Group-theoretical approach to the classical and quantum oscillator with time-dependent mass and frequency

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We investigate the classical and quantum oscillator with time-dependent mass and frequency, in the framework of the Lie group theory. Our analysis shows that both systems possess five time-dependent Noether invariants, where only two of them are functionally independent. In the classical case, three of these invariants form the su(1,1) Lie algebra under the Poisson bracket operation, while in the quantum case the corresponding invariants form the su(1,1) Lie algebra under the commutator operation. Since this algebra underlies the noncompact group SU(1,1), we are thus able to define unambiguously the dynamical group related to the generalized quantum oscillator. Furthermore, the Hamiltonian of this system can be expressed in terms of the Noether invariant operators satisfying the su(1,1) Lie algebra. This form is useful in the evaluation of the energy spectrum and in the study of coherence and squeezing. Some particular cases are briefly illustrated. The other two Noether invariants can be interpreted as the quadrature-phase amplitudes involved in the quantum nondemolition measurements and in the problem of the generation of coherent and squeezed states.

I. INTRODUCTION

In certain problems pertinent to quantum optics [1,2], acoustic and plasma physics, and other fields [3], some processes occur that are described by time-dependent Hamiltonians. These include, for example, the degenerate parametric amplifier [4], the quantum system corresponding to the Kanai-Caldirola Hamiltonian [5–8], and the quantum Morse oscillator in a single-mode classical electromagnetic field [9]. Quantum-mechanical parametric devices are of interest also because they may generate coherent and squeezed states [10]. It is therefore useful to set up a theoretical framework where a general time-dependent (mass and frequency) oscillator (TDO) may be studied.

Following this line of research, in this paper we first carry out a group analysis of a classical TDO governed by the Hamiltonian

$$H=p^2/(2m)+\tfrac{1}{2}m\omega^2q^2 , \tag{1.1}$$

where the conjugate variables q and p are c numbers, and $m=m(t)$ and $\omega=\omega(t)$ are given functions of time.

The Hamiltonian (1.1) gives rise to the equation of motion

$$\ddot{q}+M\dot{q}+\omega^2q=0 , \tag{1.2}$$

where $M=M(t)=\dot{m}/m$ (the dot means time derivative).

We find that Eq. (1.2) affords a symmetry group G of Lie point transformations which consists of eight generators. Five of these, which are of the Noether type, provide as many invariants (constants of the motion) but only two of them are functionally independent. Such invariants yield the general solution of Eq. (1.2). The remaining three generators do not lead to Noether invariants, but can be used to derive alternative Lagrangians for Eq. (1.2) [11]. The algebra defined by the generators of the symmetry group G is formally the same as that related to the oscillator (1.2) with $M=0$. The reader is referred to Ref. [11] for a discussion of the properties of such an algebra. We notice that three of the five Noether invariants admitted by Eq. (1.2) form the su(1,1) Lie algebra under the Poisson bracket operation. This fact allows us to define unambiguously the dynamical group associated with the TDO (1.2), which turns out to be the Lie group SU(1,1). This noncompact group is important in evaluating the energy spectrum as well as the degeneracy of levels of the quantum version of Eq. (1.2) [12].

Second, we extend our investigation to a quantum model represented by a Hamiltonian $\hat{H}$ of the form (1.1) where now q and p are replaced by two conjugate operators, Q and P, which can be expressed in terms of a pair of time-dependent lowering and raising operators, $a(t)$ and $a^\dagger(t)$. In correspondence to the classical Noether invariants exhibited by Eq. (1.2), we obtain a set of Hermitian operators $\hat{I}_j$ ($j=1,2,\ldots,5$) such that

$$\frac{d}{dt}\hat{I}_j=\frac{1}{i\hbar}[\hat{I}_j,\hat{H}]+\frac{\partial\hat{I}_j}{\partial t}=0 . \tag{1.3}$$

Three of these quantum invariants obey the commutation relations defining the subalgebra su(1,1) underlying the dynamical group SU(1,1). Thus, as a consequence of the symmetry properties possessed by the quantum system driven by $\hat{H}$, a natural realization of the su(1,1) algebra in terms of $a(t)$ and $a^\dagger(t)$ is furnished. The quantum Hamiltonian $\hat{H}$ can be written as a linear combination of the generators of the group SU(1,1). This feature is important in the study of the mechanism of generation of coherent and squeezed states associated with the su(1,1) Lie algebra [13,14].

Another results that seems noteworthy is that two of the five quantum invariants $\hat{I}_j$ may be identified with the so-called quadrature-phase amplitude operators, which play a basic role in the treatment of some optical devices [10,15] and in the context of quantum nondemolition measurements [16]. As a matter of fact, by means of the variances of these operators one can formulate an uncertainty relation that allows one to distinguish a squeezed state from a squeezed coherent state [17].

This paper is organized as follows. In Sec. II we develop the group-theoretical technique that is applied, in Sec. III, to derive the symmetry group and the Noether invariants for Eq. (1.2). Section IV is devoted to get a set of five invariant operators for a system related to the quantum Hamiltonian $\hat{H}$ corresponding to the classical one (1.1). The quantum Hamiltonian is expressed in terms of the invariant operators which obey the su(1,1) commutation relations, and some applications to certain physical cases are considered. In Sec. V some final comments are reported, while the Appendix contains details of the calculation.

II. THE LIE GROUP APPROACH

Let us consider a Lie group G of local point transformations, depending on one parameter ϵ and with nonzero Jacobian, acting on (t,q), namely,

$$t'=R(t,q;\epsilon) , \tag{2.1a}$$

$$q'=S(t,q;\epsilon) , \tag{2.1b}$$

where the functions R and S are differentiable with respect to t, and the value $\epsilon=0$ corresponds to the identity transformation $t=R(t,q;0)$, $q=S(t,q;0)$ [18].

The transformations (2.1) are generated by the infinitesimal operator (vector field)

$$V=\xi(t,q)\partial_t+\phi(t,q)\partial_q , \tag{2.2}$$

where $\partial_t=\partial/\partial t$, $\partial_q=\partial/\partial q$, and

$$\xi(t,q)=\left.\frac{\partial R(t,q;\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} , \quad \phi(t,q)=\left.\frac{\partial S(t,q;\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} . \tag{2.3}$$

Regarding ϵ as a perturbative parameter, Eqs. (2.1) yield the infinitesimal transformations

$$t'=t+\epsilon\xi(t,q), \quad q'=q+\epsilon\phi(t,q) , \tag{2.4}$$

at the first order in ϵ.

