(\mathbf{x}(\mathbf{j}),\omega_n).$$

The continuum limit is obtained by making the following replacements: $\mathbf{x}(\mathbf{j})\to\mathbf{x}$, $\omega_n\to\omega$, and

$$\frac{\delta^{(3)}_{\mathbf{jj}'}}{\Delta V}\to\delta^{(3)}(\mathbf{x}-\mathbf{x}') , \qquad \frac{\delta_{nn'}}{\Delta\omega}\to\delta(\omega-\omega'). \quad (3.18)$$

In order to specify a solution to Eqs. (3.15), it must be recalled that before the atoms and field begin to interact, the Heisenberg-picture operators are equal to their interaction-picture counterparts. Thus the field at the input face of the medium ($z=0$) is always equal to the free-field operator. For the case in which the excitation of the atoms sweeps through the medium with leading edge at $t=z/c$, the atomic operators $\hat{R}^{\pm,z}$ are equal to their noninteracting counterparts at $t\le z/c$. If the atoms are prepared in their excited states $|2\rangle$ at $t=z/c$, such as in superfluorescence, then the initial operators have the properties

$$\langle \hat{R}^{+}_{\mathrm{n}}\hat{R}^{-}_{\mathrm{n}'}\rangle=\frac{4\delta^{(4)}_{\mathrm{nn}'}}{N_{\mathrm{n}}} , \quad (3.19a)$$

$$\langle \hat{R}^{-}_{\mathrm{n}}\hat{R}^{+}_{\mathrm{n}'}\rangle=0 . \quad (3.19b)$$

On the other hand, if the atoms are in the their ground states at $t=z/c$, the initial operators obey

$$\langle \hat{R}^{+}_{\mathrm{n}}\hat{R}^{-}_{\mathrm{n}'}\rangle=0 , \quad (3.20a)$$

$$\langle \hat{R}^{-}_{\mathrm{n}}\hat{R}^{+}_{\mathrm{n}'}\rangle=\frac{4\delta^{(4)}_{\mathrm{nn}'}}{N_{\mathrm{n}}} . \quad (3.20b)$$

It is also known that the initial operators $\hat{R}^{+}$ and $\hat{R}^{-}$ obey the Gaussian decorrelation property, and so higher-order moments can be obtained from Eqs. (3.19) or (3.20).[8] In the case that the field is initially in a coherent state with classical amplitude $E_c(t,x,y)$ at $z=0$, the initial free-field operator at $z=0$ has the properties

$$\langle \hat{E}_f^{(+)}(t,x,y)\rangle=E_c(t,x,y)$$

and

$$\langle \hat{E}_f^{(-)}(t,x,y)\hat{E}_f^{(+)}(t',x',y')\rangle = E_c^{*}(t,x,y)E_c(t',x',y') , \quad (3.21a)$$

$$\langle \hat{E}_f^{(+)}(t,x,y)\hat{E}_f^{(-)}(t',x',y')\rangle = E_c^{*}(t,x,y)E_c(t',x',y') + [\hat{E}_f^{(+)}(t,x,y),\hat{E}_f^{(-)}(t',x',y')] . \quad (3.21b)$$

The commutator in (3.21) is a known c-number function. In addition, the free-field operator in the coherent state obeys the Gaussian decorrelation property.

Equations (3.15) were first derived by Graham and Haken in 1968.[7] Due to their nonlinear, operator nature, they have not yet been solved without the use of strong approximations. Two major approaches have been used. The first[7] involves linearization with respect to the operators around the c-number steady-state mean value, usually with just one mode present. In the presence of steady, incoherent pumping W_{12} in Eq. (3.6), a laserlike threshold was predicted, above which long-range order appears. At a still higher pumping rate, a second threshold was predicted, above which undamped spiking should occur. The equations are the quantum version of the semiclassical theory of a single-mode laser, which is formally equivalent to the Lorenz equations in chaos theory.[16]

The other approach to the solution of Eqs. (3.15) has been to retain the nonlinearities, but to treat the variables as c numbers, rather than as operators.[8] The justification of this approximation is only heuristic, relying on the intuitive argument that when the field amplitude is large, the quantum properties are relatively unimportant. In the case of superfluorescence, where $E_c=0$, this procedure still allows for fluctuations due to the Gaussian random nature of the initial conditions, characterized by Eqs. (3.19) and (3.21). Note that in simply replacing the operator equations by c-number equations, an ambiguity arises in the initial condition: Should one use Eqs. (3.19a) and (3.21a) or Eqs. (3.19b) and (3.21b)? It has been argued[20] that if one wants to obtain information about normally ordered operator expectations, such as light intensity $\langle \hat{E}^{(-)}\hat{E}^{(+)}\rangle$, one should use the normally ordered initial conditions, Eqs. (3.19a) and (3.21a) (with $E_c=0$), but if one wants information on antinormally ordered quantities, one should use as initial conditions Eqs. (3.19b) and (3.21b) (with $E_c=0$). It has also been argued that in the limit of large amplification, where the field becomes nearly classical, it does not matter, to a good ap-

proximation, which of these orderings one chooses.[20] But it should be recognized that the replacement of the operator equations by nearly equivalent c-number, stochastic equations can preclude the treatment of intrinsically quantum effects, such as squeezing and antibunching.

In fact, it is possible to obtain more rigorous c-number equations. This procedure uses the theory of representing operator evolution using Fokker-Planck equations, and will be treated in Sec. IV.

