Squeezing limits at high parametric gains

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The generation of squeezed light in a single-pass parametric amplifier pumped by Gaussian beams is analyzed. Limits to the degree of squeezing that can be obtained are found that arise from spatial distortion of the signal beam that is significant for parametric gains greater than 3 dB. It is found that squeezing is limited to approximately 6 dB for typical experimental configurations. These limits severely constrain the design of quantum noise experiments unless high finesse cavities or waveguides are used.

I. INTRODUCTION

The initial squeezed-state experiments used continuous-wave lasers to generate quadrature squeezing of the electromagnetic field, either by four-wave mixing or by parametric down-conversion [1]. Because the nonlinear susceptibilities available for both systems are small, it was necessary to enhance both the pump field and the squeezed field in a high-finesse resonant cavity in order to obtain large squeezing. The purpose of the cavity is to increase the pump intensity in the nonlinear medium, and to increase the effective path length experienced by the squeezed field. Another effect of the cavity is to enhance selected transverse and longitudinal modes that are resonant, and to suppress other modes that are not resonant. Later, it was realized that pulsed squeezed fields could be created and detected if both the pump field and the local oscillator field were pulsed [2]. Taking advantage of the extremely high intensities obtained from mode-locked [3] and Q-switched [4] Nd:YAG (yttrium aluminum garnet) lasers, experiments using a potassium-titanyl-phosphate (KTP) parametric amplifier demonstrated large parametric gains from a single-pass, traveling-wave configuration. This system is pumped by a second harmonic beam with a Gaussian spatial profile and acts on an input mode with a corresponding spatial profile. At large gains, and in the absence of the spatial filtering effect of a resonant cavity, the spatial variation of the pump mode causes the parametrically amplified and deamplified quadrature field components of the signal mode to assume strongly non-Gaussian spatial profiles and to be distorted significantly with respect to each other [5]. We analyze this distortion in the classical limit for an input beam with a Gaussian intensity profile and find that the gain measured by direct or homodyne detection differs from that predicted by a plane-wave analysis. In the quantum limit, the case of squeezing is considered. It is found that in addition to the classical effects, noise from higher-order spatial modes is coupled into the Gaussian mode, which is measured by a homodyne detection scheme using a Gaussian-profile local oscillator beam.

The final goal of the work is to calculate the gain, and the level of squeezing that can be obtained in a thick parametric amplifier, i.e., one in which the pump and the signal fields diffract appreciably as they propagate through the nonlinear medium. The results depend critically on the relative sizes and alignments of the pump and signal fields. For this analysis, we assume that the second harmonic pump is a Gaussian spatial mode which comes to a minimum focus at the center of the nonlinear medium. The input signal field is taken to be a Gaussian mode focused at the same location in the crystal. The minimum beam radius of the signal beam is taken to be a factor of $\sqrt{2}$ larger than that of the pump beam, so that the diffraction of the two beams will be the same, despite the difference in wavelengths. In this configuration both the $\sqrt{2}$ ratio of the mode sizes and the exact matching of the phase fronts of the two modes are maintained throughout the interaction region, as shown in Fig. 1. This results in optimum coupling between Gaussian pump and signal modes. For the homodyne measure-

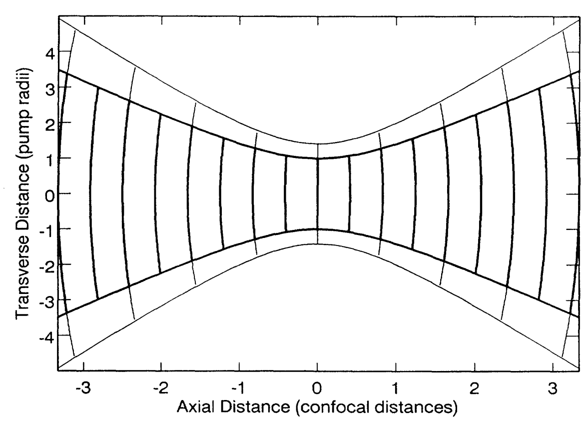

FIG. 1. A map of the confocal region of a parametric amplifier in the x-z plane is shown for the pump and signal fields. The x axis gives the axial distance in terms of the confocal distance $z_0 = \pi\omega_{p0}^2 n/\lambda$. The y axis gives the transverse distance in terms of the minimum pump radius ω_{p0}. The horizontal curved contours are lines of $1/e^2$ power for the signal (outer curve) and pump (inner curve) beams. Wave-front contours at wavelength periods match throughout the focal volume in order to optimize the parametric gain. The thin sample analysis applies only for the nearly planar wave-front region near the focus.

ment analysis, it is assumed that the local oscillator is matched to this input signal mode, and therefore measures the noise and amplitude of this mode. We will consider degenerate squeezing so dispersion may be neglected [6], and we assume that the signal mode is weak compared with the pump mode, so that the pump power is constant throughout the interaction region.

The calculations reported in this article show that there are severe limits to the amount of squeezing that can be obtained for a single-pass parametric amplifier pumped by a beam with a Gaussian spatial profile. If the nonlinear medium fills more than an infinitesimal fraction of the confocal mode volume the amount of squeezing that can be obtained is limited to approximately a factor of 4 in power. This will significantly impact the design of quantum light experiments such as quantum nondemolition measurements [7], "Schrödinger's cat" generators [8], and quantum frequency converters [9]. It also indicates the importance of cavities and waveguide structures in obtaining high gains and large squeezing. We first describe the case for a thin interaction region in Sec. II, and then the thick interaction region, that includes diffraction in Sec. III.

