Theory of the fundamental laser linewidth

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The theory of the laser linewidth is formulated to account for arbitrarily large output couplings and spatial hole burning. We show explicitly that the linewidth can be interpreted in terms of either spontaneous-emission noise or the amplification of vacuum field modes leaking into the cavity, depending on the ordering of operators in the correlation function determining the laser spectrum. This allows us to derive the Petermann K factor associated with "excess spontaneous-emission noise" in a physically transparent and mathematically simple way, without the need to introduce adjoint modes of the resonator. It also allows us to straightforwardly include spatial-hole-burning effects, which are found to increase the K factor and the linewidth in high-gain systems appreciably.

I. INTRODUCTION

It has been known since the earliest days of the laser that spontaneous emission both initiates laser oscillation and fundamentally limits the degree of monochromaticity. Since spontaneous emission cannot be fully described semiclassically, a rigorous theory of the fundamental linewidth demands field quantization. Such a theory has been available in various versions for many years [1,2].

In most lasers the fundamental quantum linewidth $\Delta\omega$ is too small to be of practical interest. In semiconductor lasers, however, $\Delta\omega$ is of considerable concern, largely because they have very small lengths compared with other lasers, and also because they often have small mirror reflectivities, thus giving them large cavity bandwidths and, therefore, sizable quantum linewidths. Linewidths $\approx 10-100$ MHz are typical of semiconductor lasers. For this reason there has in recent years been a resurgence of interest in the theory of the laser linewidth.

Much of this interest has focused on deviations of $\Delta\omega$ from the so-called Schawlow-Townes linewidth, $\Delta\omega_{\mathrm{ST}}$. Petermann [3] deduced, from classical considerations of a radiating dipole in a gain medium, that in gain-guided lasers, where transverse gain variations serve to confine the field, the spontaneous emission into a laser mode is enhanced by an "astigmatism parameter" K which multiplies $\Delta\omega_{\mathrm{ST}}$. The essence of Petermann's conclusion has been confirmed by several other authors [4], and most recently Siegman [5] has generalized the (semiclassical) theory and related the K factor to the so-called adjoint modes of lossy optical cavities. The existence of the K factor, which has come to be associated with "excess spontaneous-emission noise," is no longer controversial, but in our opinion a clear physical understanding of its origin has been lacking.

Another correction to the Schawlow-Townes linewidth was suggested by Henry [6] in order to account for unexpectedly large linewidths observed in semiconductor lasers. This correction involves the change in the refractive index associated with a change in the gain coefficient. Henry defined a parameter α as the ratio of the changes in the real and imaginary parts of the refractive index, and showed semiclassically that the Schawlow-Townes linewidth should be multiplied by the factor $1+\alpha^2$. The theory of this enhancement of the laser linewidth will be extended in a forthcoming paper based on the formalism developed here.

The standard quantum theories of the laser linewidth are restricted to the case of small output couplings (i.e., mirror reflectivities near unity). In this limit the field and the gain coefficient are approximately spatially uniform, thus simplifying the theory, and one obtains the Schawlow-Townes linewidth. It is necessary to consider arbitrary output coupling and nonuniform fields to go beyond this limit, and to arrive at a physical and fully quantum-mechanical understanding of the enhancement of the Schawlow-Townes linewidth.

Another effect that is ignored in the standard quantum theory of the linewidth is the spatial variation of the saturated gain arising from the interference of counterpropagating waves in a standing-wave laser. Spatial hole burning causes a reduction in the output power, and its inclusion in early semiclassical laser theory, but not in quantum theory, was initially misperceived as a difference between the semiclassical and quantum theories, and led to at least one experimental investigation [7] before it was realized that the difference was artificial [8]. To the best of our knowledge the effect of spatial hole burning has not previously been included in theories of the linewidth. We find that spatial hole burning increases the linewidth, as might be expected from the fact that it decreases the output power and therefore the signal-to-noise ratio. Of course spatial hole burning is washed out by atomic motion in gas lasers, and is certainly mitigated by carrier diffusion in semiconductor lasers. Such motional effects are not included here but will be dealt with later.