IV. MASTER EQUATION AND REPRESENTATIONS

We now turn to the approach of using density matrices. The time evolution of an initial density matrix $\hat{\rho}$ in the interaction picture is determined by the interaction Hamiltonian. For each spatial cell, the interaction terms and damping terms correspond precisely to the known Hamiltonian for a single-mode interferometer with an intracavity two-level medium. It is therefore straightforward to use existing master-equation formalisms to treat this problem. This simplification is possible only because of the use of the transformed field operators which behave in each small region or cell as a single-mode field. Propagation is achieved by the coupling of each $\hat{\alpha}_{\mathrm{j}}$ to neighboring cells, through the spatial coupling term $\omega(\mathrm{j},\mathrm{j}')$.

In the case of the present Hamiltonian, the term $\hat{H}'_I$ includes the system variables $\hat{\sigma}$ and $\hat{\alpha}$. The reservoir interaction can be written as $\hbar\sum_i \hat{\Gamma}_i \hat{A}_i$, where $\hat{A}_i$ is a system operator. The damping rates are determined by the interaction correlations via[15]

$$C_{ij}^{+}=\int_0^{\infty} e^{i\omega_i\tau}\langle \hat{\Gamma}_i(\tau)\hat{\Gamma}_j(0)\rangle_R d\tau\ ,$$
$$C_{ji}^{-}=\int_0^{\infty} e^{i\omega_i\tau}\langle \hat{\Gamma}_j(0)\hat{\Gamma}_i(\tau)\rangle_R d\tau\ . \tag{4.1}$$

The reservoir operators $\hat{\Gamma}_i(\tau)$ can be regarded as being either in the interaction picture or the Heisenberg picture, because the coupling between reservoir and system is weak. Here ω_i is the characteristic frequency of the ith system operator $\hat{A}_i$ in the free Hamiltonian $\hat{H}_0$. Clearly, from Eq. (2.12), the only possibilities are that $\omega_i=0,\pm\omega_0$. Using standard techniques, the overall master equation in the interaction picture is[15]

$$\frac{\partial\hat{\rho}}{\partial t}=\frac{1}{i\hbar}[\hat{H}_R+\hat{H}'_I,\hat{\rho}]$$
$$-\sum_{i,j}\delta(\omega_i,-\omega_j)[(\hat{A}_i\hat{A}_j\hat{\rho}-\hat{A}_j\hat{\rho}\hat{A}_i)C_{ij}^{+}$$
$$-(\hat{A}_i\hat{\rho}\hat{A}_j-\hat{\rho}\hat{A}_j\hat{A}_i)C_{ji}^{-}]\ . \tag{4.2}$$

This equation of motion of $\hat{\rho}$ for the atom-field system follows from using the technique of the Markovian approximation, and projecting out the reservoir variables. The result can be rewritten in the current case as

$$\frac{\partial}{\partial t}\hat{\rho}=\frac{1}{i\hbar}[\hat{H}_R+\hat{H}'_I,\hat{\rho}]+L_5[\hat{\rho}]+L_6[\hat{\rho}]\ , \tag{4.3}$$

where

$$\hat{H}'_I=\hbar\left[\sum_{\mathrm{j}}\sum_{\mathrm{j}'}\Delta\omega(\mathrm{j},\mathrm{j}')\hat{\alpha}^{\dagger}_{\mathrm{j}}\hat{\alpha}_{\mathrm{j}'}+\sum_{\mathrm{n}}\Delta\omega_n\hat{J}^{z}_{\mathrm{n}}\right.$$
$$\left.+g'\left(\sum_{\mathrm{j}}\hat{\alpha}^{\dagger}_{\mathrm{j}}\sum_{n}\hat{J}^{-}_{\mathrm{n}}+\mathrm{H.c.}\right)\right]$$

The operator $\hat{H}_R$ is a renormalization of the original Hamiltonian $\hat{H}'_I$, which gives the Lamb-shift terms for the radiative fluorescence reservoirs, as well as refractive index shift terms due to the background atoms. For simplicity, we suppose these terms are small enough to be neglected in the present treatment, so that $\hat{H}_R\approx 0$. These frequency and refractive index shifts can readily be included in general. The terms L_5 and L_6 describe relaxation into the reservoir modes. These are given in the atomic case by

$$L_5[\hat{\rho}]=\sum_{\mu}\left[\frac{W_{21}}{2}([\hat{\sigma}^{-}_{\mu}\hat{\rho},\hat{\sigma}^{+}_{\mu}]+[\hat{\sigma}^{-}_{\mu},\hat{\rho}\hat{\sigma}^{+}_{\mu}])\right.$$
$$\left.+\frac{W_{12}}{2}([\hat{\sigma}^{+}_{\mu}\hat{\rho},\hat{\sigma}^{-}_{\mu}]+[\hat{\sigma}^{+}_{\mu},\hat{\rho}\hat{\sigma}^{-}_{\mu}])\right]$$
$$+\frac{\gamma_P}{4}\sum_{\mu}([\hat{\sigma}^{z}_{\mu},\hat{\rho}\hat{\sigma}^{z}_{\mu}]+[\hat{\sigma}^{z}_{\mu}\hat{\rho},\hat{\sigma}^{z}_{\mu}]) \tag{4.4}$$

and in the field case by

$$L_6[\hat{\rho}]=\frac{c\kappa}{2}\sum_{\mathrm{k}}[(1+\bar{n})([\hat{a}_{\mathrm{k}}\hat{\rho},\hat{a}^{\dagger}_{\mathrm{k}}]+[\hat{a}_{\mathrm{k}},\hat{\rho}\hat{a}^{\dagger}_{\mathrm{k}}])$$
$$+\bar{n}([\hat{a}^{\dagger}_{\mathrm{k}}\hat{\rho},\hat{a}_{\mathrm{k}}]+[\hat{a}^{\dagger}_{\mathrm{k}},\hat{\rho}\hat{a}_{\mathrm{k}}])]\ . \tag{4.5}$$

Recall that W_{21} is the energy damping rate, W_{12} the optical pumping rate, while γ_P is the collisional phase-damping rate. The coefficient that describes absorption of the field is κ. This will be given in terms of observed absorption lengths in a subsequent section, where we find that κ^{-1} is the intensity absorption length.