II. THIN INTERACTION REGION

Before studying an experimentally realistic parametric amplifier, we first consider the simplest case, where both the pump and the signal fields are taken to be infinite plane waves, so that all transverse variation of the pump and signal fields is neglected. It is convenient to write the classical optical field in terms of its phase quadrature amplitudes E_x and E_y,

$$E(t)=E_x\sin(\Omega t+\phi)+E_y\cos(\Omega t+\phi) , \qquad (1)$$

where Ω is the optical frequency. For proper choice of ϕ, the classical parametric gain in a nonlinear medium of length L can be described by [10]

$$E_x(L)=GE_x(0) ,$$
$$E_y(L)=\frac{1}{G}E_y(0) , \qquad (2)$$

$$G=e^{kE_pL}=e^{\Phi_{\rm nl}} , \qquad (3)$$

where G is the parametric field gain, k is the nonlinear coupling constant, E_p is the pump field, and $\Phi_{\rm nl}=kE_pL$ is the nonlinear phase shift. The plane-wave limit is the most basic form of the parametric interaction, where the gain is the same everywhere in the interaction region. In this case the gain increases exponentially as the pump field is increased. For an analysis of the parametric gain of the system it is sufficient to consider the quadrature amplitudes to be classical variables. A quantum analysis in which the variables are replaced by corresponding quantum field operators shows that the parametric amplifier is capable of transforming the *quantum fluctuations* of the electromagnetic field quadratures according to the same equations, producing a state in which one quadrature has smaller fluctuations than the vacuum state, i.e., a squeezed state. If squeezed state is observed with a detector of perfect quantum efficiency, the observed suppression of the quantum field fluctuations would be equal to the parametric gain parameter G.

We can easily generalize to the case of pump and signal modes having a Gaussian intensity profile if we assume that the interaction region is sufficiently thin that no diffraction of the beams takes place. This condition is fulfilled if the propagation distance is very small compared to $\pi l^2 n/\lambda$, where l is the width over which the intensity of the mode varies significantly. For the input mode, l is equal to the beam radius ω_0. If the output mode becomes distorted, l is the size of the smallest feature of the distorted mode. If we consider a sufficiently thin region at the focus of the mode volume, the wave fronts will again be planar, but the pump and signal fields vary as a function of the transverse radius [11]:

$$E_p(r)=E_{p0}e^{-(r/\omega_p)^2} , \qquad (4)$$

$$E_s(r)=E_{s0}e^{-(r/\omega_s)^2} , \qquad (5)$$

$$\omega_s=\sqrt{2}\omega_p , \qquad (6)$$

where ω_p and ω_s are the minimum beam waists of the pump and signal beams and E_{s0} and E_{p0} are the pump and signal fields at the center of the mode. In this configuration the pump field and the corresponding parametric gain vary over the area of the signal mode, resulting in an inhomogeneous amplification of the mode. The gain still has the same dependence on the pump intensity as shown in Eq. (3), but E_p varies as a function of radius, so that the gain is given by

$$G^{\pm}(r)=\exp[\pm kLE_p(r)]=\exp(\pm kLE_{p0}e^{-(r/\omega_p)^2})$$
$$=\exp(\pm\Phi_{\rm nl}e^{-(r/\omega_p)^2}) , \qquad (7)$$

where $\Phi_{\rm nl}$ is the nonlinear phase shift at the center of the mode, ω_p is the Gaussian beam waist of the pump, and the signs specify the gain for the amplified $(+)$ and deamplified $(-)$ quadratures. Since we assume the amplifier is sufficiently thin that no diffraction occurs, the amplification is local, and the output field may be found by pointwise multiplication of the input field by the gain.

$$E_{\rm out}(r)=E_{\rm in}(r)G(r,\Phi_{\rm nl}) , \qquad (8)$$

$$E_{\rm out}(r)=(E_{s0}e^{-(1/2)(r/\omega_p)^2})\exp(\pm\Phi_{\rm nl}e^{-(r/\omega_p)^2}) . \qquad (9)$$

$E_{\rm out}(r)$ is the amplitude of the signal field as it leaves the parametric amplifier, where we have used the fact that the beam waist of the signal field mode is assumed to be $\sqrt{2}$ larger than that of the pump field. This output field is shown as a function of radius for gains of 2 and 4 in Fig. 2. For $G(0)\geq 2$, the signal mode is severely distorted, and the amplified mode is no longer Gaussian. It is noteworthy that the two quadratures of the mode are distorted differently, so that the distortion is unlike any that could be produced by a lens, or index gradient, or any other optical device which is not phase sensitive. Since the signal mode is not homogeneously amplified, (i.e., the

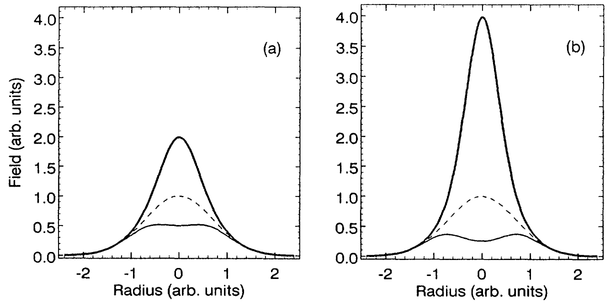

FIG. 2. Transverse profiles for the input signal (dashed) and output signal (solid) fields are shown for gains $G(0)$ of (a) 2 and (b) 4. The sharply peaked signal curves correspond to the amplified signal and the curves with the dips at zero radius are deamplified signals.

entire signal field is not multiplied by a constant factor) the meaning of the *gain* of the system is no longer clear. The apparent gain of the system will depend on the method used to measure it. We will consider two possible methods of measuring the gain, direct detection and homodyne detection.

For direct detection, the entire signal mode is collected on a photodetector, and the total photocurrent is measured. This gives a current which is proportional to the photon flux, or the total power of the beam. This quantity is calculated by integrating the optical intensity over the area of the beam. The power gain may be defined as the ratio of the photocurrents of the input and output modes, and the effective field gain G_d is the square root of the power gain. The field may therefore be written

$$G_d=\left[\frac{\int I_{\rm out}\,dA}{\int I_{\rm in}\,dA}\right]^{1/2}=\left[\frac{\int_0^\infty E_{\rm out}^2(r)r\,dr}{\int_0^\infty E_{\rm in}^2(r)r\,dr}\right]^{1/2}. \tag{10}$$

Using the input and output fields from Eqs. (9) and (5), we obtain

$$G_d^\pm=\left[\frac{2}{\omega_p^2}\int_0^\infty (e^{-(r/\omega_p)^2})\exp(\pm 2\Phi_{\rm nl}e^{-(r/\omega_p)^2})r\,dr\right]^{1/2}, \tag{11}$$

$$G_d^+=\left[\frac{e^{2\Phi_{\rm nl}}-1}{2\Phi_{\rm nl}}\right]^{1/2},\quad G_d^-=\left[\frac{e^{2\Phi_{\rm nl}}-1}{e^{2\Phi_{nl}}2\Phi_{\rm nl}}\right]^{1/2}, \tag{12}$$

where G_d^+ and G_d^- are the gains for the amplified and deamplified quadratures. Clearly the gain as measured by direct detection does not exhibit the simple exponential dependence on the pump field that is obtained from the plane-wave analysis. Furthermore, the gains for amplification and deamplification are no longer symmetric. Particular attention should be paid to the deamplification gain G_d^-, since this is closely related to the expected squeezing. In the limit that $\Phi_{\rm nl}$ is small G_d^- can be approximated as

$$\lim_{\Phi_{\rm nl}\to 0} G_d^-\simeq e^{-(1/2)\Phi_{\rm nl}}=e^{-(1/2)kLE_{p0}}\ . \tag{13}$$