In recent work [9] the authors presented several results concerning the quantum limit to the laser linewidth for a one-dimensional resonator with arbitrary output coupling. In particular, it was shown that the "excess" spontaneous emission into a longitudinal mode of the resonator, which has been related to the nonorthogonal nature of these modes and yields the Petermann K factor [4,5], is given simply by the amplification of vacuum fluctuations leaking into the cavity from the outside world. We then showed how the K factor is modified when the effects of saturation and spatial hole burning are included.

In this paper we provide the theoretical foundation for these results by presenting a general quantum-mechanical calculation of the laser linewidth, valid for arbitrarily large output couplings. We explore and exploit the intimate connection between spontaneous emission and the vacuum electromagnetic field, emphasizing the physical origin of contributions to the linewidth.

We first give a derivation of $\Delta\omega_{\rm ST}$ and show explicitly that it applies in the limit of small output coupling and nearly uniform intracavity fields. We then go beyond this limit, extending the analysis to arbitrary output couplings and allowing for the amplification of both the field and source fluctuations. We show that the quantum linewidth may be attributed to spontaneous emission alone or to both spontaneous emission and vacuum field fluctuations. This is shown to be a consequence of various possible orderings of operators in the first-order field correlation function, which lead to different explicit contributions from "external" and "internal" noise sources. In Sec. III we choose a symmetric ordering and study the external and internal contributions in both linear and saturated regimes, allowing for directional gain and interference between counterpropagating cavity fields.

In Sec. IV we demonstrate that the contributions to the linewidth from vacuum field fluctuations (external) and from source noise (internal) are equal in the linear regime. Saturation of the gain medium modifies the internal contribution and increases the linewidth. Spatial hole burning introduces directional gains that modify both contributions and further increase the linewidth. We show that measurements of the linewidth at either end of a laser with two-sided output will yield the same result, in spite of the spatial variations of the field amplitude and phase within the gain medium.

Our approach allows the excess spontaneous-emission factor (K) to be derived in a physically transparent manner. In Sec. V we derive a modified enhancement factor K' that accounts for saturation and spatial hole burning. In high-gain systems K' may differ appreciably from the Petermann K factor.

In Sec. VI we focus attention on the physical origin of the laser linewidth, and show that different operator orderings in the first-order field correlation function lead to exactly the same result for the calculated linewidth. We remark further on the physical origin of the K factor, and point out that there is no "excess spontaneous emission" in the case of a single atom in a lossy cavity; the existence of the K factor requires the presence of a background gain medium, as in Petermann's classical calculation or Siegman's semiclassical approach based on an "equivalent noise" treatment of spontaneous emission. Section VII is a brief summary, and Appendices A and B contain brief derivations of field and atomic noise correlation functions used in the text of the paper. In Appendix C we present a simple semiclassical argument for the reduction of the above-threshold linewidth due to stabilization of the field amplitude.

II. THE SCHAWLOW-TOWNES LINEWIDTH

As noted in the Introduction, the Schawlow-Townes formula for the fundamental laser linewidth applies when the field and the gain coefficient in a saturable medium may be assumed to be spatially uniform. This is a good approximation when the output coupling is small, i.e., when $R_1, R_2 \cong 1$ in Fig. 1 [10].