With respect to (2.4), $\dot{q}$ changes as

$$\dot{q}'=\dot{q}+\epsilon\phi^t , \tag{2.5}$$

where $\dot{q}'=dq'/dt'$,

$$\phi^t=\phi_t+[\phi_q-(\xi_t+\xi_q\dot{q})]\dot{q} , \tag{2.6}$$

and subscripts denote partial derivatives [11].

The finite transformations corresponding to Eqs. (2.4) and (2.5) take the form

$$t'=[\exp(\epsilon V)]t, \quad q'=[\exp(\epsilon V)]q ,$$
$$\dot{q}'=[\exp(\epsilon \mathrm{pr}^{(1)}V)]\dot{q} , \tag{2.7}$$

where

$$\mathrm{pr}^{(1)}V=V+\phi^t\partial_{\dot{q}} \tag{2.8}$$

is the first prolongation [19] of the vector field V.

The group G of transformations (2.1) is called a symmetry group of a second-order ordinary differential equation,

$$\ddot{q}=F(t,q,\dot{q}) , \tag{2.9}$$

where F is a given function, if $q'(t')=g\circ q(t')$ is a solution of Eq. (2.9) for $g\in G$ so that $g\circ q$ is defined whenever $q(t)$ satisfies Eq. (2.9). (The symbol $\circ$ denotes composition of functions.)

The symmetry group G, which transforms solutions of Eq. (2.9) to other solutions, can be obtained via an algorithmic procedure. This allows us to write down the Lie algebra of vector fields as Eq. (2.2), underlying the Lie group G. In such a way we can determine the coefficients ξ and ϕ, which come from the relation

$$\mathrm{pr}^{(2)}V[\ddot{q}-F(t,q,\dot{q})]=0 , \tag{2.10}$$

whenever $\ddot{q}-F(t,q,\dot{q})=0$, for every infinitesimal generator V of G.

The operator $\mathrm{pr}^{(2)}V$, defined by

$$\mathrm{pr}^{(2)}V=V+\phi^t\partial_{\dot{q}}+\phi^{tt}\partial_{\ddot{q}} , \tag{2.11}$$

is the second prolongation of V, where ϕ^t is given by (2.6) and

$$\phi^{tt}=\frac{d^2}{dt^2}(\phi-\xi\dot{q})+\xi\dddot{q} . \tag{2.12}$$

Equation (2.10) can be explicited to yield

$$(\phi_q-2\xi_t-3\xi_q\dot{q})F-\xi F_t-\phi F_q$$
$$-[\phi_t+(\phi_q-\xi_t)\dot{q}-\xi_q\dot{q}^2]F_{\dot{q}}+\phi_{tt}$$
$$+(2\phi_{qt}-\xi_{tt})\dot{q}+(\phi_{qq}-2\xi_{qt})\dot{q}^2-\xi_{qq}\dot{q}^3=0 . \tag{2.13}$$

Equation (2.13) can be regarded as the starting point to obtain all the Lie point symmetries for a given differential equation of the form (2.9). We notice that these comprise also the so-called divergence symmetries, which lead to the constants of motion of the Noether type for the equation under consideration [19].

The divergence symmetries, whose generators form a subalgebra of the Lie algebra corresponding to G, called Noether symmetry algebra, can be singled out in the following manner. Let us suppose that Eq. (2.9) coincides with the Euler-Lagrange equation which can be derived from the variational integral (action)

$$S=\int_{t_1}^{t_2}\mathcal{L}(t,q,\dot{q})dt , \tag{2.14}$$

where $\mathcal{L}$ is the (density) Lagrangian. Then, a vector field V of the form (2.2) is said to be an infinitesimal divergence symmetry of S if there exists a function $B(t,q)$ so that

$$(\not{h}^{(1)}V)\mathcal{L}+\mathcal{L}\frac{d}{dt}\xi=\frac{d}{dt}B \tag{2.15}$$

for all t,q.

The set of all the vector fields V obtained by (2.15) is called the Noether symmetry group G_N exhibited by Eq. (2.9). This is a subgroup of the (complete) symmetry group G. If Eq. (2.9) admits a Noether symmetry group, then the conservation equation

$$\frac{d}{dt}I=0\ , \tag{2.16}$$

where

$$I=(\xi\dot{q}-\phi)\partial_{\dot{q}}\mathcal{L}-\xi\mathcal{L}+B \tag{2.17}$$

holds, if and only if the action integral (2.14) is invariant with respect to G_N. This is an extended version of the original Noether theorem [20]. The conserved quantities (2.17) are called Noether invariants.

III. CLASSICAL THEORY

In this section we shall look for the (complete) Lie point symmetry algebra associated with Eq. (1.2). To this aim, we notice first that such an equation can be derived by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}m(\dot{q}^{\,2}-\omega^2q^2)\ . \tag{3.1}$$

Then, substituting the quantity $F=-\omega^2q-M\dot{q}$ into (2.13) and equating the coefficients of powers of $\dot{q}$ to zero, we are led to the expressions

$$\xi=a_1q+a_2\ , \tag{3.2}$$

$$\phi=(\dot{a}_1-Ma_1)q^2+b_1q+b_2\ , \tag{3.3}$$

where a_1, a_2, b_1, and b_2 are time-dependent functions of integration which obey the constraints

$$\ddot{a}_1-M\dot{a}_1+(\omega^2-\dot{M})a_1=0\ , \tag{3.4a}$$

$$2\dot{b}_1-\ddot{a}_2+\dot{M}a_2+M\dot{a}_2=0\ , \tag{3.4b}$$

$$\ddot{b}_1+M\dot{b}_1+2\omega^2\dot{a}_2+2\omega\dot{\omega}a_2=0\ , \tag{3.4c}$$

$$\ddot{b}_2+M\dot{b}_2+\omega^2b_2=0\ . \tag{3.4d}$$

In order to handle the system of linear differential equations (3.4), let us put in Eq. (3.4a)