The thermal temperatures of the radiative reservoirs for the atoms and of the absorbing reservoirs for the paraxial mode are given by T_σ and T_a, respectively, with photon occupation numbers of $\bar{n}_\sigma,\bar{n}$ where

$$\bar{n}_\sigma=[\exp(\hbar\omega_0/k_BT_\sigma)-1]^{-1}\ ,$$
$$\bar{n}=[\exp(\hbar\omega_0/k_BT_a)-1]^{-1}\ . \tag{4.6}$$

These reservoirs could therefore have different temperatures. In most cases at optical wavelengths, the reservoirs have a negligible effect since $k_BT\ll\hbar\omega_0$. These should be included in calculations at infrared wavelengths, however, as $k_BT\sim\hbar\omega_0$ at typical laboratory temperatures for mid-infrared wavelengths. When incoherent pumping is not present, the rates W_{12},W_{21} can be calculated from the temperature and from the Einstein A coefficient, to give

$$W_{21}=\gamma_0(1+\bar{n}_\sigma)\ ,$$
$$W_{12}=\gamma_0\bar{n}_\sigma\ , \tag{4.7}$$
$$\gamma_{\parallel}=\gamma_0(1+2\bar{n}_\sigma)\ .$$

With incoherent pumping this result would then be modified as in Eq. (3.6), which includes the possibility of arbitrary pumping or relaxation rates.

We next introduce a normally ordered operator representation, by defining a correspondence between complex c numbers written as A, and system operators $\hat{A}$:

$$\begin{aligned}
&\alpha_j^\dagger \leftrightarrow \hat{\alpha}_j^\dagger ,\\
&\alpha_j \leftrightarrow \hat{\alpha}_j ,\\
&J_n^+ \leftrightarrow \hat{J}_n^+ ,\\
&J_n^z \leftrightarrow \hat{J}_n^z ,\\
&J_n^- \leftrightarrow \hat{J}_n^- .
\end{aligned} \tag{4.8}$$

This correspondence is defined by equating operator moments of $\hat{\rho}$ with the c-number moments of a positive distribution $P(\boldsymbol{\alpha})$. That is, let the characteristic function be[11,15]

$$\hat{\chi}(\boldsymbol{\lambda}) = \prod_j e^{i\lambda_j^\dagger \hat{\alpha}_j^\dagger} e^{i\lambda_n \hat{\alpha}_j} \prod_n e^{i\lambda_n^+ \hat{J}_n^+} e^{i\lambda_n^z \hat{J}_n^z} e^{i\lambda_n^- \hat{J}_n^-} , \tag{4.9}$$

where

$$\boldsymbol{\lambda} = \{\lambda_j^\dagger, \lambda_j, \lambda_n^+, \lambda_n^z, \lambda_n^-\} .$$

Then the expectation value of $\hat{\chi}(\boldsymbol{\lambda})$ is defined as being equal to a corresponding characteristic of $P(\boldsymbol{\alpha})$, thus extending the technique of Haken and Louisell[15] to a complex phase space:

$$\mathrm{Tr}[\hat{\rho}\hat{\chi}(\boldsymbol{\lambda})] = \int P(\boldsymbol{\alpha}) e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\alpha}} \prod_i d^2\alpha_i , \tag{4.10}$$

where

$$\boldsymbol{\alpha} = \{\alpha_j^\dagger, \alpha_j, J_n^+, J_n^z, J_n^-\} .$$

Here the vector $\boldsymbol{\alpha}$ includes all the relevant c-number variables listed in Eq. (4.8). The integration in Eq. (4.10) covers all the complex phase space of each variable. Note that $\alpha, \alpha^\dagger$ are treated as distinct complex variables, as are J^-, J^+, and J^z. Thus, J^- and J^+ are not complex conjugates. We will use the term Hermitian conjugation to mean the replacement of J^- with J^+ and α with $\alpha^\dagger$, together with complex conjugation of constants, even though these are not operators.

This master equation can now be transformed to an equivalent Fokker-Planck equation for $P(\boldsymbol{\alpha})$, following standard operator algebra techniques. The result is similar to that for the single-mode laser, with the addition of extra coupling terms that describe propagation and diffraction:

$$\begin{aligned}
\frac{\partial}{\partial t} P(\boldsymbol{\alpha}) = &\left[\sum_n \mathcal{L}(\boldsymbol{\alpha}_n, N_n) + \left[i \sum_n \frac{\partial}{\partial J_n^-} \Delta\omega_n J_n^- + \mathrm{H.c.} \right] \right.\\
&\left. + \left[i \sum_j \sum_{j'} \frac{\partial}{\partial \alpha_j} \Delta\omega(\mathbf{j},\mathbf{j}') \alpha_{j'} + \mathrm{H.c.} \right] \right] P(\boldsymbol{\alpha}) ,
\end{aligned} \tag{4.11}$$

where $\mathcal{L}(\boldsymbol{\alpha}_n, N_n)$ is the standard laser Fokker-Planck operator,[15] with all variables evaluated at the nth lattice location:

$$\begin{aligned}
\mathcal{L}(\boldsymbol{\alpha}, N) = W_{12} &\left\{ \frac{N}{2} \left[e^{-\partial/\partial J^z} - 1 + e^{\partial/\partial J^z} \left(\frac{\partial^4}{\partial J^{-2} \partial J^{+2}} \right) + 2 \frac{\partial^2}{\partial J^- \partial J^+} \right] \right.\\
&+ \left[\frac{\partial}{\partial J^-} \left(\frac{\partial^2}{\partial J^- \partial J^+} + e^{-\partial/\partial J^z} - \tfrac{1}{2} \right) J^- + \mathrm{H.c.} \right]\\
&\left. - \left[e^{-\partial/\partial J^z} - 1 - e^{\partial/\partial J^z} \left(\frac{\partial^4}{\partial J^{-2} \partial J^{+2}} \right) \right] J^z \right\}\\
&+ W_{21} \left[(e^{\partial/\partial J^z} - 1) \left(J^z + \frac{N}{2} \right) + \tfrac{1}{2} \left(\frac{\partial}{\partial J^-} J^- + \frac{\partial}{\partial J^+} J^+ \right) \right]\\
&+ ig' \left[\left[(e^{-\partial/\partial J^z} - 1) J^- \alpha^\dagger - \frac{\partial^2}{\partial J^{+2}} J^+ \alpha^\dagger + 2 \frac{\partial}{\partial J^+} J^z \alpha^\dagger + \frac{\partial}{\partial \alpha} J^- \right] - \mathrm{H.c.} \right]\\
&+ \frac{c\kappa}{2} \left[\frac{\partial}{\partial \alpha} \alpha + \frac{\partial}{\partial \alpha^\dagger} \alpha^\dagger + 2\bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^\dagger} \right] + \gamma_P \left[\frac{\partial}{\partial J^-} J^- + \frac{\partial}{\partial J^+} J^+ + 2 \frac{\partial^2}{\partial J^+ \partial J^-} e^{\partial/\partial J_z} \left[J^z + \frac{N}{2} \right] \right] .
\end{aligned} \tag{4.12}$$

This equation is valid only when the distribution $P(\boldsymbol{\alpha})$ is sufficiently rapidly vanishing at the boundaries as $|\boldsymbol{\alpha}| \to \infty$ to allow the use of partial integration. Similar restrictions are required in most quasiprobability treatments of quantum systems. In practical applications, it is usually the case that the damping terms provide a strong bound at infinity on the distribution function. We note that our equations are essentially a generalization of the P distribution originally introduced by Glauber and Sudarshan,[21] which use a coherent state[22] basis for the radiation field. Other equivalent representations are possible,[23] including the Wigner distribution.[24] However, all these techniques require some care in their

application to atomic systems. In particular, a second-order equation with a positive-definite diffusion term is required when stochastic methods are utilized. This is not the case with Eq. (4.11), although this equation is exact as it stands.

V. STOCHASTIC EQUATIONS

While Eq. (4.11) is an exact partial differential equation, it has infinite-order derivatives, and is not guaranteed to result in a positive-definite distribution. However, to a good approximation, the terms of higher than second order in the derivatives may be neglected when the number of atoms per lattice point is large. The reason for this is that these terms are scaled by a relative factor of order $(1/N)$ when compared with lower-order derivatives. This argument, however, requires some care in its application. It is not necessarily valid in tunneling situations (where the distribution is rapidly varying) or were a lattice cell contains only a small number of atoms. When the truncation is permissible, Eq. (4.12) reduces to a second-order differential operator:

$$\begin{aligned}\lim_{N\to\infty}\mathcal{L}(\boldsymbol{\alpha},N)=&\left[\frac{\partial}{\partial\alpha}\left[\frac{c\kappa\alpha}{2}+ig'J^-\right]+\text{H.c.}\right]+c\kappa\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^\dagger}\\&+\left[\frac{\partial}{\partial J^-}\{[\gamma_P+\tfrac{1}{2}(W_{12}+W_{21})]J^--2ig'J^z\alpha\}+\text{H.c.}\right]+\left[ig'\frac{\partial^2}{\partial J^{-2}}J^-\alpha+\text{H.c.}\right]\\&+\frac{\partial^2}{\partial J^-\partial J^+}[W_{12}N+\gamma_P(N+2J^z)]-W_{12}\frac{\partial}{\partial J^z}\left[\frac{\partial}{\partial J^-}J^-+\frac{\partial}{\partial J^+}J^+\right]\\&+\frac{\partial}{\partial J_z}\left[ig'(J^+\alpha-J^-\alpha^\dagger)+W_{12}\left[J^z-\frac{N}{2}\right]+W_{21}\left[J^z+\frac{N}{2}\right]\right]\\&+\tfrac{1}{2}\frac{\partial^2}{\partial J_z^2}\left[ig'(J^-\alpha^+-J^+\alpha)+W_{12}\left[\frac{N}{2}-J^z\right]+W_{21}\left[J^z+\frac{N}{2}\right]\right].\end{aligned}\tag{5.1}$$