Let the positive-frequency (annihilation) part of the intracavity field operator propagating to the right in Fig. 1 be denoted $A_R(z,t)e^{-i\omega t}$, where A_R is slowly varying compared with the sinusoidal oscillation at frequency ω. For simplicity we assume for now that ω is exactly resonant with the transition frequency of a uniform distribution of two-level atoms comprising the gain medium. We assume also that the gain medium fills the entire length d of the cavity. Consider the field $A_R(d_<, t+2d/c)$ at the right mirror:

$$A_R\left[d_<, t+\frac{2d}{c}\right] = A_L(d_<, t)(GR_1G)^{1/2} + A_{R,\rm vac}(-d,t)(T_1G)^{1/2} + A_{L,\rm vac}(d_>,t)(T_2GR_1G)^{1/2} + A_{\rm sp}(d_<,t) \tag{1}$$

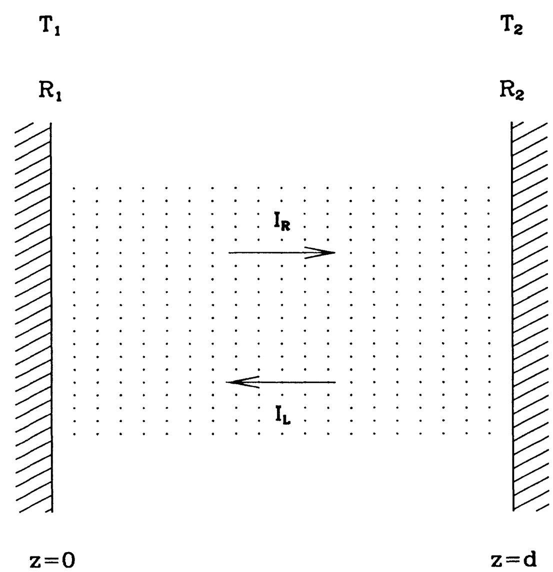

FIG. 1. Schematic of the resonator.

Here $d_<$ and $d_>$ correspond to points just inside and outside, respectively, the mirror at d. R_j and T_j denote mirror power reflection and transmission coefficients, and $\sqrt{G}$ is the amplitude amplification factor associated with a single pass through the gain cell. Subscripts R and L label right- and left-going fields, while "vac" labels source-free vacuum fields. Finally $A_{\rm sp}$ stands for the contribution from spontaneous emission, as opposed to the contributions from stimulated emission involving the factor $\sqrt{G}$.

The first term on the right-hand side of (1) arises from the propagation of the left-going field at $d_<$ through the gain medium ($\sqrt{G}$), reflection off the mirror at $z=0$ $[(R_1)^{1/2}]$, and a second pass through the gain medium ($\sqrt{G}$). The second term arises from the transmission of the external vacuum field through the mirror at $z=0$ $[(T_1)^{1/2}]$, followed by amplification of this field as it propagates to the mirror at $z=d$ ($\sqrt{G}$). The minus sign in $A_{R,\rm vac}(-d,t)$ merely indicates that the right-propagating field reaching $d_<$ at time $t+2d/c$ is, except for the effects of transmission and gain, the right-propagating field at $d_< -2d=-d$ at the retarded time $(t+2d/c)-2d/c$. Similarly the third term results from the transmission of the left-going vacuum field through the mirror at $z=d$ $[(T_2)^{1/2}]$, amplification ($\sqrt{G}$) and reflection $[(R_1)^{1/2}]$, and a second pass through the gain cell ($\sqrt{G}$). The term $A_{\rm sp}$ is discussed below.

Equation (1) expresses a basic kinematical relationship that holds regardless of whether the field is quantized or treated semiclassically. In the latter case the last three terms on the right may be treated as classical noise sources with statistical properties chosen to mimic their quantum counterparts. Throughout most of this paper the field is treated fully quantum mechanically.

Equation (1) has contributions only from vacuum fields that have passed at least once through the medium. This expresses the fact [or, at the heuristic level of Eq. (1), the assumption] that the vacuum fields are transmitted and amplified just as are fields of "real" photons. Similarly, successive passes of the vacuum fields through the medium need not be explicitly accounted for: because of the threshold condition for steady-state oscillation, there are no modifications of (1) from successive round-trip passes.