$$a_1=m^{1/2}A\ , \tag{3.5}$$

where $A=A(t)$. Then Eq. (3.4a) reproduces the Hill equation

$$\ddot{A}+\Omega^2A=0\ , \tag{3.6}$$

with $\Omega=\Omega(t)$ given by

$$\Omega^2=\tfrac{1}{4}(4\omega^2-2\dot{M}-M^2)\ . \tag{3.7}$$

Equation (3.6) is formally solved by

$$A=\eta(c_1\cos\alpha+c_2\sin\alpha)\ , \tag{3.8}$$

where c_1 and c_2 are arbitrary constants, and the functions $\eta=\eta(t)$ and $\alpha=\alpha(t)$ are defined by

$$\ddot{\eta}+\Omega^2\eta=\frac{1}{\eta^3}\ , \tag{3.9}$$

$$\alpha=\int\frac{1}{\eta^2}dt\ . \tag{3.10}$$

Taking account of (3.8), after some manipulations Eqs. (3.4) yield the following expressions for the quantities a_1, a_2, b_1, and b_2:

$$a_1=m^{1/2}\eta(c_1\cos\alpha+c_2\sin\alpha)\ , \tag{3.11a}$$

$$a_2=\sigma^2(c_5\cos\theta+c_6\sin\theta+c_7)\ , \tag{3.11b}$$

$$\begin{aligned}b_1=c_5&\left[\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]\cos\theta-\frac{1}{2}\sin\theta\right]\\&+c_6\left[\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]\sin\theta+\frac{1}{2}\cos\theta\right]\\&+c_7\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]+c_8\ ,\end{aligned} \tag{3.11c}$$

$$b_2=m^{-1/2}\eta(c_3\cos\alpha+c_4\sin\alpha)\ , \tag{3.11d}$$

where $c_3,\ldots,c_8$ are arbitrary constants and the functions $\sigma=\sigma(t)$, $\theta=\theta(t)$ are such that

$$\ddot{\sigma}+\Omega^2\sigma=\frac{1}{4\sigma^3}\ , \tag{3.12}$$

$$\theta=\int\frac{1}{\sigma^2}dt\ . \tag{3.13}$$

Inserting (3.11) into (3.2) and (3.3), we deduce

$$\xi=m^{1/2}\eta(c_1\cos\alpha+c_2\sin\alpha)q+\sigma^2(c_5\cos\theta+c_6\sin\theta+c_7)\ , \tag{3.14}$$

$$\begin{aligned}\phi=m^{1/2}&\left\{c_1\left[\left[\dot{\eta}-\frac{M}{2}\eta\right]\cos\alpha-\frac{1}{\eta}\sin\alpha\right]+c_2\left[\left[\dot{\eta}-\frac{M}{2}\eta\right]\sin\alpha+\frac{1}{\eta}\cos\alpha\right]\right\}q^2\\&+\left\{c_5\left[\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]\cos\theta-\frac{1}{2}\sin\theta\right]+c_6\left[\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]\sin\theta+\frac{1}{2}\cos\theta\right]\right.\\&\left.+c_7(\sigma\dot{\sigma}-\frac{M}{2}\sigma^2)+c_8\right\}q+m^{-1/2}\eta(c_3\cos\alpha+c_4\sin\alpha)\ .\end{aligned} \tag{3.15}$$

Looking at Eqs. (3.9) and (3.12), we infer that the functions α,θ and η,σ are mutually dependent, in the sense that

$$\alpha=\theta/2, \quad \eta=\sqrt{2}\sigma \ . \tag{3.16}$$

Thus, by introducing Eqs. (3.14) and (3.15) into Eq. (2.2) we can write the general expression for the generator of the (complete) Lie point symmetries related to Eq. (1.2). Then, by choosing $c_1=1$, $c_j=0$ $(j\neq 1)$, $c_2=1$, $c_j=0$ $(j\neq 2)$, and so on, we obtain that the Lie point symmetry algebra for Eq. (1.2) is spanned by the eight vector fields

$$V_1=\sigma^2(\cos\theta)\partial_t+\left[\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]\cos\theta-\frac{1}{2}\sin\theta\right]q\partial_q \ , \tag{3.17a}$$

$$V_2=\sigma^2(\sin\theta)\partial_t+\left[\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]\sin\theta+\frac{1}{2}\cos\theta\right]q\partial_q \ , \tag{3.17b}$$

$$V_3=\sigma^2\partial_t+\left[\sigma\dot{\sigma}-\frac{M}{2}\sigma^2\right]q\partial_q \ , \tag{3.17c}$$

$$V_4=m^{-1/2}\sqrt{2}\sigma\left[\cos\frac{\theta}{2}\right]\partial_q \ , \tag{3.17d}$$

$$V_5=m^{-1/2}\sqrt{2}\sigma\left[\sin\frac{\theta}{2}\right]\partial_q \ , \tag{3.17e}$$

$$V_6=\sqrt{2m}\left\{\sigma\left[\cos\frac{\theta}{2}\right]q\partial_t+\left[\left[\dot{\sigma}-\frac{M}{2}\sigma\right]\cos\frac{\theta}{2}-\frac{1}{2\sigma}\sin\frac{\theta}{2}\right]q^2\partial_q\right\} , \tag{3.17f}$$

$$V_7=\sqrt{2m}\left\{\sigma\left[\sin\frac{\theta}{2}\right]q\partial_t+\left[\left[\dot{\sigma}-\frac{M}{2}\sigma\right]\sin\frac{\theta}{2}+\frac{1}{2\sigma}\cos\frac{\theta}{2}\right]q^2\partial_q\right\} , \tag{3.17g}$$

$$V_8=q\partial_q \ . \tag{3.17h}$$

Each operator (3.17) generates a one-parameter subgroup of Lie point symmetries for Eq. (1.2). The commutation relations satisfied by $V_1,\ldots,V_8$ are the same as those corresponding to the oscillator (1.2) with $M=0$ (see Ref. [11]). Here we omit them for the sake of brevity. Among the vector fields (3.17), we can select those of the Noether type by resorting to the relation (2.15). In doing so, let us introduce in Eq. (2.15) the density Lagrangian (3.1). We find that the generators of the Noether type are V_1, V_2, V_3, V_4, and V_5. These yield the constants of the motion (see Appendix):

$$I_1=\frac{m}{2}\left\{\left[\sigma\dot{q}-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q\right]^2-\frac{q^2}{4\sigma^2}\right\}\cos\theta+\frac{m}{2}\left[q\dot{q}-\frac{1}{\sigma}\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q^2\right]\sin\theta \ , \tag{3.18a}$$