This second-order differential operator has, in general, a non-positive-definite diffusion term. It is often useful to be able to treat the system as behaving nearly classically in terms of a stochastic process on a generalized phase space. While stochastic processes on the usual semiclassical phase space cannot represent this quantum system, there are stochastic processes on the extended or complex phase space that are equivalent to Eq. (5.1). These are obtained on transforming the diffusion operator into an equivalent positive-definite form, giving rise to a Fokker-Planck equation. The resulting solution is called a positive P representation,[11] as an initially positive distribution remains so in time. The crucial step in obtaining a positive-definite diffusion involves the replacement of the analytic derivatives in Eq. (5.1) with the appropriate derivatives of the real and imaginary components of the variables $\boldsymbol{\alpha}$. This can always be carried out in a way that leaves the analytic moments—which are the measurable quantities—invariant. While the general proof is well known, specific examples in simple cases of parametric amplifiers[25] and optical solitons[26] are given in the literature. In cases where comparisons are possible, this gives identical predictions to those made using other representations.[27] The new Fokker-Planck equation generates stochastic driving terms F^α having correlation coefficients that are equal to the diffusion term in the Fokker-Planck equation, as usual. The resulting stochastic equations involve only analytic functions of $\boldsymbol{\alpha}$.

The stochastic equations that correspond to the transformed Fokker-Planck equations can now be calculated.[28] They have a form similar to the stochastic equations normally found in the theory of single-mode lasers or optical bistability.[11,15] The chief difference is the existence of the additional coupling term $\Delta\omega(\mathbf{j},\mathbf{j}')$, which couples adjacent spatial regions, causing propagation to occur. The stochastic equations depend on modified relaxation terms, which are defined as in Eqs. (3.6) and (3.7). These give the transverse and longitudinal relaxation rates of the atom, in the presence of the radiative and collisional reservoirs. Because of the reservoirs, the equilibrium inversion is modified. From Eq. (5.1), the original master equation in the limit of large N is equivalent to the following c-number stochastic equations:

$$\begin{aligned}\frac{d}{dt}\alpha_{\mathrm{j}}&=-\tfrac{1}{2}c\kappa\alpha_{\mathrm{j}}-i\sum_{\mathrm{j}'}\Delta\omega(\mathbf{j},\mathbf{j}')\alpha_{\mathrm{j}'}-i\sum_n g'J_{\mathrm{n}}^-+F_{\mathrm{j}}^\alpha(t)\ ,\\\frac{d}{dt}\alpha_{\mathrm{j}}^\dagger&=-\tfrac{1}{2}c\kappa\alpha_{\mathrm{j}}^\dagger+i\sum_{\mathrm{j}'}\Delta\omega(\mathbf{j},\mathbf{j}')\alpha_{\mathrm{j}'}^\dagger+i\sum_n g'J_{\mathrm{n}}^++F_{\mathrm{j}}^{\alpha^\dagger}(t)\ ,\\\frac{d}{dt}J_{\mathrm{n}}^-&=-(\gamma_\perp+i\Delta\omega_n)J_{\mathrm{n}}^-+2ig'\alpha_{\mathrm{j}}J_{\mathrm{n}}^z+F_{\mathrm{n}}^J(t)\ ,\\\frac{d}{dt}J_{\mathrm{n}}^+&=-(\gamma_\perp-i\Delta\omega_n)J_{\mathrm{n}}^+-2ig'\alpha_{\mathrm{j}}^\dagger J_{\mathrm{n}}^z+F_{\mathrm{n}}^{J^\dagger}(t)\ ,\\\frac{d}{dt}J_{\mathrm{n}}^z&=-\gamma_\parallel(J_{\mathrm{n}}^z-\tfrac{1}{2}\sigma^{\mathrm{SS}}N_{\mathrm{n}})-ig'(J_{\mathrm{n}}^+\alpha_{\mathrm{j}}-J_{\mathrm{n}}^-\alpha_{\mathrm{j}}^\dagger)+F_{\mathrm{n}}^z(t)\ .\end{aligned}\tag{5.2}$$

Here the nonstochastic part of the equations embodies the standard results of semiclassical theory. The

quantum-mechanical fluctuations are all contained in the stochastic terms Γ, which depend on the characteristics of the reservoirs, and of the nonlinear coupling between the atoms and the radiation field. These terms can be written as follows, in terms of elementary Gaussian stochastic processes:

$$
\begin{aligned}
&F_{\mathrm{j}}^{\alpha}(t)\equiv\xi_{\mathrm{j}}^{\alpha}(t)\sqrt{c\kappa\bar{n}}=[F_{\mathrm{j}}^{\alpha\dagger}(t)]^{*}\ ,\\
&F_{\mathrm{n}}^{J}(t)\equiv\xi_{\mathrm{n}}^{J}(t)\sqrt{2ig'\alpha_{\mathrm{j}}J_{\mathrm{n}}^{-}}+\xi_{\mathrm{n}}^{P}(t)\sqrt{\gamma_{P}(2J_{\mathrm{n}}^{z}+N_{\mathrm{n}})}\\
&\qquad+\xi_{\mathrm{n}}^{0}(t)\sqrt{W_{12}N_{\mathrm{n}}}\ ,\\
&F_{\mathrm{n}}^{J\dagger}(t)=\xi_{\mathrm{n}}^{J\dagger}(t)\sqrt{-2ig'\alpha_{\mathrm{j}}^{\dagger}J_{\mathrm{n}}^{+}}+\xi_{\mathrm{n}}^{P*}(t)\sqrt{\gamma_{P}(2J_{\mathrm{n}}^{z}+N_{\mathrm{n}})}\\
&\qquad+\xi_{\mathrm{n}}^{0*}(t)\sqrt{W_{12}N_{\mathrm{n}}}\ ,\qquad(5.3)\\
&F_{\mathrm{n}}^{z}(t)=\xi_{\mathrm{n}}^{z}(t)[(\gamma_{\parallel}/2)(N_{\mathrm{n}}-2\sigma^{\mathrm{SS}}J_{\mathrm{n}}^{z})\\
&\qquad+ig'(J_{\mathrm{n}}^{-}\alpha_{\mathrm{j}}^{\dagger}-J_{\mathrm{n}}^{+}\alpha_{\mathrm{j}})-2W_{12}J_{\mathrm{n}}^{+}J_{\mathrm{n}}^{-}/N_{\mathrm{n}}]^{1/2}\\
&\qquad-[\xi_{\mathrm{n}}^{0}(t)J_{\mathrm{n}}^{+}+\xi_{\mathrm{n}}^{0*}(t)J_{\mathrm{n}}^{-}]\sqrt{W_{12}/N_{\mathrm{n}}}.
\end{aligned}
$$