It is convenient to convert (1) to a first-order differential equation by making the approximation [11]

$$A_R\left(d_<,t+\frac{2d}{c}\right)\cong A_R(d_<,t)+\frac{2d}{c}\dot{A}_R(d_<,t)\ . \tag{2}$$

In Eq. (1) we can also write $A_L(d_<,t)=(R_2)^{1/2}A_R(d_<,t)$; note that the contribution from the transmitted vacuum field is already included in the third term in Eq. (1). Thus we can replace (1) by

$$\begin{aligned}\dot{A}_R(d_<,t)\cong{}&\frac{c}{2d}[G(R_1R_2)^{1/2}-1]A_R(d_<,t)\\&+\frac{c}{2d}[A_{R,\rm vac}(-d,t)(GT_1)^{1/2}\\&\qquad+A_{L,\rm vac}(d_>,t)G(R_1T_2)^{1/2}]\\&+\frac{c}{2d}A_{\rm sp}(d_<,t)\ .\end{aligned} \tag{3}$$

In the "Schawlow-Townes limit" of small output coupling and nearly uniform intracavity fields we have

$$\begin{aligned}G(R_1R_2)^{1/2}-1&=(R_1R_2)^{1/2}e^{\bar{g}d}-1\cong(R_1R_2)^{1/2}+\bar{g}d(R_1R_2)^{1/2}-1\\&\cong\ln(R_1R_2)^{1/2}+\bar{g}d=-\frac{2d}{c}(\tfrac{1}{2}\gamma_c)+\bar{g}d=\frac{2d}{c}(\tfrac{1}{2}c\bar{g}-\tfrac{1}{2}\gamma_c)\ .\end{aligned} \tag{4}$$

Here $\bar{g}\equiv(1/d)\ln G$ is a "mean" power gain coefficient and $\gamma_c\equiv-(c/2d)\ln(R_1R_2)$ is the cavity power damping rate. Thus

$$\begin{aligned}\dot{A}_R(d_<,t)\cong{}&\tfrac{1}{2}(c\bar{g}-\gamma_c)A_R(d_<,t)\\&+\frac{c}{2d}[A_{R,\rm vac}(-d,t)(GT_1)^{1/2}\\&\qquad+A_{L,\rm vac}(d_>,t)G(R_1T_2)^{1/2}]\\&+\frac{c}{2d}A_{\rm sp}(d_<,t)\ .\end{aligned} \tag{5}$$

We show in Appendix A that

$$\langle A_{\rm sp}^\dagger(t')A_{\rm sp}(t'')\rangle=\frac{1}{2}\left[\frac{2d}{c}\right]^2\frac{\gamma_cP_2}{P_2-P_1}\delta(t'-t'')\ , \tag{6}$$

$$\langle A_{\rm sp}(t')A_{\rm sp}^\dagger(t'')\rangle=\frac{1}{2}\left[\frac{2d}{c}\right]^2\frac{\gamma_cP_1}{P_2-P_1}\delta(t'-t'')\ , \tag{7}$$

where P_2 and P_1 are the steady-state upper- and lower-level probabilities, respectively.

A. Laser linewidth as a consequence of spontaneous emission

We are interested in expectation values of Heisenberg-picture operators over an initial state $|\psi\rangle$ in which there are no photons in the field. For such a state $A_{R,\rm vac}(-d,t)|\psi\rangle=A_{L,\rm vac}(d_>,t)|\psi\rangle=A_R(d_<,0)|\psi\rangle=0$ and therefore, from (5),

$$\langle A_R^\dagger(t)A_R(t)\rangle=\left[\frac{c}{2d}\right]^2\int_0^t dt'\int_0^t dt''\langle A_{\rm sp}^\dagger(t')A_{\rm sp}(t'')\rangle\times e^{\gamma(t'+t''-2t)}\ , \tag{8}$$

where $\gamma \equiv \frac{1}{2}(\gamma_c - c\bar{g})$. Since the intracavity field is essentially uniform in the "Schawlow-Townes limit," we have written $A_R(t)$ instead of $A_R(d_<, t)$. Equation (6) implies