In the limit that the gain is small, it appears that the gain follows the plane-wave result, with an effective pump field which is half E_{p0}, the field at the center of the mode. However, in the limit that $\Phi_{\rm nl}$ is large, the plane-wave approximation breaks down, and G_d^- approaches the asymptotic form

$$\lim_{\Phi_{\rm nl}\to\infty} G_d^-\simeq\left[\frac{1}{2\Phi_{\rm nl}}\right]^{1/2}=\left[\frac{1}{2kLE_{p0}}\right]^{1/2}. \tag{14}$$

As the pump becomes large, it is found that the deamplified signal, as measured by direct detection, does not decay exponentially with E_{p0}, but decreases much more slowly with the squeezed root of E_{p0}.

In squeezed-state experiments, the gain is more often measured by balanced homodyne detection. In this case, a 50% beam splitter is used to interfere the signal mode with a strong local oscillator field whose spatial mode is adjusted to correspond with that of the signal mode. The mixed fields are measured with photodetectors and the currents from the detectors at the two output ports of the beam splitter are subtracted. The homodyne signal is the beat between the signal and the local oscillator, and is a measure of the wave-front overlap of the signal field with the local oscillator mode. It can be interpreted as the projection of the signal mode on the spatial mode of the local oscillator. The balanced homodyne signal is given by the integral

$$\begin{aligned}\int E_{\det A}^2\,dA-\int E_{\det B}^2\,dA &=\frac{1}{2}\left[\int (E_s+E_{\rm LO})^2dA-\int (E_s-E_{\rm LO})^2dA\right]\\ &=2\int_0^\infty E_sE_{\rm LO}r\,dr\ .\end{aligned} \tag{15}$$

$E_{\det A}$ and $E_{\det B}$ are the amplitudes of the total fields incident on the two photodetectors. E_s and $E_{\rm LO}$ are the complex amplitudes of the signal and local oscillator fields incident on the homodyne beam splitter. Note that the complex product of E_s and $E_{\rm LO}$ only measures the component of E_s which is in phase with $E_{\rm LO}$, so by varying the phase of $E_{\rm LO}$ the two quadrature amplitudes of E_s can be independently measured. The gain as measured by homodyne detection can be defined as the ratio of the homodyne signals obtained for the input and output signal fields.

$$G_h=\frac{\int_0^\infty E_{\rm out}E_{\rm LO}r\,dr}{\int_0^\infty E_{\rm in}E_{\rm LO}r\,dr}\ , \tag{16}$$

where it has been assumed that the local oscillator phase has been adjusted to give the maximum signal. Substituting as before for the input and output fields, one obtains

$$G_h^\pm=\frac{2}{\omega_p^2}\int_0^\infty (e^{-(r/\omega_p)^2})\exp(\pm\Phi_{\rm nl}e^{-(r/\omega_p)^2})r\,dr\ , \tag{17}$$

$$G_h^+=\frac{e^{\Phi_{\rm nl}}-1}{\Phi_{\rm nl}},\quad G_h^-=\frac{e^{\Phi_{\rm nl}}-1}{e^{\Phi_{\rm nl}}\Phi_{\rm nl}}\ , \tag{18}$$

where G_h^+ and G_h^- are the gains obtained when the parametric amplifier pump phase is set for amplification and deamplification, respectively. Again, the gain as measured by homodyne detection has lost its simple exponential form. In the low gain limit, the homodyne gain is the same as the direct gain

$$\lim_{\Phi_{\rm nl}\to 0} G_h^- \simeq e^{-(1/2)\Phi_{\rm nl}} = e^{-(1/2)kLE_{p0}} . \tag{19}$$

However, in the limit that $\Phi_{\rm nl}$ is large the asymptotic limit of the homodyne gain is given by

$$\lim_{\Phi_{\rm nl}\to \infty} G_h^- \simeq \frac{1}{\Phi_{\rm nl}} = \frac{1}{kLE_{p0}} . \tag{20}$$

As before, the plane-wave analysis is approximately correct at low gains, but breaks down when the gain becomes large. Figure 3 shows the gains calculated for the homodyne and direct measurements [Eqs. (12) and (18)], along with the plane-wave extrapolation from the low gain limit [Eq. (13)]. For gains less than 3 dB, all three calculations of the gain are indistinguishable. For gains greater than 6 dB, there are substantial differences between the actual gains and the extrapolation of the plane-wave result. The differences between these gains is very significant in the design of experiments. The plane-wave calculation predicts that a factor of 3 increase in pump field, a factor of 9 in pump power, is required to increase the gain from 3 to 10 dB. In the case of measurement by homodyne detection, a factor of 25 increase in power is actually needed; for direct detection a factor of 50 in power is needed. The plane-wave analysis underestimates the pump power requirements in this example by factors of 2.8 and 5.5.

Up to this point we have only considered the evolution of the semiclassical amplitude of a coherent state. We now consider how the parametric amplifier acts on the quantum fluctuations of the electromagnetic field to produce a squeezed state. In order to do this, we must attach a new interpretation to the distortion of the signal beam which occurs in the parametric amplifier. The distorted, non-Gaussian output mode of the parametric amplifier must be interpreted as a superposition of higher-order generalized Gaussian beam modes [11]. These are defined by

$$E_{mn}(x,y) = \frac{2}{\omega} H_m\left[\frac{\sqrt{2}x}{\omega}\right] H_n\left[\frac{\sqrt{2}y}{\omega}\right] e^{-[(x^2+y^2)/\omega^2]} , \tag{21}$$