Here the noise terms F^{J} have been reduced to stochastic terms $\xi(t)$, which are independent, Gaussian processes. The terms $\xi_{\mathrm{j}}^{\alpha},\xi_{\mathrm{n}}^{0},\xi_{\mathrm{n}}^{P}$ are complex, while $\xi_{\mathrm{n}}^{J},\xi_{\mathrm{n}}^{J\dagger},\xi_{n}^{z}$ are real. Their nonvanishing correlation properties are

$$
\begin{aligned}
&\langle\xi_{\mathrm{j}}^{\alpha}(t)\xi_{\mathrm{j}'}^{\alpha*}(t')\rangle=\delta(t-t')\delta_{\mathrm{j},\mathrm{j}'}^{(3)}\ ,\\
&\langle\xi_{\mathrm{n}}^{0}(t)\xi_{\mathrm{n}'}^{0*}(t')\rangle=\delta(t-t')\delta_{\mathrm{n},\mathrm{n}'}^{(4)}\ ,\\
&\langle\xi_{\mathrm{n}}^{P}(t)\xi_{\mathrm{n}'}^{P*}(t')\rangle=\delta(t-t')\delta_{\mathrm{n},\mathrm{n}'}^{(4)}\ ,\\
&\langle\xi_{\mathrm{n}}^{J}(t)\xi_{\mathrm{n}'}^{J}(t')\rangle=\delta(t-t')\delta_{\mathrm{n},\mathrm{n}'}^{(4)}\ ,\\
&\langle\xi_{\mathrm{n}}^{J\dagger}(t)\xi_{\mathrm{n}'}^{J\dagger}(t')\rangle=\delta(t-t')\delta_{\mathrm{n},\mathrm{n}'}^{(4)}\ ,\\
&\langle\xi_{\mathrm{n}}^{z}(t)\xi_{\mathrm{n}'}^{z}(t')\rangle=\delta(t-t')\delta_{\mathrm{n},\mathrm{n}'}^{(4)}\ .
\end{aligned}
\qquad(5.4)
$$

These results comprise a set of ordinary stochastic differential equations, which can be relatively straightforwardly treated to obtain fully quantum-mechanical results for propagation. It is noteworthy that the solutions to these equations only involve c-number variables, not operators. This greatly simplifies the treatment of the problem, allowing direct computer simulation of the equations when necessary. The current results are valid only for large N, as terms of order $(1/N)$ relative to the included terms were omitted to obtain the appropriate stochastic equations.

It is also possible to obtain exact stochastic equations without truncation.[29] These can be calculated using the SU(2) coherent-state representation in each spatial cell, thus utilizing the spin-$\frac{1}{2}$ operator algebra. The use of SU(2) coherent states results in an exact equation, and can be extended to SU(N) coherent states for N levels. However, the SU(2) technique does not permit the treatment of collisional or fluorescent relaxation of the individual atoms, which is why the alternative technique of normal ordering is used here.

Equivalence of the Fokker-Planck equation to the original master equation requires the distribution to vanish as $|\boldsymbol{\alpha}|\rightarrow\infty$, which implies that all trajectories must be bounded. Certain master equations with nonlinear damping are known to generate boundary terms when the nonlinear noise terms are large compared with the linear damping.[30] This is not anticipated to occur in cases of interest here, with linear damping, and only moderate coupling strength between individual atoms in the field. However, the requirement that trajectories be bounded can be checked by investigating the stochastic trajectories numerically, where finite damping may be required to obtain stable (i.e., bounded) trajectories. In solving the equations numerically, we note that our equations are rigorously of the Ito[28] type of stochastic differential equation. The equations can be transformed to Stratonovich form, leading to additional terms of relative order $(1/N)$ in the deterministic part of Eq. (5.2), for numerical solution.[31]

We next wish to generate equations of motion for the field similar to the usual semiclassical Maxwell-Bloch equations and to the previous operator equations. To achieve this, the mode equations are transformed into propagation equations for a slowly varying field envelope. The new feature introduced here is the inclusion of the stochastic terms in the propagation equations, which gives results that otherwise have great similarities to the operator equations of Sec. III.