$$\langle A_R^\dagger A_R \rangle_{ss} = \frac{1}{2} \frac{P_2}{P_2 - P_1} \left[\frac{\gamma_c}{2\gamma} \right] \tag{9}$$

for the steady-state value of $\langle A_R^\dagger A_R \rangle$. Thus

$$c\bar{g} = \gamma_c - \frac{1}{2} \frac{P_2}{P_2 - P_1} \frac{\gamma_c}{\langle A_R^\dagger A_R \rangle_{ss}} , \tag{10}$$

so that the steady-state gain coefficient is actually slightly less than the loss coefficient [12]. The difference between $c\bar{g}$ and γ_c is obviously a consequence of spontaneous emission, which puts photons into the lasing mode (at a rate proportional to P_2) and therefore lowers the gain necessary to realize a steady-state cavity photon number.

From (5) and (6) we also obtain

$$\langle A_R^\dagger(t) A_R(t+\tau) \rangle = \langle A_R^\dagger A_R \rangle_{ss} e^{-\gamma\tau} , \tag{11}$$

in steady-state laser oscillation ($\gamma t \gg 1$). This implies the linewidth [full width at half maximum (FWHM)]

$$\Delta\omega = 2\gamma = \gamma_c - c\bar{g} = \frac{1}{2} \frac{P_2}{P_2 - P_1} \frac{\gamma_c}{\langle A_R^\dagger A_R \rangle_{ss}} . \tag{12}$$

In the Schawlow-Townes limit $\langle A_R^\dagger A_R \rangle_{ss} \cong \langle A_L^\dagger A_L \rangle_{ss} \cong \frac{1}{2} n_{ss}$, where n_{ss} is the steady-state intracavity photon number expectation value, and the output power $P_{out} = \gamma_c \hbar\omega n_{ss}$. Then the fundamental laser linewidth due to spontaneous emission is

$$\Delta\omega_{ST} = \frac{P_2}{P_2 - P_1} \left[\frac{\hbar\omega}{P_{out}} \right] \gamma_c^2 . \tag{13}$$

Actually this linewidth is reduced by a factor of $\frac{1}{2}$ when the laser is oscillating in the nonlinear, above-threshold regime. Since this correction has no real import for our purposes in this paper, we relegate further discussion of this point to Appendix C.

B. Role of vacuum field fluctuations

The first-order correlation function (11) is normally ordered, but it is not difficult to conceive of detection schemes that would measure, say, the antinormally ordered correlation function [13]

$$\begin{aligned} \langle A_R(t) A_R^\dagger(t+\tau) \rangle = & \left[\frac{c}{2d} \right]^2 \int_0^t dt' \int_0^{t+\tau} dt'' \langle V(t') V^\dagger(t'') \rangle e^{\gamma(t'+t''-2t-\tau)} \\ & + \left[\frac{c}{2d} \right]^2 \int_0^t dt' \int_0^{t+\tau} dt'' \langle A_{sp}(t') A_{sp}^\dagger(t'') \rangle e^{\gamma(t'+t''-2t-\tau)} . \end{aligned} \tag{14}$$

Here

$$\begin{aligned} V(t) \equiv & A_{R,vac}(-d,t)(GT_1)^{1/2} \\ & + A_{L,vac}(d_>,t) G (R_1 T_2)^{1/2} . \end{aligned} \tag{15}$$

Since $\langle A_{R,vac}(-d,t) A_{R,vac}^\dagger(-d,t) \rangle$ and $\langle A_{L,vac}(d_>,t) \times A_{L,vac}^\dagger(d_>,t) \rangle$ are not zero, the vacuum field leaking into the cavity from the outside contributes *explicitly* to (14). This is in sharp contrast to the previous calculation of the linewidth based on the normally ordered correlation function, where the vacuum field made no explicit contribution and the linewidth could be interpreted solely in terms of spontaneous emission.