where $H_m(z)$ is the Hermite polynomial of order m, x and y are the two transverse dimensions, and ω is the mode radius. The prefactor has been chosen to satisfy the normalization condition $\int E^2 dA = 1$. This set of spatial modes is sufficient to express the distorted signal mode because the inhomogeneous amplification which takes place in the parametric amplifier modifies only the intensity profile of the signal mode; the phase fronts of the signal beam remain planar. The symmetrical geometry of the parametric amplifier implies that the output mode must be radially symmetric. It would therefore be useful to have a more restricted basis of modes which depend only on r and which are complete and orthogonal on the interval $\int r\,dr$. Such a set of modes $E_q(r)$ can be constructed from linear combinations of the generalized Gaussian beam modes. The lowest-order mode is, of course, the fundamental Gaussian mode

$$E_0(r) = E_{00}(x,y) . \tag{22}$$

The next few modes are

$$E_2(r) = E_{20}(x,y) + E_{02}(x,y) , \tag{23}$$

$$E_4(r) = E_{40}(x,y) + 2E_{22}(x,y) + E_{04}(x,y) . \tag{24}$$

The mode of order $2n$ is given by

$$E_{(2n)}(r) = \sum_{a=0}^{n} \begin{bmatrix} a \\ n \end{bmatrix} E_{(2a)(2n-2a)}(x,y) , \tag{25}$$

where $\binom{a}{n}$ is the binomial coefficient. (There are no modes of odd order, since these modes would be antisymmetric in radius.) If the input and output signal modes are expanded in this basis of spatial modes, the gain process can be thought of as a transformation that scatters the modes of the input field into the modes of the output field. Although the parametric amplification arises from a nonlinear coupling of the signal modes to the pump mode, the transformation of the signal mode is *linear*, and may be represented by the following matrix equation:

$$\begin{bmatrix} E_0 \\ E_2 \\ E_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} g_{00} & g_{02} & g_{04} & \cdots \\ g_{20} & g_{22} & g_{24} & \\ g_{40} & g_{42} & g_{44} & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} E_{\rm in} \\ 0 \\ 0 \\ \vdots \end{bmatrix} . \tag{26}$$

Here the components of the input and output field vectors are the amplitudes of the various spatial modes for one quadrature of the field. The matrix element g_{mn} gives the coupling between the nth-order input mode and the mth-order output mode, and is defined by

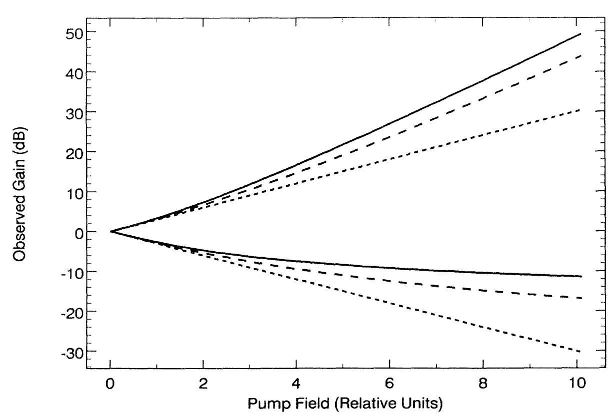

FIG. 3. The gain measured by direct (solid lines) and homodyne (long-dashed lines) detection is compared to that obtained from a plane-wave (uniform gain) analysis. The pump field is normalized so that a field of 1 unit corresponds to a gain of 3 dB in the plane-wave case.

$$g_{mn}=\int E_m(r)G(r,\Phi_{\rm nl})E_n(r)r\,dr\ , \tag{27}$$

where $G(r,\phi_{\rm nl})$ depends on the pump field, and is taken from Eq. (7) above. For example, the first three matrix elements would be given by

$$g_{00}=\frac{2}{\omega_p^2}\int_0^\infty (e^{-(r/\omega_p)^2})\exp(\Phi_{\rm nl}e^{-(r/\omega_p)^2})r\,dr\ , \tag{28}$$

$$g_{02}=\frac{2}{\omega_p^2}\int_0^\infty (e^{-(r/\omega_p)^2})\left[1-\left[\frac{r}{\omega_p}\right]^2\right] \times\exp(\Phi_{\rm nl}e^{-(r/\omega_p)^2})r\,dr\ , \tag{29}$$

$$g_{04}=\frac{2}{\omega_p^2}\int_0^\infty (e^{-(r/\omega_p)^2})\left[1-2\left[\frac{r}{\omega_p}\right]^2+\frac{1}{2}\left[\frac{r}{\omega_p}\right]^4\right] \times\exp(\Phi_{\rm nl}e^{-(r/\omega_p)^2})r\,dr\ , \tag{30}$$

where we have used the fact that $\omega_s=\sqrt{2}\omega_p$. It is clear from the definition that $g_{mn}=g_{nm}$, and the gain matrix is symmetric. Furthermore, as long as the alignment of the pump and signal modes is preserved, the gain matrix depends only on $\Phi_{\rm nl}=kLE_{p0}$. As written, the equation describes the parametric *amplification* of the signal beam; if the pump phase were shifted by π, we would take $\Phi_{\rm nl}=-kLE_{p0}$, and the signal field would be *deamplified.* The vector on the right side in Eq. (26) represents the input field, and has only one nonzero element because the input mode is assumed to be a coherent state with a pure Gaussian spatial mode, which has no higher-order spatial components. The other matrix elements are simply set to zero because in the analysis of the parametric gain we are only concerned with the coherent amplitudes of the fields. The direct and homodyne measurements of the parametric gain may be identified in this matrix formulation. The homodyne detector measures the amplitude of the lowest-order spatial mode, and the gain is the ratio of the output amplitude to the input amplitude of this mode so that the gain is given by $G_h=g_{00}$. The direct detector measures the total output power in all spatial modes. Since the spatial modes are orthogonal, this is equal to the sum of the squares of the component amplitudes $(E_0^2+E_2^2+E_4^4+\cdots)$. The gain as measured by direct detection is then the sum of the contributions of the matrix elements of the first column, since

$$\begin{aligned}G_d&=\left[\frac{g_{00}^2E_0^2+g_{20}^2E_0^2+g_{40}^2E_0^2+\cdots}{E_0^2}\right]^{1/2}\\&=(g_{00}^2+g_{20}^2+g_{40}^2+\cdots)^{1/2}\\&=\left[\frac{e^{2\Phi_{\rm nl}}-1}{2\Phi_{\rm nl}}\right]^{1/2},\ \left[\frac{e^{2\Phi_{\rm nl}}-1}{e^{2\Phi_{\rm nl}}2\Phi_{\rm nl}}\right]^{1/2}.\end{aligned} \tag{31}$$

The integral calculation of the gain by direct detection [from Eq. (12) above] then gives an analytic solution for the quadrature sum of the first row of the amplified or deamplified gain matrix.