The terms involving $\Delta\omega(\mathbf{j},\mathbf{j}')$ are the result of taking a discretized approximation to a differential operator. This term corresponds to the usual slowly varying envelope approximation of Maxwell's equations in the limit of $\Delta V\rightarrow 0$. Hence, the term can be rewritten formally in the limit of small cell size. Using differential operators again, one obtains, where $\nabla_{\perp}$ is the transverse Laplacian,

$$
i\sum_{\mathbf{j}'}\Delta\omega(\mathbf{j},\mathbf{j}')\alpha_{\mathbf{j}'}(t)\simeq c\frac{\partial}{\partial z}\alpha(t,\mathbf{x})-\frac{ic^{2}}{2\omega_{0}}\nabla_{\perp}\alpha(t,\mathbf{x})\ .\qquad(5.5)
$$

It is useful at this point to scale the variables as in Sec. III. We therefore introduce stochastic field variables denoted $\Omega,\Omega^{\dagger}$ (Rabi frequency), R^{+},R^{-} (polarization), and R^{z} (inversion). The length of the Bloch vector with these varibles is $(R^{z})^{2}+(R^{+}R^{-})$, and is limited to a maximum expected value of unity. The resulting equations are then (omitting Hermitian conjugate equations for simplicity)

$$
\begin{aligned}
&\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}-\frac{i}{2k_{0}}\nabla_{\perp}\right]\Omega(t,\mathbf{x})\\
&\qquad=-\tfrac{1}{2}\kappa\Omega(t,\mathbf{x})+G\int\rho(\mathbf{x},\omega)R^{-}(t,\mathbf{x},\omega)d\omega+F^{\Omega}(t,\mathbf{x})\ ,\\
&\frac{\partial}{\partial t}R^{-}(t,\mathbf{x},\omega)=-(\gamma_{\perp}+i\Delta\omega)R^{-}(t,\mathbf{x},\omega)\\
&\qquad+\Omega(t,\mathbf{x})R^{z}(t,\mathbf{x},\omega)+F^{R}(t,\mathbf{x},\omega)\ ,\qquad(5.6)\\
&\frac{\partial}{\partial t}R^{z}(t,\mathbf{x},\omega)=-\gamma_{\parallel}[R^{z}(t,\mathbf{x},\omega)-\sigma^{\mathrm{SS}}]\\
&\qquad-\tfrac{1}{2}[\Omega(t,\mathbf{x})R^{+}(t,\mathbf{x},\omega)\\
&\qquad+\Omega^{\dagger}(t,\mathbf{x})R^{-}(t,\mathbf{x},\omega)]+F^{z}(t,\mathbf{x},\omega)\ ,
\end{aligned}
$$

where

$$\Omega(t,\mathbf{x},(\mathbf{j}))=2ig'\alpha_{\mathbf{j}} ,$$
$$R^{\pm}(t,\mathbf{x}(\mathbf{j}),\omega_n)=2J_{\mathbf{n}}^{\pm}/N_{\mathbf{n}} ,$$
$$R^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n)=2J_{\mathbf{n}}^{z}/N_{\mathbf{n}} .$$

The gain $G=Vg^2/c$ is defined as before. Noting that $g'=g(V/\Delta V)^{1/2}$, the noise sources F are obtained as follows:

$$\begin{aligned}
F^{\Omega}(t,\mathbf{x}(\mathbf{j}))&=2\xi_{\mathbf{j}}^{\alpha}(t)\left[\frac{G\kappa\bar{n}}{\Delta V}\right]^{1/2}=[F^{\Omega\dagger}(t,\mathbf{x}(\mathbf{j}))]^* ,\\
F^{R}(t,\mathbf{x}(\mathbf{j}),\omega_n)&=2\xi_{\mathbf{n}}^{P}(t)\left[\frac{\gamma_P(1+R^z)}{N_{\mathbf{n}}}\right]^{1/2}+\xi_{\mathbf{n}}^{J}(t)\left[\frac{2\Omega R^-}{N_{\mathbf{n}}}\right]^{1/2}+2\xi_{\mathbf{n}}^{0}(t)\sqrt{W_{12}/N_{\mathbf{n}}} ,\\
F^{R\dagger}(t,\mathbf{x}(\mathbf{j}),\omega_n)&=2\xi_{\mathbf{n}}^{P*}(t)\left[\frac{\gamma_P(1+R^z)}{N_{\mathbf{n}}}\right]^{1/2}+\xi_{\mathbf{n}}^{J\dagger}(t)\left[\frac{2\Omega^{\dagger}R^+}{N_{\mathbf{n}}}\right]^{1/2}+2\xi_{\mathbf{n}}^{0*}(t)\sqrt{W_{12}/N_{\mathbf{n}}} ,\\
F^{z}(t,\mathbf{x}(\mathbf{j}),\omega_n)&=\xi_{\mathbf{n}}^{z}(t)\left[\frac{2\gamma_{\parallel}(1-\sigma^{\mathrm{SS}}R^z)-(\Omega R^++\Omega^{\dagger}R^-)-2W_{12}R^+R^-}{N_{\mathbf{n}}}\right]^{1/2}-[\xi_{\mathbf{n}}^{0}(t)R^++\xi_{\mathbf{n}}^{0*}(t)R^-]\sqrt{W_{12}/N_{\mathbf{n}}} ,
\end{aligned} \tag{5.7}$$

where

$$N_{\mathbf{n}}=\rho(\mathbf{x}(\mathbf{j}),\omega_n)\Delta V\Delta\omega .$$

These scaled equations and their Hermitian conjugates completely define the spatial propagation of the quantum fluctuations in the system of interest. The noise terms are defined in terms of the lattice size to emphasize that this equation strictly holds only on a discretized lattice with $N_{\mathbf{n}}\gg 1$. We note that it is often useful to transform to a moving reference frame, thus removing the term in $(1/c)(\partial/\partial t)$ from Eq. (5.6).