We show in Appendix B that

$$\begin{aligned} & \langle A_{R,vac}(-d,t') A_{R,vac}^\dagger(-d,t'') \rangle \\ & \quad = \langle A_{L,vac}(d_>,t') A_{L,vac}^\dagger(d_>,t'') \rangle = \frac{d}{c} \delta(t'-t'') , \end{aligned} \tag{16}$$

so that

$$\langle V(t') V^\dagger(t'') \rangle = (GT_1 + G^2 R_1 T_2) \frac{d}{c} \delta(t'-t'') , \tag{17}$$

while the corresponding normally ordered correlation function of course vanishes. Since they correspond to different, uncorrelated vacuum field modes, the left- and right-going vacuum fields do not make an "interference" contribution to (17). In the Schawlow-Townes limit,

$$GT_1 = e^{\bar{g}d}(1-R_1) \cong 1 - R_1 \cong -\ln R_1 , \tag{18}$$

$$G^2 R_1 T_2 = e^{2\bar{g}d} R_1 (1-R_2) \cong 1 - R_2 \cong -\ln R_2 , \tag{19}$$

and

$$\begin{aligned} \langle V(t') V^\dagger(t'') \rangle & \cong -\frac{d}{c} (\ln R_1 R_2) \delta(t'-t'') \\ & = \frac{1}{2} \left[\frac{2d}{c} \right]^2 \gamma_c \delta(t'-t'') . \end{aligned} \tag{20}$$

Equations (7), (14), and (20) then give

$$\begin{aligned} \langle A_R(t) A_R^\dagger(t+\tau) \rangle & = \left[\frac{1}{2} + \frac{1}{2} \frac{P_1}{P_2 - P_1} \right] \left[\frac{\gamma_c}{2\gamma} \right] e^{-\gamma\tau} \\ & = \frac{1}{2} \frac{P_2}{P_2 - P_1} \left[\frac{\gamma_c}{2\gamma} \right] e^{-\gamma\tau} , \end{aligned} \tag{21}$$

where the two terms in large parentheses in the first equality arise from V and $A_{\rm sp}$, respectively, i.e., from vacuum and source fields. Since $\langle A_R(t)A_R^\dagger(t)\rangle = \langle A_R^\dagger A_R\rangle + 1 = n_{\rm ss}+1 \cong n_{\rm ss}$ in steady-state laser oscillation, Eq. (21) implies the result (10) and therefore the "Schawlow-Townes" linewidth (13). But according to this calculation, the fundamental laser linewidth is attributable to *both* vacuum and source fields.

C. Remarks

It is interesting to consider also the "symmetrically" ordered correlation function

$$X(\tau)\equiv\tfrac{1}{2}\langle A_R^\dagger(t)A_R(t+\tau)+A_R(t)A_R^\dagger(t+\tau)\rangle :$$

$$\begin{aligned}
X(\tau)&=\frac{1}{2}\left[\frac{c}{2d}\right]^2\int_0^t dt'\int_0^{t+\tau}dt''[\langle V^\dagger(t')V(t'')\rangle+\langle V(t')V^\dagger(t'')\rangle]e^{\gamma(t'+t''-2t-\tau)}\\
&\quad+\frac{1}{2}\left[\frac{c}{2d}\right]^2\int_0^t dt'\int_0^{t+\tau}dt''[\langle A_{\rm sp}^\dagger(t')A_{\rm sp}(t'')\rangle+\langle A_{\rm sp}(t')A_{\rm sp}^\dagger(t'')\rangle]e^{\gamma(t'+t''-2t-\tau)}\\
&\cong\frac{1}{2}\left[\frac{1}{2}\left[\frac{\gamma_c}{2\gamma}\right]e^{-\gamma\tau}+\frac{1}{2}\left[\frac{P_2+P_1}{P_2-P_1}\right]\left[\frac{\gamma_c}{2\gamma}\right]e^{-\gamma\tau}\right]\\
&=\frac{1}{4}\left[\frac{P_2-P_1+P_2+P_1}{P_2-P_1}\right]\left[\frac{\gamma_c}{2\gamma}\right]e^{-\gamma\tau}=\frac{1}{2}\frac{P_2}{P_2-P_1}\left[\frac{\gamma_c}{2\gamma}\right]e^{-\gamma\tau}\ ,
\end{aligned} \tag{22}$$