To determine the squeezing which can be obtained from the parametric amplifier we must determine how the gain matrix acts on the vacuum state. Squeezing in a parametric amplifier is described by

$$\begin{bmatrix}E_0\\E_2\\E_4\\\vdots\end{bmatrix}=\begin{bmatrix}g_{00}&g_{02}&g_{04}&\cdots\\g_{20}&g_{22}&g_{24}&\\g_{40}&g_{42}&g_{44}&\\\vdots&&&\ddots\end{bmatrix}\begin{bmatrix}V_0\\V_2\\V_4\\\vdots\end{bmatrix}. \tag{32}$$

The input vector on the right represents the vacuum state incident on the parametric amplifier. The amplitude of each spatial mode of the vacuum is represented by an independent noise term V_n. Each amplitude has a mean value of zero, but is assumed to be fluctuating independently of the other amplitudes with a variance σ_V, corresponding to the quantum uncertainty of the vacuum field. When we measure the squeezing, we measure the variance of the spatial mode that matches the spatial mode of the local oscillator, which in this case is the variance of the amplitude E_0. The primary function of the parametric amplifier is to deamplify the fluctuations of the fundamental mode. However, the inhomogeneity of the gain distorts the fundamental mode, scattering its amplitude and fluctuations into higher-order modes by the elements of the first column of the gain matrix. But, the parametric amplifier also acts on the higher-order modes, deamplifying them, distorting them so they are no longer orthogonal to the fundamental mode, scattering their amplitude and fluctuations into the fundamental mode by the elements of the first row of the gain matrix. The variance of the fundamental mode is therefore found by combining the fluctuations which are coupled into this mode from all of the spatial modes. Since the noise sources which model quantum fluctuations of the input modes are independent, the noise fields add incoherently. The variance of E_0, denoted by σ_{E_0}, is related to the variance of the vacuum field σ_V by

$$\begin{aligned}\sigma_{E_0}&=(\sigma_V^2g_{00}^2+\sigma_V^2g_{02}^2+\sigma_V^2g_{04}^2+\cdots)^{1/2}\\&=\sigma_V(g_{00}^2+g_{02}^2+g_{04}^2+\cdots)^{1/2}\ .\end{aligned} \tag{33}$$

The noise power squeezing S is given by the square of the ratio of the variance of the output state to the variance of the vacuum state.

$$S=\frac{(\sigma_{E_0})^2}{(\sigma_V)^2}=g_{00}^2+g_{02}^2+g_{04}^2+\cdots\ . \tag{34}$$

Comparison of Eq. (31) with Eq. (34), using the symmetry of the gain matrix ($g_{mn}=g_{nm}$), reveals that the expression for the squeezing is identical to the square of the expression for the parametric gain measured by direct detection. The direct measurement of gain measures the coupling of the fundamental mode into all of the modes, the squeezing measures the coupling of all the modes into the fundamental modes, and the symmetry of the gain matrix corresponds to the fact that these reciprocal couplings are equal. As a consequence, the expression which was derived above for the gain by direct detection also applies to the observable squeezing. To reiterate, the quadrature power squeezing which can be observed from a parametric amplifier pumped by a Gaussian mode with the symmetry we have specified is given by

$$S=\frac{e^{2\Phi_{\rm nl}}-1}{e^{2\Phi_{\rm nl}}2\Phi} . \tag{35}$$

There are several points that should be noted here. For nonlinear phase shifts of more than 1 rad the observable squeezing is dramatically degraded from what is predicted by a plane-wave analysis. Furthermore, even with a perfect homodyne detector, the degree of squeezing predicted is substantially less than the parametric gain as measured by homodyne detection.

III. THICK NONLINEAR INTERACTION REGION

The method which we have applied to the parametric amplifier is not valid if the propagation distance is sufficiently large to allow diffraction of the signal field to occur in the interaction region. In this case the profile of the output mode cannot be found by pointwise multiplication of the input mode and the gain. This is because the effect of a change in field at a point on the wave front will not be confined to that point after the wave has propagated a finite distance. In fact the effect of a change in a region of width l will only remain confined to that region after a propagation distance short compared to $\pi l^2 n/\lambda$. The behavior of a thick parametric amplifier can be calculated by dividing the interaction region into layers which are sufficiently thin to be approximated as diffractionless and compounding the effect of the layers. For each layer, the input state must be transformed to reflect the parametric gain which occurs in the layer, and to reflect the evolution of the non-Gaussian signal mode as it propagates from layer to layer.

The gain matrix described above specifies how the various spatial modes evolve under the influence of the gain interaction. We must now describe the evolution of the spatial modes as they propagate from layer to layer. This evolution is found by considering the generalized Gaussian mode in three dimensions [8].

$$E_{mn}(x,y,z)=\frac{2}{\omega(z)}H_m\left[\frac{\sqrt{2}x}{\omega(z)}\right]H_n\left[\frac{\sqrt{2}y}{\omega(z)}\right]\times\exp\left[-\left[\frac{x^2+y^2}{\omega^2(z)}+\frac{ik(x^2+y^2)}{2R(z)}+ikz-i(m+n+1)\eta(z)\right]\right] , \tag{36}$$

where

$$\eta(z)=\tan^{-1}\left[\frac{z\lambda}{\pi\omega_0^2 n}\right] ,$$

$$R(z)=z\left[1+\left[\frac{\pi\omega_0^2 n}{\lambda z}\right]^2\right] ,$$

$$\omega^2(z)=\omega_0^2\left[1+\left[\frac{\lambda z}{\pi\omega_0^2 n}\right]^2\right] .$$