$$I_2=\frac{m}{2}\left\{\left[\sigma\dot{q}-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q\right]^2-\frac{q^2}{4\sigma^2}\right\}\sin\theta-\frac{m}{2}\left[q\dot{q}-\frac{1}{\sigma}\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q^2\right]\cos\theta \ , \tag{3.18b}$$

$$I_3=\frac{m}{2}\left\{\left[\sigma\dot{q}-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q\right]^2+\frac{q^2}{4\sigma^2}\right\} , \tag{3.18c}$$

$$I_4=\sqrt{2m}\left\{\left[-\sigma\dot{q}+\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q\right]\cos\frac{\theta}{2}-\frac{q}{2\sigma}\sin\frac{\theta}{2}\right\} , \tag{3.18d}$$

$$I_5=\sqrt{2m}\left\{\left[-\sigma\dot{q}+\left[\dot{\sigma}-\frac{M}{2}\sigma\right]q\right]\sin\frac{\theta}{2}+\frac{q}{2\sigma}\cos\frac{\theta}{2}\right\} . \tag{3.18e}$$

In other words $I_1,\ldots,I_5$ are such that

$$\frac{dI_j}{dt}=\{I_j,H\}+\frac{\partial I_j}{\partial t}=0 \ , \tag{3.19}$$

where the symbol { } denotes the Poisson bracket

$$\{A,B\}=\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \ , \tag{3.20}$$

with A and B any pair of dynamical variables depending on q, p, and t.

We point out that even if the vector fields V_6, V_7, and V_8 do not lead to Noether invariants, they are nevertheless important in the construction of alternative Lagrangians for Eq. (1.2). All these Lagrangians give rise to the same classical equation of motion, but may correspond to different quantum-mechanical versions of the system under investigation. A recent account on this interesting problem, which is beyond the purposes of the present paper, is contained in Ref. [21].

We observe that only two of the Noether invariants (3.18) are functionally independent, in the sense that the relations

$$I_1=\tfrac{1}{2}(I_4^2-I_5^2), \quad I_2=I_4I_5, \quad I_3=\tfrac{1}{2}(I_4^2+I_5^2) \tag{3.21}$$

hold. Furthermore, Eqs. (3.18d) and (3.18e) provide the general solution of the equation of motion (1.2), namely

$$q=\sqrt{2/m}\,\sigma\left[I_5\cos\frac{\theta}{2}-I_4\sin\frac{\theta}{2}\right] , \tag{3.22}$$

while the conjugate momentum $p=m\dot{q}$ can be written as

$$p=\left[\frac{m}{2}\right]^{1/2}\frac{1}{\sigma}\left\{\left[2\sigma\left[\dot{\sigma}-\frac{M}{2}\sigma\right]I_5-I_4\right]\cos\frac{\theta}{2}-\left[2\sigma\left[\dot{\sigma}-\frac{M}{2}\sigma\right]I_4+I_5\right]\sin\frac{\theta}{2}\right\} . \tag{3.23}$$

The constants of the motion (3.18) fulfill the relations

$$\{I_1,I_2\}=2I_3,\quad \{I_2,I_3\}=-2I_1,\quad \{I_3,I_1\}=-2I_2\ , \tag{3.24a}$$

$$\{I_4,I_5\}=1\ , \tag{3.24b}$$

$$\{I_1,I_4\}=I_5,\quad \{I_1,I_5\}=I_4,\quad \{I_2,I_4\}=-I_4\ ,$$
$$\{I_2,I_5\}=I_5,\quad \{I_3,I_4\}=-I_5,\quad \{I_3,I_5\}=I_4\ . \tag{3.24c}$$

The main result emerging from the analysis of Eqs. (3.24) is that the Noether invariants I_1, I_2, and I_3 form a Lie algebra under the Poisson bracket operation [see (3.24a)]. The algebra turns out to be su(1,1), underlying the noncompact group SU(1,1). As we shall show in Sec. IV, this fact is important because it allows us to define unambiguously the dynamical group associated with the TDO system (2.1).

IV. QUANTUM THEORY

A. Noether invariant operators and dynamical group

The quantum theory of the classical TDO (1.2) can be described by the Hamiltonian operator

$$\hat{H}=P^2/(2m)+\tfrac{1}{2}m\omega^2Q^2\ , \tag{4.1}$$

where Q is a canonical coordinate, P is its conjugate momentum, and

$$[Q,P]=i\hbar\ . \tag{4.2}$$

To build up a set of invariant operators of the Noether type for the quantum oscillator governed by (4.1), it is convenient to introduce the time-dependent lowering and raising operators $a=a(t)$ and $a^\dagger=a^\dagger(t)$, defined by

$$a=\left[\frac{m}{\hbar}\right]^{1/2}\left\{\frac{Q}{2\sigma}+i\left[\frac{\sigma}{m}P-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q\right]\right\}, \tag{4.3a}$$

$$a^\dagger=\left[\frac{m}{\hbar}\right]^{1/2}\left\{\frac{Q}{2\sigma}-i\left[\frac{\sigma}{m}P-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q\right]\right\}, \tag{4.3b}$$

where σ fulfills the nonlinear ordinary differential equation (3.12) with Ω given by (3.7). The operators a and $a^\dagger$ obey the commutation relation

$$[a,a^\dagger]=1\ , \tag{4.4}$$

because of (4.2).