where we have used the approximations (18) and (19) and the fact that $P_2+P_1=1$. The resulting expression (22) leads once again to the condition (10) and the linewidth (13).

This symmetric ordering is noteworthy because it underlies standard approaches to the theory of the laser linewidth, where the intracavity field operator $A(t)$ is written as $(n_{\rm ss})^{1/2}e^{-i\phi}$ and the phase is taken to have the equation of motion (see Appendix C)

$$2i\dot{\phi}=\frac{1}{A^\dagger}\dot{A}^\dagger-\frac{1}{A}\dot{A}\ . \tag{23}$$

One then calculates the phase diffusion coefficient and finds that both field and atomic noise sources contribute to the laser linewidth. Thus Lax, for instance, obtained the bracketed term in the last line of (22) and concluded from this that the linewidth "depends on *both* photon and atomic noise sources" [1]. Such an interpretation is implicit also in the treatment of Sargent, Scully, and Lamb [2]. We have shown, however, that it is possible to eliminate any explicit contribution from the field noise source by choosing a normal ordering.

Our approach also enables us to identify the field noise source as precisely the vacuum field leaking into the cavity from the outside. This vacuum field can be amplified by the gain medium but, in the "Schawlow-Townes limit" that we have just treated, amplification of the vacuum field is in effect neglected by the replacement of G by 1 in (18) and (19). In the following section we go beyond this approximation and allow for the amplification of vacuum field fluctuations or equivalently, when a normal ordering is employed, the amplification of spontaneous-emission noise.

The interpretation of various operator orderings has an important (and related) antecedent in the theory of spontaneous emission [14]. There, and in related problems of radiative corrections and intermolecular and Casimir forces [15], the normal ordering procedure eliminates explicit contributions from the vacuum, but there is no ordering in which the source contribution can be completely suppressed. The same statements apply in the theory of the laser linewidth just presented.

III. BEYOND THE SCHAWLOW-TOWNES LIMIT

For the purpose of treating spatial hole burning it will be necessary to distinguish between gain factors for left- and right-going waves, denoting them by G_L and G_R, respectively. Thus we rewrite (1) as

$$\begin{aligned}
A_R\left[d_<,t+\frac{2d}{c}\right]&=A_L(d_<,t)(G_LG_RR_1)^{1/2}\\
&\quad+A_{R,\rm vac}(-d,t)(G_RT_1)^{1/2}\\
&\quad+A_{L,\rm vac}(d_>,t)(G_LG_RR_1T_2)^{1/2}\\
&\quad+A_{\rm sp}(d_<,t)
\end{aligned} \tag{24}$$

and consequently replace (3) by

$$\begin{aligned}
\dot{A}_R(d_<,t)&\cong-\gamma A_R(d_<,t)\\
&\quad+\frac{c}{2d}[A_{R,\rm vac}(-d,t)(G_RT_1)^{1/2}\\
&\qquad+A_{L,\rm vac}(d_>,t)(G_LG_RR_1T_2)^{1/2}]\\
&\quad+\frac{c}{2d}A_{\rm sp}(d_<,t)\ ,
\end{aligned} \tag{25}$$

where now $\gamma\equiv(c/2d)[1-(G_LG_RR_1R_2)^{1/2}]$.