As the modes propagate along the z axis, the parameters $\eta(z)$, $R(z)$, and $\omega(z)$ evolve. The intensity profiles of the modes are determined by the Hermite polynomials and the real part of the argument of the exponentiation. In both of these factors, the transverse dimension scales linearly with the beam waist $\omega(z)$ without any dependence on the mode order. As a result the intensity profiles of all the modes retain a fixed relationship and the amplitudes of the modes do not mix during propagation. The curvatures of the wave fronts all evolve with $R(z)$, and also remain equal as the modes propagate. However, the term $i(m+n+1)\eta(z)$ in the exponential introduces an overall phase shift between the spatial modes as they propagate through the interaction region. In the distance that it takes for the mode to expand by a factor of $\sqrt{2}$ from its minimum beam radius, the second-order mode has undergone a phase shift of π with respect to the fundamental mode. The result of this is that as the non-Gaussian signal mode propagates through the interaction region, its intensity profile will change because the phases of the spatial components are constantly shifting with respect to each other. The key point is that only the phases of the modes evolve; there is no exchange of energy among the spatial modes. In the parametric amplification process, the consequence of this mode dispersion is that the quadratures of the spatial mode of order m are rotated with respect to the fundamental mode by an angle $\Psi_m=m[\eta(z)-\eta(z+\delta z)]$ as the field propagates through a slice of thickness δz. The quadratures of the various spatial modes are therefore transformed according to

$$\begin{bmatrix}\cos\Psi_m & \sin\Psi_m\\ -\sin\Psi_m & \cos\Psi_m\end{bmatrix}\begin{bmatrix}E_{mX}\\ E_{mY}\end{bmatrix} , \tag{37}$$

where E_{mX} and E_{mY} are the X and Y quadrature amplitudes of the spatial mode of order m.

Now we can specify an algorithm for calculating the gain and squeezing of a thick layer. First we divide the interaction region into n layers, such that diffraction is negligible in each layer. Since the modal dispersion introduces a coupling between the two phase quadratures, we can no longer write separate equations for each quadrature. We therefore define a state vector which specifies the field amplitudes for both phase quadratures of each spatial mode. To represent the gain of the ith layer we multiply the input state by a matrix which transforms the amplitudes of one quadrature by a single quadrature gain matrix calculated using $\Phi_{\rm nl}=k\delta zE_p(z_i)$, and transforms the amplitudes of the other quadrature by a single quadrature gain matrix using $\Phi_{\rm nl}=-k\delta zE_p(z_i)$, where z_i and δz_i are the center and thickness of the layer, and $E_p(z)$ is the center mode pump field as a function of z. The two phase quadratures remain segregated during this operation. This matrix is given by

$$\underline{E}'=\underline{G}(\Phi_{\rm nl})/\underline{E}\ , \tag{38}$$

$$\begin{bmatrix} E'_{X0}\\ E'_{X2}\\ E'_{X4}\\ \vdots\\ E'_{Y0}\\ E'_{Y2}\\ E'_{Y4}\\ \vdots \end{bmatrix} = \begin{bmatrix} g_{00} & g_{02} & g_{04} & \cdots & 0 & 0 & 0 & \cdots\\ g_{20} & g_{22} & g_{24} & \cdots & 0 & 0 & 0 & \\ g_{40} & g_{42} & g_{44} & \cdots & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & \ddots\\ 0 & 0 & 0 & \cdots & h_{00} & h_{02} & h_{04} & \cdots\\ 0 & 0 & 0 & & h_{20} & h_{22} & h_{24} & \\ 0 & 0 & 0 & & h_{40} & h_{42} & h_{44} & \\ \vdots & & & \ddots & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} E_{X0}\\ E_{X2}\\ E_{X4}\\ \vdots\\ E_{Y0}\\ E_{Y2}\\ E_{Y4}\\ \vdots \end{bmatrix}, \tag{39}$$

where g_{mn} and h_{mn} are the single quadrature gain matrix elements for $\Phi_{\rm nl}=\pm k\,\delta z E_p(z_i)$. The number of terms which must be retained in this calculation will depend on the strength of the interaction. For small squeezing, only the lowest-order term is significant, and this corresponds to the reversion to the plane-wave result which is found in the low gain limit. For observed squeezing exceeding 3 dB additional terms are needed. In general, we must retain enough terms that the highest-order spatial mode amplitude remains negligible compared to the other amplitudes throughout the propagation. After the action of the gain matrix, the state vector is multiplied by a matrix that rotates the quadratures of each spatial mode by an angle $\Psi_m=m[\eta(z_i)-\eta(z_{i+1})]$, where m is the order of the mode. The spatial mode remains segregated during this operation. The transformation may be written

$$\underline{E}''=\underline{R}(\Psi)\underline{E}'\ , \tag{40}$$

$$\begin{bmatrix} E''_{X0}\\ E''_{X2}\\ E''_{X4}\\ \vdots\\ E''_{Y0}\\ E''_{Y2}\\ E''_{Y4}\\ \vdots \end{bmatrix} = \begin{bmatrix} C_0 & 0 & 0 & \cdots & S_0 & 0 & 0 & \cdots\\ 0 & C_2 & 0 & & 0 & S_2 & 0 & \\ 0 & 0 & C_4 & & 0 & 0 & S_4 & \\ \vdots & & & \ddots & \vdots & & & \ddots\\ -S_0 & 0 & 0 & \cdots & C_0 & 0 & 0 & \cdots\\ 0 & -S_2 & 0 & & 0 & C_2 & 0 & \\ 0 & 0 & -S_4 & & 0 & 0 & C_4 & \\ \vdots & & & \ddots & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} E'_{X0}\\ E'_{X2}\\ E'_{X4}\\ \vdots\\ E'_{Y0}\\ E'_{Y2}\\ E'_{Y4}\\ \vdots \end{bmatrix}, \tag{41}$$

where

$$C_m=\cos(\Psi_m),\quad S_m=\sin(\Psi_m)\ .$$

The final output of the parametric amplifier is calculated by multiplying the state vector by the gain and dispersion matrices computed for each layer. Of course, it is convenient to take advantage of the associative property of matrix multiplication and multiply the matrices representing the gain and dispersion of all the layers together to form a combined matrix $\underline{P}$ representing the thick parametric amplifier.

$$\underline{P}(P_p,\omega_0,L)=\prod_{i=0}^{n}\underline{R}(\Psi(z_i))\underline{G}(\Phi_{\rm nl}(z_i))\ . \tag{42}$$