A set of time-dependent invariant operators of the Noether type for the quantum-mechanical system driven by the Hamiltonian (4.1) can be determined from the classical constants of the motion (3.18) by adopting the prescription $q\to Q$, $\dot{q}\to P/m$. In doing so, we obtain

$$\hat{I}_1=m\left\{\left[\frac{\sigma}{m}P-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q\right]^2-\frac{Q^2}{4\sigma^2}\right\}\cos\theta$$
$$+m\left[\frac{1}{m}QP-\frac{1}{\sigma}\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q^2-\frac{i\hbar}{2m}\right]\sin\theta\ , \tag{4.5a}$$

$$\hat{I}_2=m\left\{\left[\frac{\sigma}{m}P-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q\right]^2-\frac{Q^2}{4\sigma^2}\right\}\sin\theta$$
$$-m\left[\frac{1}{m}QP-\frac{1}{\sigma}\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q^2-\frac{i\hbar}{2m}\right]\cos\theta\ , \tag{4.5b}$$

$$\hat{I}_3=m\left\{\left[\frac{\sigma}{m}P-\left[\dot{\sigma}-\frac{M}{2}\sigma\right]Q\right]^2+\frac{Q^2}{4\sigma^2}\right\}, \tag{4.5c}$$

$$\hat{I}_4=\sqrt{2m}\left\{\left[\left[-\frac{\sigma}{m}P+\dot{\sigma}Q\right]\cos\frac{\theta}{2}-\frac{Q}{2\sigma}\sin\frac{\theta}{2}\right]-\tfrac{1}{2}M\sigma Q\cos\frac{\theta}{2}\right\}, \tag{4.5d}$$

$$\hat{I}_5=\sqrt{2m}\left\{\left[\left[-\frac{\sigma}{m}P+\dot{\sigma}Q\right]\sin\frac{\theta}{2}+\frac{Q}{2\sigma}\cos\frac{\theta}{2}\right]-\tfrac{1}{2}M\sigma Q\sin\frac{\theta}{2}\right\}. \tag{4.5e}$$

A direct calculation shows that the quantities (4.5) satisfy Eq. (1.3), where $\hat{H}$ is given by (4.1). The operators $\hat{I}_j$ ($j=1,\dots,5$) can be written in a simpler form by using [see (4.3)]

$$Q=(\hbar/m)^{1/2}\sigma(a+a^\dagger)\ , \tag{4.6a}$$

$$P=\frac{\sqrt{m\hbar}}{\sigma}\left\{\left[-\frac{i}{2}+\sigma\left[\dot{\sigma}-\frac{M}{2}\sigma\right]\right]a+\left[\frac{i}{2}+\sigma\left[\dot{\sigma}-\frac{M}{2}\sigma\right]\right]a^\dagger\right\}. \tag{4.6b}$$

In fact, inserting (4.6) into (4.5) we obtain

$$\hat{I}_1=-(\hbar/2)(e^{i\theta}a^2+e^{-i\theta}a^{\dagger 2})\ , \tag{4.7a}$$

$$\hat{I}_2=(i\hbar/2)(e^{i\theta}a^2-e^{-i\theta}a^{\dagger 2})\ , \tag{4.7b}$$

$$\hat{I}_3=\hbar(a^\dagger a+\tfrac{1}{2})\ , \tag{4.7c}$$

$$\hat{I}_4=i(\hbar/2)^{1/2}(e^{i(\theta/2)}a-e^{-i\theta/2}a^\dagger)\ , \tag{4.7d}$$

$$\hat{I}_5=(\hbar/2)^{1/2}(e^{i\theta/2}a+e^{-i\theta/2}a^\dagger)\ . \tag{4.7e}$$

Similarly to what happens for the classical case, the invariant operators $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ can be expressed in terms of $\hat{I}_4$ and $\hat{I}_5$, namely

$$\hat{I}_1=\tfrac{1}{2}(\hat{I}_4^2-\hat{I}_5^2),\quad \hat{I}_2=\tfrac{1}{2}(\hat{I}_4\hat{I}_5+\hat{I}_5\hat{I}_4)\ ,$$
$$\hat{I}_3=\tfrac{1}{2}(\hat{I}_4^2+\hat{I}_5^2)\ . \tag{4.8}$$

We can see straightforwardly that $\hat{I}_1,\dots,\hat{I}_5$ obey the commutation rules

$$[\hat{I}_1,\hat{I}_2]=2i\hbar\hat{I}_3,\quad [\hat{I}_2,\hat{I}_3]=-2i\hbar\hat{I}_1,\quad [\hat{I}_3,\hat{I}_1]=-2i\hbar\hat{I}_2\ , \tag{4.9a}$$

$$[\hat{I}_4,\hat{I}_5]=i\hbar\ , \tag{4.9b}$$

$$[\hat{I}_1,\hat{I}_4]=i\hbar\hat{I}_5,\quad [\hat{I}_1,\hat{I}_5]=i\hbar\hat{I}_4,\quad [\hat{I}_2,\hat{I}_4]=-i\hbar\hat{I}_4\ ,$$
$$[\hat{I}_2,\hat{I}_5]=i\hbar\hat{I}_5,\quad [\hat{I}_3,\hat{I}_4]=-i\hbar\hat{I}_5,\quad [\hat{I}_3,\hat{I}_5]=i\hbar\hat{I}_4\ . \tag{4.9c}$$

We note that Eqs. (4.9) are the quantum-mechanical versions of Eqs. (3.24), i.e., the former come formally from the latter by the substitution

$$\{I_j,I_k\}\rightarrow\frac{1}{i\hbar}[\hat{I}_j,\hat{I}_k]\ . \tag{4.10}$$

Looking at Eqs. (4.9a), we see that the Noether invariant operators $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ satisfy a set of commutation relations defining the noncompact Lie algebra su(1,1). As we have already established in Sec. III, the corresponding classical Noether invariants I_1, I_2, and I_3 expressed by (3.18a), (3.18b), and (3.18c) are also elements of an su(1,1) algebra (under the Poisson bracket operation). This property is noteworthy in as much as the su(1,1) algebra is related to the noncompact group SU(1,1), which can be identified as the dynamical group associated with the system under consideration [12,22]. This statement is motivated by the following reasons. Dynamical groups are essentially noninvariance groups whose generators do not all commute with the Hamiltonian of a dynamical system. They yield the energy spectrum and the degeneracy of levels and can be used to build up the transition probabilities between states [12]. Dynamical groups and their algebras are important in many branches of physics, ranging from nuclear and particle physics to condensed-matter physics, quantum optics, and field theory [23]. Notwithstanding, the concept of a dynamical group does not appear to be defined uniquely in the literature. A way to remove this ambiguity was proposed by Dotan in Ref. [24], where the definition of the dynamical group of a given system is based on the symmetry group of the corresponding quantum-mechanical equation of motion for the system. Recently, Castaños, Frank, and Lopez-Peña [23] showed that Dotan's definition arises naturally from the quantum version of time-dependent Noether symmetry transformations. The starting point of the above considerations is a remark by Malkin and Man'ko [25], on the ground of which if $\psi(q,t)$ is a solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi(q,t)}{\partial t}=\hat{H}\psi(q,t)\ , \tag{4.11}$$

then $\hat{K}(q,p,t)\psi(q,t)$ is also a solution, where $\hat{K}$ is a generally time-dependent conserved quantity, i.e., it is such that

$$\frac{d\hat{K}}{dt}=\frac{1}{i\hbar}[\hat{K},\hat{H}]+\frac{\partial\hat{K}}{\partial t}=0\ . \tag{4.12}$$