We will now outline the appropriate initial conditions to use in the solution of Eq. (5.6). These are simple for a coherent field, which is accurately represented by the coherent-state representation. In this case, the initial variables $\Omega,\Omega^{\dagger}$ at location $\mathbf{x}_{\mathbf{j}}$ are distributed as

$$P(t,\Omega,\Omega^{\dagger})=\delta^{(2)}(\Omega^*-\Omega^{\dagger})\delta^{(2)}\left[\Omega-\left[\frac{2|\mathbf{d}_{12}\cdot\mathbf{e}|}{\hbar}\right]\times\langle\hat{E}^{(+)}(t,\mathbf{x}(\mathbf{j}))\rangle\times e^{i[\omega_0 t-k_0 z(\mathbf{j})]}\right] . \tag{5.8}$$

This is applied to the field in the region of $t,z\rightarrow-\infty$, before interaction with the atoms or reservoirs. We note, for example, that a vacuum state corresponds to $\Omega=0$, although interaction with the thermal reservoirs will turn this field into a thermal field if $T_a>0$.

For the atoms, the initial distribution is easily found by solving the stochastic equation in the absence of the field, thus giving the appropriate distribution due to the pump alone. For the case of a ground-state or inverted atomic population, respectively, we find

(a) Ground state,

$$P(\mathbf{R})=\delta^{(2)}(R^+)\delta^{(2)}(R^-)\delta^{(2)}(R^z+\tfrac{1}{2}) ,$$

(b) Excited state,

$$P(\mathbf{R})=\frac{N}{4\pi}\delta^{(2)}(R^+-(R^-)^*)\delta^{(2)}(R^z-\tfrac{1}{2})\times\exp(-NR^+R^-/4), \tag{5.9}$$

where $N=N_{\mathbf{n}}$ is the cell population.

Since these results are obtained for reservoir couplings alone, the nonclassical phase space is not involved. This is evidenced by the fact that, in these distributions, $\Omega^{\dagger}=\Omega^*$, $R^+=(R^-)^*$, and $\mathrm{Im}(R^z)=0$. We note that the distribution is a smooth function of R^+R^-, in the inverted state. This is due to the truncation of the Fokker-Planck equation to second order in its derivatives. It is expected that the solution to the complete equation to all orders in $(1/N)$ would show singularities. These are not evident here, since Eq. (5.9) is consistent with the truncations used to obtain the Fokker-Planck equation (5.2), and hence involves a large-N approximation.

VI. DISCUSSION

It is interesting to compare the stochastic equations (5.6) and (5.7) with the scaled operator equations (3.15)–(3.17) obtained previously. If we take expectation values, and assume that operator and stochastic products factorize, then either set of equations gives the usual Maxwell-Bloch result. These are valid in the semiclassical limit, to zeroth order in the $(1/N)$ expansion. Next, the noise terms can be compared. It is necessary to compare normally ordered products as the stochastic equations correspond to normal operator orderings. Calculating the corresponding correlation functions using Eq. (5.7), it is seen that there is complete agreement in the reservoir-related terms, but there are now additional stochastic terms involving the coupling g that correspond to

operator products appearing in the time-evolution equations.

The reason for this difference is straightforward. When operator products occur, the ordering of the initial conditions gradually becomes altered through time evolution. Thus, the approach discussed in Sec. III of replacing the operators by c numbers is no longer valid. In this nonlinear regime, it is not possible to maintain a consistent replacement of operators by c-number quantities using just a classical phase space. Instead, a nonclassical phase space is required. The additional stochastic terms are exactly of the correct size to generate nonclassical trajectories corresponding to operator products.

While the two approaches are essentially equivalent, there are some differences. We first notice that the operator equations have no noise attributed to operator products. However, these are usually only soluble when linearization assumptions are used. The stochastic equations have the advantage that they can be numerically simulated. This approach is used in a companion paper, in which good agreement between theory and experiment is found in treating the transition of superfluorescence to amplified spontaneous emission.[6] In this treatment the nonclassical quantum-noise terms were dropped since collisional and fluorescence fluctuations were larger than any nonclassical effects.

Although simpler to treat numerically, the stochastic equations are derived from a Fokker-Planck equation whose equivalence to the original master equation is exact only when boundary terms vanish. We emphasize that this is a universal criterion in developing quasiprobability distributions, and remains an open mathematical problem. In specific cases, it is necessary to solve the resulting equations and demonstrate that boundary terms vanish sufficiently rapidly at large radius in phase space. This implies that all stochastic trajectories must be bounded for the results to be reliable, as they are in the case of the transition from superfluorescence to amplified spontaneous emission.[6]

Stochastic equations like (5.2) have usually been treated by neglecting the nonclassical components of the noise terms. An exception is theoretical work on optical bistability and four-wave mixing in which nonclassical terms were retained, although only one spatial mode was treated.[11,32] These calculations resulted in predictions of photon antibunching and squeezing, respectively, which are verified experimentally. Here, boundary terms in the equations vanished exponentially, owing to the use of linearization approximations. However, it is clear that the nonclassical theory is not restricted to linearized one-mode problems. Other applications include the study of quantum effects in nonlinear dynamical equations, as an example of quantum chaos.

It is more difficult to treat nonlinear propagation problems in which quantum effects are significant, although nonclassical effects were predicted[33] and recently observed[34] in coherently generated solitons using stochastic proagation equations. The techniques given here provide a mathematical tool for progress toward solving this type of problem in atomic systems. We note, in particular, that obvious methods like photon-number-state expansions do *not* appear to be useful, owing to the enormous basis sets required. The transition matrices for just the field part of the density operator typically scale as $(1+N_P)^{4M}$ for a cutoff at N_P photons per lattice point. This means that the problem diverges exponentially with the total photon-number cutoff of MN_P. Stochastic or path-integral methods therefore are preferable when photon numbers are large.

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