The elements of this matrix would give the coupling of each quadrature of each spatial mode of the input state to each quadrature of each mode of the output state. Although the combined matrix explicitly depends on the pump power, the minimum beam waist, and the interaction length, there are in fact only two independent parameters. The dispersive phase shift (which determines $\underline{R}$) and the spreading of the Gaussian mode (which determines $\underline{G}$) scale identically with the confocal distance $\pi\omega_0^2 n/\lambda$. All that is needed to specify the matrix is the pump power at the center of the mode, and the ratio of the length to the confocal distance. Once the combined matrix has been computed the observables of the amplifier can be immediately obtained. The squeezing, in particular, is again the quadrature sum of the first row of the total matrix, but in this case there are twice as many terms as there are for the thin parametric amplifier. This corresponds to the fundamental difference between the thick and the thin amplifier. In the thin amplifier the spatial modes of each quadrature were mixed, but the two quadratures remained uncoupled. In the thick amplifier, the modal dispersion mixes the two quadratures, and the amplitudes of both quadratures of all of the spatial modes are coupled into the squeezed quadrature. An additional subtlety is introduced as a result of the mixing of quadratures in the thick parametric amplifier. In this case it is not necessarily true that the maximally squeezed quadra-

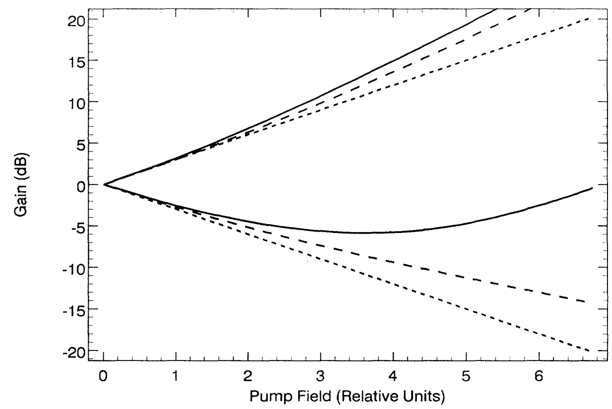

FIG. 4. Parametric gain and squeezing in a thick medium are shown as a function of pump field normalized as in Fig. 3. The solid curve applies for squeezing while the long-dashed line and short-dashed lines correspond to a homodyne measurement of the gain and the plane-wave (uniform gain) case, respectively. A half interaction length of 20% of the confocal distance is used in this illustration.

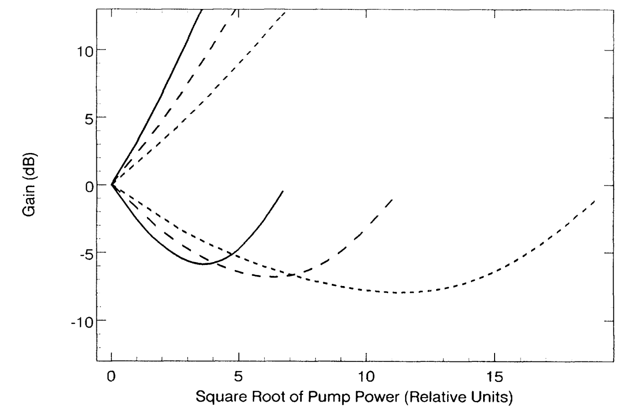

FIG. 5. A comparison of squeezing as a function of the square root of the pump power is shown for three geometries. The half interaction lengths are 20% (solid line), 10% (long-dashed line), and 5% (short-dashed line) of the confocal distance. As the pump focus expands and the confocal distance becomes larger compared to the interaction region, the degree of squeezing that can be achieved increases, but higher pump power is required.

ture is the one with zero phase difference from the pump field. However, if it is required that the modes reach their minimum radius at the center of the interaction region, the symmetry of the dispersive phase shifts implies that the quadrature with zero phase shift is the maximally squeezed one, as is verified by computation.

A calculation of parametric gain and squeezing in a thick parametric amplifier is shown in Fig. 4. The parameters have been chosen to represent a realistic KTP squeezer where the interaction length is 20% of the confocal distance. The finely dashed line is the extrapolation from the small gain limit based on the plane-wave analysis. The dashed line is the homodyne measurement of the parametric gain, which is not significantly affected by the modal dispersion. The solid line is the squeezing which is predicted for this geometry. Note that the squeezing reaches an optimum level of 5.8 dB at a relative pump field of 2.3, and becomes *worse* if the pump intensity further increased, despite the fact that the parametric gain is still improving. This is because the mode dispersion of the thick amplifier mixes the squeezed mode with both the squeezed and the *antisqueezed* quadratures of the higher-order modes. This is different from the thin amplifier, where the squeezed field is coupled only to weakly squeezed higher-order modes of the same quadrature. At low gains, the mixing is small, and the effect is negligible. As the gain is increased, noise power in the antisqueezed higher-order modes increases rapidly, and the strength of the coupling to these modes also increases. The product of these two factors increases faster than the deamplification of the fundamental mode, resulting in a degradation in squeezing. Eventually, both quadratures will become antisqueezed. Figure 5 compares squeezing calculations for several geometries. The three sets of curves represent the squeezing predicted for successively shorter interaction distances. As the interaction length is reduced the quadrature mixing becomes weaker and the optimum squeezing improves, although more pump power is needed. A similar result is obtained if the confocal distance is increased by enlarging the pump beam radius. In the limit that the interaction region is infinitely thin compared with the confocal distance, the curve will revert to the thin parametric amplifier calculation, shown in Fig. 3, and the obtainable squeezing will again become unbounded. Figure 6 shows the optimum squeezing level as a function of interaction length normalized to the confocal distance. The optimum squeezing is at its worst when the interaction length is on the order of the confocal distance. The optimum squeezing increases painfully slowly as the amplifier is made thinner. In order to obtain 10 dB squeezing, the thickness can be no more than 1% of the confocal thickness.

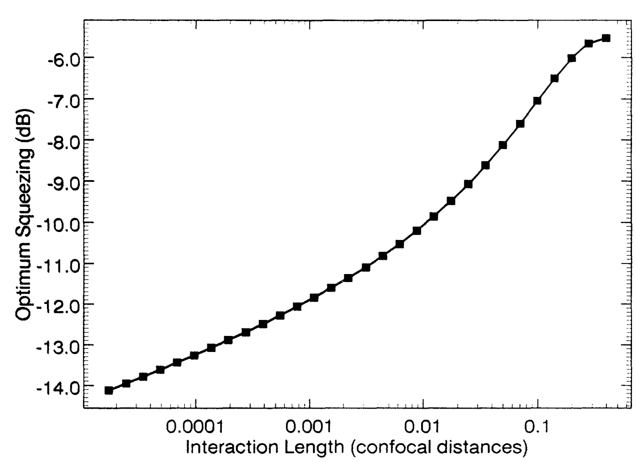

FIG. 6. Optimum squeezing levels are shown as a function of the log of the interaction length normalized to confocal distance. The power required to achieve optimum squeezing increases dramatically as the optimum squeezing increases.