Dotan proposed to adopt as a definition of dynamical group, that group whose generators are provided by Eq. (4.12). Anyway, one had to solve the problem of finding explicitly the invariant operators $\hat{K}$ related to the system under investigation. In Ref. [23] this question was handled resorting to a technique proposed by D'Hoker and Vinet [26] in the context of spectrum-generating superalgebras. In this paper we have followed a more direct procedure, which can be exploited algorithmically and can be applied to any system of the form (1.2) and its quantum version. Our main result is that the existence of the dynamical group SU(1,1) is dictated by the Noether symmetry properties of Eq. (1.2), and the invariant operators of the Noether type $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ [see (4.7a)–(4.7c)], represent a natural realization of the Lie algebra of SU(1,1).

B. The TDO quantum Hamiltonian in terms of SU(1,1) Noether invariant operators

For practical purposes, such as for instance in the problem of evaluating the energy spectrum and in the study of SU(1,1) coherent and squeezed states of quantum devices described by the Hamiltonian (4.1), it is convenient to introduce the notation

$$X_1=-\frac{\hat{I}_1}{2\hbar}=\tfrac{1}{2}(J_-+J_+),\quad X_2=\frac{\hat{I}_2}{2\hbar}=\left[\frac{i}{2}\right](J_--J_+)\ ,$$

$$X_3=\frac{\hat{I}_3}{2\hbar}=J_0\ , \tag{4.13a}$$

$$X_4=-\tfrac{1}{2}(2/\hbar)^{1/2}\hat{I}_4=-(i/2)(e^{i\theta/2}a-e^{-i\theta/2}a^\dagger)\ ,$$

$$X_5=\tfrac{1}{2}(2/\hbar)^{1/2}\hat{I}_5=\tfrac{1}{2}(e^{i\theta/2}a+e^{-i\theta/2}a^\dagger)\ , \tag{4.13b}$$

from which

$$J_+=X_1+iX_2=\tfrac{1}{2}e^{-i\theta}a^{\dagger 2},\quad J_-=X_1-iX_2=\tfrac{1}{2}e^{i\theta}a^2\ ,$$

$$J_0=X_3=\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2})\ . \tag{4.14}$$

Then Eqs. (4.9a) and (4.9b) imply

$$[J_+,J_-]=-2J_0,\quad [J_0,J_+]=J_+,\quad [J_0,J_-]=-J_- \tag{4.15}$$

and

$$[X_5,X_4]=i/2\ . \tag{4.16}$$

By virtue of (4.6a) and (4.6b), the Hamiltonian (4.1) can be written in terms of the SU(1,1) Noether invariant operators J_0, J_+, and J_-, namely

$$\hat{H}=\gamma_1J_0+\gamma_2J_++\gamma_2^*J_-\ , \tag{4.17}$$

where γ_1 and γ_2 are the time-dependent functions

$$\gamma_1=2\hbar\left\{\frac{1}{4\sigma^2}\left[1+4\sigma^2\left[\dot{\sigma}-\frac{M}{2}\sigma\right]^2\right]+\omega^2\sigma^2\right\}, \tag{4.18}$$

$$\gamma_2=\hbar\left\{\frac{1}{4\sigma^2}\left[i+2\sigma\left[\dot{\sigma}-\frac{M}{2}\sigma\right]\right]^2+\omega^2\sigma^2\right\}e^{i\theta}\ . \tag{4.19}$$

The form (4.17) of the Hamiltonian (4.1), which is Hermitian, is that usually employed to solve the energy spectrum problem [12]. In addition, we point out that (4.17) belongs to the class of the most general Hamiltonian preserving an arbitrary initial SU(1,1) coherent state under time evolution, that is [14]

$$\hat{H}=f_1(t)K_0+f_2(t)K_++f_2^*(t)K_-+f_3(t)\ , \tag{4.20}$$

where $f_1(t),f_2(t),f_3(t)$ are arbitrary functions, and K_0,K_+,K_- are the generators of the SU(1,1) group, satisfying the commutation relations

$$[K_+,K_-]=-2K_0,\quad [K_0,K_+]=K_+,\quad [K_0,K_-]=-K_-, \tag{4.21}$$

which define the Lie algebra su(1,1). This algebra admits the Casimir operator

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+)\ . \tag{4.22}$$

Thus, the operators J_0, J_+, and J_- expressed by (4.14) constitute an invariant realization of the su(1,1) Lie algebra (4.21). In this case the Casimir operator (4.22) takes the constant value $C=-\frac{3}{16}$. We recall that in the study of SU(1,1) coherent states exhibited by quantum models described by the Hamiltonian (4.17), one is interested mainly in the unitary irreducible representations $D^+(k)$, known as the positive discrete series, where k is the Bargmann index. Since the Casimir operator (4.22) has the eigenvalues $k(k-1)$, we obtain $k(k-1)=-\frac{3}{16}$, yielding $k=\frac{1}{4}$ and $k=\frac{3}{4}$, which corresponds to the even states and odd-parity states, respectively [14].

Some cases of particular physical relevance are contained in the generic Hamiltonian (4.17). For example, let us choose (i) $M=0$, $\omega=\omega_0=\text{const}$; (ii) $M=M_0=\text{const}$, $\omega=\omega_0$ (with $M_0<2\omega_0$); and (iii) $M=0$ ($m=m_0=\text{const}$), $\omega^2(t)=\omega_0^2[1+\lambda\cos(2\nu_0 t)]$ where λ and ν_0 are constants.

In case (i), $\Omega=\omega_0$, $\sigma^2=1/2\omega_0$, $\theta=2(\omega_0 t+\theta_0)$, where θ_0 is a constant of integration and $\gamma_1=2\hbar\omega_0$, $\gamma_2=0$. The Hamiltonian (4.17) becomes that of the simple quantum harmonic oscillator, i.e., $\hat{H}=\gamma_1 J_0=\hbar\omega_0(a^\dagger a+\frac{1}{2})$, where the operators a and $a^\dagger$ are furnished by (4.3a) and (4.3b) with $\sigma=1/\sqrt{2\omega_0}$.