This implies that the beam can only be allowed to spread by one part in 10^5 to preserve 10 dB squeezing. For a 35-μm pump, this would require to an interaction length of only 120 μm.

IV. DISCUSSION

The results presented in this paper should be viewed as an upper limit on the squeezing that can be observed in a single-pass parametric amplifier. There are other effects which can degrade the squeezing obtainable in a pulsed experiment. In order to obtain optimum squeezing it is necessary to have a local oscillator field which is temporally much shorter than the pump pulse [2]. This condition is typically not satisfied in experiments. Unless some method is employed to shorten the local oscillator pulse, the green pump pulses which are produced by the second harmonic generator and hence the squeezed pulses will be shorter than infrared local oscillator pulses. In addition, parametric oscillators based on KTP suffer beam walkoff. This occurs because the pump and signal fields must propagate at an angle with respect to the principle axes of the crystal lattice in order to fulfill the phase matching condition. For KTP used with a Nd:YAG laser the phase matching angle is 26° and the two polarization components of the signal beam separate by an angle of 0.23°. As a result of this the squeezing obtained would be further degraded if the mode separation due to walkoff is not negligible compared with the mode size. Finally, the linear loss in the beam transport optics and deviation of the photodetector from unity quantum efficiency will also degrade the squeezed state.

Another important issue is whether a different geometry could be devised to avoid or mitigate the effects described in Sec. III. One strategy might be to try to find a different spatial mode for the local oscillator which would yield better squeezing. In fact, the extra noise which is coupled into the fundamental spatial mode by the parametric amplifier is partially correlated with the noise which is present in the higher-order spatial modes. In principle, it would be possible to build a local oscillator out of a superposition of spatial modes which would cancel this correlated extra noise. In this way, it may be possible to measure squeezing levels closer to the homodyne gain measurement. If the diffraction effects in the amplifier are not negligible it would be necessary to manipulate both the phase and the amplitude of the different modes that would constitute the compound local oscillator. Making such a local oscillator would be technically difficult, and the adjustment of the amplitudes and phases of the spatial components would likely be very sensitive to changes in the parametric amplifier gain and alignment. An alternate approach would be to use a non-Gaussian pump to optimally squeeze the fundamental Gaussian spatial mode, but this would be a much more difficult problem to solve. The gain is exponential in the pump field, so finding the optimum pump is not a linear problem.

Another possibility is to relax the requirement that the pump and signal modes come to identical Gaussian focuses in the interaction medium. One strategy which has been tried [12] is to make the pump beam very large compared with the signal beam, making both beams sufficiently large that the diffraction in the interaction region is weak. In this case the intensity of the pump does not vary substantially over the signal mode and uniform amplification is expected. However, in order to achieve the greater uniformity of pump intensity, the absolute phase front matching of the pump and signal is sacrificed. In the thin parametric amplifier limit the improved pump uniformity of this scheme will make the difference between the squeezing and the plane-wave result smaller. But for finite thickness the quadrature mixing will be stronger and the optimum squeezing level will be poorer than if the pump were matched to the signal. The method we have described can be generalized to determine the squeezing for this alternate geometry, although the calculation of the gain matrix will be more complicated, since the function $G(r,\Phi_{\rm nl})$ must be generalized to take into account the relative phase of the signal and the pump. Although we plan to perform the calculation for other pump geometries, we anticipate that for a given signal mode size the most favorable squeezing limit will be obtained by matching the pump mode to the signal mode.

We believe the strong constraint on the interaction length is weakened when using a high finesse resonator to enhance the squeezed state. In this case the squeezed state effectively makes many passes through the nonlinear medium, coherently accumulating a small parametric gain for each pass. The higher-order spatial modes are still present, but unless the resonator is exactly confocal they will not be resonant at the squeezed frequency. As a result, even though these modes are still coupled into the squeezed mode by the inhomogeneous pump intensity, their noise content will not be high and they will cause a much less severe degradation of the squeezing than occurs in the single-pass configuration. Efforts are in progress to generalize this method to the calculation of squeezing in a cavity. Waveguides also offer the potential for improvement. Higher-order spatial modes in a waveguide would be coupled together in a similar way, but the modal structure depends on the details of the waveguide. It is possible that the waveguide could be designed to have a modal structure which minimizes this coupling.

It should also be noted that the effects described here are applicable to experiments which use a local oscillator to measure electric field quadratures. We have not considered experiments in which the photon counting statistics of the parametric amplifier output are measured. In photon counting experiments one normally considers the *signal* and *idler* channels, whose field amplitudes evolve as hyperbolic sine and cosine functions of the pump field. The quantum parametric amplifier emits one photon into the idler channel for each photon which it adds to the signal channel, making it possible to observe nonclassical photon correlations between the beams. The channel which we refer to as the signal channel is actually a linear superposition of the signal and idler channels. Our signal is quadrature squeezed by the parametric amplifier, and its field amplitude evolves as a pure exponential function of the pump field. We expect the effective gain experienced by the signal and idler channels in a photon counting experiment to behave in a manner similar to the quadrature amplification gain, which continues to increase exponentially with the pump field even in the high gain limit. The dramatic loss of correlation that we predict at high gain in quadrature squeezing experiments arises from the phase sensitivity of the homodyne detector, and would not be observed in photon counting experiments.

V. CONCLUSION

It has been demonstrated in this paper that the amount of quadrature squeezing that can be generated in a single-pass parametric amplifier is absolutely bounded. Under the constraint that the pump and signal modes are matched, the bound depends only on the ratio of the interaction length to the confocal distance of the pump beam. If the interaction length is of the same order as the confocal distance, the squeezing cannot exceed a factor of 4 in power (6 dB). To substantially improve the squeezing limit the interaction length must be made very short compared with the confocal distance, and this makes the single-pass pulsed squeezer a less attractive alternative to a cavity resonator squeezer. If the parametric amplifier is used in other nonclassical light experiments, such as back action evading measurements, generation of "Schrödinger's cat" states, or quantum frequency conversion, similar limits will apply to the correlations that can be observed.

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