In case (ii), we obtain $\Omega=\Omega_0=\frac{1}{2}(4\omega_0^2-M_0^2)^{1/2}$, $\sigma^2=1/2\Omega_0$, $\theta=2(\Omega_0+\theta_0)$, $\gamma_1=2\hbar(\omega_0^2/\Omega_0)$, and

$$\gamma_2=-i(\hbar M_0/2)\exp[2i(\Omega_0 t+\theta_0)]\ .$$

The Hamiltonian (4.17) takes the form

$$\hat{H}=\hbar\frac{\omega_0^2}{\Omega_0}(a^\dagger a+\tfrac{1}{2})-i\frac{\hbar M_0}{4}(a^{\dagger 2}-a^2)\ , \tag{4.23}$$

whose classical partner reads

$$H=e^{-M_0 t}\frac{p^2}{2m_0}+e^{M_0 t}\frac{m_0\omega_0^2}{2}q^2\ , \tag{4.24}$$

where

$$m(t)=m_0 e^{M_0 t}\ . \tag{4.25}$$

This Hamiltonian, which gives the equation of motion of the ordinary damped harmonic oscillator

$$\ddot{q}+M_0\dot{q}+\omega_0^2 q=0\ , \tag{4.26}$$

is a particular case of the Kanai-Caldirola Hamiltonian

$$H=\frac{p^2}{2m(t)}+\frac{1}{2}m(t)\omega_0^2 q^2\ , \tag{4.27}$$

with $m(t)$ given by (4.25), where the damping is expressed by $M_0 t$. Finally, we notice that (4.23) can be considered as a prototype for the generation of squeezed states [27].

Case (iii) is of interest in many physical areas, for example, in quantum optics and generally in the field of parametric interactions [1,28]. The classical Hamiltonian

$$H=\frac{p^2}{2m_0}+\frac{1}{2}m_0\omega_0^2(1+\lambda\cos 2\nu_0 t)q^2 \tag{4.28}$$

provides the equation of motion

$$\ddot{q}+\omega_0^2[1+\lambda\cos(2\nu_0 t)]q=0\ , \tag{4.29}$$

which is of the Mathieu type. The quantum-mechanical Hamiltonian corresponding to (4.28), in terms of the operators a and $a^\dagger$, is [see (4.17)]

$$\hat{H}=\left[\frac{\hbar}{2}\right]\left[\left[\frac{1}{2\sigma^2}(1+4\sigma^2\dot{\sigma}^2)+2\omega^2\sigma^2\right](a^\dagger a+\tfrac{1}{2})+\left[\frac{1}{4\sigma^2}(i+2\sigma\dot{\sigma})^2+\omega^2\sigma^2\right]a^{\dagger 2}+\left[\frac{1}{4\sigma^2}(-i+2\sigma\dot{\sigma})^2+\omega^2\sigma^2\right]a^2\right]\ , \tag{4.30}$$

where $\omega^2=\omega_0^2[1+\lambda\cos(2\nu_0 t)]$, and σ fulfills the nonlinear ordinary differential equation [see (3.12)]

$$\ddot{\sigma}+\omega_0^2[1+\lambda\cos(2\nu_0 t)]\sigma=\frac{1}{4\sigma^3}\ . \tag{4.31}$$

In order to find an approximate solution to this equation, we shall regard λ as a perturbation parameter. By working to first order, we let

$$\sigma(t)=\sigma_0(t)+\lambda\sigma_1(t)\ . \tag{4.32}$$

Substituting (4.32) into (4.31) and equating the coefficients of λ^0 and λ to zero, yields

$$\ddot{\sigma}_0+\omega_0^2\sigma_0=\frac{1}{4\sigma_0^3}\ , \tag{4.33}$$

$$\ddot{\sigma}_1+4\omega_0^2\sigma_1=-\frac{1}{\sqrt{2}}\omega_0^{3/2}\cos(2\nu_0 t)\ . \tag{4.34}$$

Solving these equations, we find

$$\sigma_0=\frac{1}{\sqrt{2\omega_0}},\quad \sigma_1=-\frac{\omega_0^{3/2}}{4\sqrt{2}(\omega_0^2-\nu_0^2)}\cos(2\nu_0 t)\ . \tag{4.35}$$

Therefore, from (4.32) we have

$$\sigma=\frac{1}{\sqrt{2\omega_0}}\left[1-\lambda\frac{\omega_0^2}{4(\omega_0^2-\nu_0^2)}\cos(2\nu_0 t)\right]\ , \tag{4.36}$$

and [see (3.13)]

$$\theta=2(\omega_0 t+\theta_0)+\lambda\frac{\omega_0^3}{2\nu_0(\omega_0^2-\nu_0^2)}\sin(2\nu_0 t)\ , \tag{4.37}$$

where θ_0 is a constant of integration. Furthermore, inserting (4.36) into (4.18) and (4.19) we obtain

$$\gamma_1=2\hbar\omega_0[1+\tfrac{1}{2}\lambda\cos(2\nu_0 t)]\ , \tag{4.38a}$$

$$\gamma_2=-\lambda\hbar\frac{\omega_0\nu_0}{2(\omega_0^2-\nu_0^2)}[\nu_0\cos(2\nu_0 t)-i\omega_0\sin(2\nu_0 t)]e^{i\theta}, \tag{4.38b}$$

at the first order in λ.

Hence, the Hamiltonian (4.30) takes the approximate form [see also (4.17)]

$$\hat{H}\cong\hbar\omega(a^\dagger a+\tfrac{1}{2})-\lambda\hbar\frac{\omega_0\nu_0}{\omega_0^2-\nu_0^2}\times\{[\nu_0\cos(2\nu_0 t)-i\omega_0\sin(2\nu_0 t)]a^{\dagger 2}+[\nu_0\cos(2\nu_0 t)+i\omega_0\sin(2\nu_0 t)]a^2\}, \tag{4.39}$$

where

$$\omega\cong\omega_0[1+\tfrac{1}{2}\lambda\cos(2\nu_0 t)], \tag{4.40}$$

and the operators a and $a^\dagger$ are expressed by (4.3a) and (4.3b) with σ given by (4.36).

The Hamiltonian (4.39) resembles that appearing in the context of electromagnetic and acoustic parametric interactions [28]. In this case, $2\nu_0$ plays the role of a driven frequency [see Eq. (4.34)].

C. Generalized quadrature phase amplitude operators

In Sec. IV A we saw that the Noether invariants $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$ can be regarded as a natural realization of the generators of the dynamical group for the quantum system described by the Hamiltonian (4.1). Here we show that also the remaining Noether invariants $\hat{I}_4$ and $\hat{I}_5$ [or, equivalently, X_4 and X_5 given by (4.13b)], have a physical interpretation. Precisely, X_4 and X_5 can be identified with a generalized version of the operators called "quadrature-phase amplitudes," which are involved, for instance, in the quantum nondemolition (QND) measurements and in the problem of the generation of squeezed states by certain quantum-mechanical systems and optical devices [10]. In fact, in QND measurements a fundamental class of oscillator variables are the "quadrature-phase" operators, defined by

$$a=(A_1+iA_2)e^{-i\omega_0 t}, \tag{4.41}$$

where ω_0 is a constant, and correspond to the real and imaginary parts of the oscillator complex amplitudes. The quadrature-phase operators are Hermitian, are constants of the motion, and obey the commutation relation

$$[A_1,A_2]=i/2. \tag{4.42}$$

Since from (4.13b) we have

$$a=(X_5+iX_4)e^{-i\theta/2}, \tag{4.43}$$

where a is defined by (4.3a), and X_5 and X_4 satisfy condition (4.16), it follows that X_5 and X_4 can be considered as a generalization of the operators A_1 and A_2. In particular, if we refer to the oscillator with constant mass and frequency, X_5 and X_4 coincide just with A_1 and A_2, respectively.

The quadrature-phase operators are also important in the theory of squeezing exhibited by certain quantum-mechanical systems and optical devices, such as, for example, the two-photon device [10]. At this stage we point out that in dealing with quantum oscillators in which the mass and frequency may be time dependent, definition (4.41) should be replaced by (4.43). In this case, one of the basic properties of the theory of squeezing, i.e., the uncertainty relation involving the variances of the quadrature-phase amplitudes [10, 14, 29], should be formulated generally in such a way that A_1 and A_2 are replaced by X_5 and X_4, respectively.

V. CONCLUDING REMARKS

We have obtained the symmetry properties of both the classical and quantum oscillator with time-dependent mass and frequency, using an algorithm based on the Lie group theory of point transformations. We have found that the classical system allows a set of five time-dependent Noether invariants, where three of them, I_1, I_2, and I_3 given by (3.18a)–(3.18c), form the su(1,1) Lie algebra under the Poisson bracket operation. This feature is shared by the corresponding Noether invariant operators $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ [see (4.7a)–(4.7c)], related to the quantum version of the generalized oscillator under consideration. In this case $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ form the su(1,1) Lie algebra under the commutator operation. This result is noteworthy because the su(1,1) Lie algebra underlies the noncompact group SU(1,1), which in this way can be identified unambiguously with the dynamical group admitted by the generalized oscillator. In other words, the existence of the dynamical group SU(1,1) is a natural consequence of the Noether symmetry properties of the system.

Also the Noether invariants $\hat{I}_4$ and $\hat{I}_5$ have an interesting physical interpretation. In fact, they can be regarded as the quadrature-phase amplitude operators that occur, for example, in the context of quantum nondemolition measurements and in the theory of squeezing.

Furthermore, our formalism can be used to write the Hamiltonian of the generalized quantum oscillator in terms of (Noether invariant) generators of the dynamical group. This is important in the evaluation of the energy spectrum of the system and in the study of coherent and squeezed states.

To conclude, we observe that the group approach we have followed is quite generally valid and could be applied, say, to coupled time-dependent oscillators as well as to nonlinear systems in more than one dimension. This might be a challenging subject of a future investigation.

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APPENDIX: THE NOETHER INVARIANTS

Here we outline the procedure to derive the Noether invariants (3.18). To this aim, we need to exploit Eq. (2.15) by using the expressions (2.8) and (3.1). In this way we find

$$(\tfrac{1}{2}\dot{m}\dot{q}^2-\tfrac{1}{2}\dot{m}\omega^2q^2-m\omega\dot{\omega}q^2)\xi-m\omega^2q\phi+\{\phi_t+[\phi_q-(\xi_t+\xi_q\dot{q})]\dot{q}\}m\dot{q}+(\tfrac{1}{2}m\dot{q}^2-\tfrac{1}{2}m\omega^2q^2)\times(\xi_t+\xi_q\dot{q})=B_t+B_q\dot{q}\ . \quad \text{(A1)}$$

Equating the coefficients of power of $\dot{q}$ to zero, Eq. (A1) provides

$$\phi=\left[\frac{1}{2}\dot{\xi}-\frac{M}{2}\xi\right]q+\chi\ , \quad \text{(A2)}$$

$$B=\frac{m}{4}(\ddot{\xi}-\dot{M}\xi-M\dot{\xi})q^2+m\dot{\chi}q\ , \quad \text{(A3)}$$

where the functions ξ and $\chi=\chi(t)$ satisfy the constraints

$$\dddot{\xi}+4\Omega^2\dot{\xi}+4\Omega\dot{\Omega}\xi=0\ , \quad \text{(A4)}$$

$$\ddot{\chi}+M\dot{\chi}+\omega^2\chi=0\ , \quad \text{(A5)}$$

with Ω defined by (3.7).

The general solutions of Eqs. (A4) and (A5) can be written as

$$\xi=\sigma^2(k_1\cos\theta+k_2\sin\theta+k_3)\ , \quad \text{(A6)}$$

$$\chi=\left[\frac{2}{m}\right]^{1/2}\sigma\left[k_4\cos\frac{\theta}{2}+k_5\sin\frac{\theta}{2}\right]\ , \quad \text{(A7)}$$

where σ obeys Eq. (3.12), θ is given by (3.13), and $k_1,\ldots,k_5$ are arbitrary constants. The invariants (3.18) arise from (2.17) taking account of (A6), (A7), (3.1), and (A3), by setting $k_1=1$, $k_j=0$ $(j\neq 1)$, $k_2=1$, $k_j=0$ $(j\neq 2)$, and so on.

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