In this section we will work with the symmetrically ordered correlation function $X(\tau)$ used in Eq. (22). From (25) we obtain formally the same expression as in the first equality of (22):

$$X(\tau)=\frac{1}{2}\left[\frac{c}{2d}\right]^2\int_0^t dt'\int_0^{t+\tau}dt''[\langle V^\dagger(t')V(t'')\rangle+\langle V(t')V^\dagger(t'')\rangle]e^{\gamma(t'+t''-2t-\tau)}$$
$$+\frac{1}{2}\left[\frac{c}{2d}\right]^2\int_0^t dt'\int_0^{t+\tau}dt''[\langle A_{\rm sp}^\dagger(d_<,t')A_{\rm sp}(d_<,t'')\rangle+\langle A_{\rm sp}(d_<,t')A_{\rm sp}^\dagger(d_<,t'')\rangle]e^{\gamma(t'+t''-2t-\tau)}$$
$$\equiv X_{\rm ext}(\tau)+X_{\rm int}(\tau)\ , \tag{26}$$

with

$$V(t)\equiv A_{R,{\rm vac}}(-d,t)(G_RT_1)^{1/2}+A_{L,{\rm vac}}(d_>,t)(G_LG_RR_1T_2)^{1/2}\ . \tag{27}$$

Since the first term on the right side of (26) arises from the vacuum field external to the laser cavity, we shall in this section call it "external." Similarly we call the second term "internal."

The evaluation of the external contribution to (26) proceeds as in the preceding section, using the correlation functions (16) and assuming the steady-state limit $\gamma t \gg 1$:

$$X_{\rm ext}(\tau)=\frac{1}{8\gamma}\left[\frac{c}{2d}\right](G_RT_1+G_LG_RR_1T_2)e^{-\gamma\tau}\ . \tag{28}$$

Note that, unlike the treatment of the "Schawlow-Townes limit," we now allow for the *amplification of the vacuum field by the gain medium.* That is, we do not assume that the gain factors G_L and G_R are close to unity.

The evaluation of the internal contribution to (26) is more complicated than in the preceding section because now we also want to allow for the amplification of the *source* noise fields generated within the gain medium. We begin by writing $A_{\rm sp}(d_<,t)$ as the sum of contributions from all the atoms comprising the gain medium:

$$A_{\rm sp}(d_<,t)=\sum_j\left[A_{\rm sp}\left[z_j,t-\frac{d-z_j}{c}\right][G_R(d,z_j)]^{1/2}+A_{\rm sp}\left[z_j,t-\frac{d+z_j}{c}\right]\times[G_L(0,z_j)]^{1/2}(R_1G_R)^{1/2}\right]\ . \tag{29}$$

$A_{\rm sp}(z_j,t-(d-z_j)/c)$ is the contribution from an atom at z_j, whose field at $z=d$ involves the retardation time $(d-z_j)/c$ as well as the amplification factor $[G_R(d,z_j)]^{1/2}$ associated with propagation from z_j to d. Similarly $A_{\rm sp}(z_j,t-(d+z_j)/c)$ is the contribution from an atom at z_j whose field propagates to the left, towards the mirror at $z=0$, and in the process undergoes amplification by the factor $[G_L(0,z_j)]^{1/2}$. After reflection from the mirror at $z=0$, it undergoes amplification by the factor $(G_R)^{1/2}=[G_R(d,0)]^{1/2}$ as it propagates to the mirror at d. This field involves the retardation time $2z_j/c+(d-z_j)/c=(d+z_j)/c$.

From Eq. (A8) of Appendix A we have $(c/2d)A_{\rm sp}(z_j,t)=(D/\beta)F_S(z_j,t)$, and we assume that the noise operators $F_S(z_j,t)$ for different atoms are uncorrelated. Then [16]

$$\left[\frac{c}{2d}\right]^2\langle A_{\rm sp}^\dagger(d_<,t')A_{\rm sp}(d_<,t'')\rangle=\left[\frac{c}{2d}\right]^2\frac{1}{2}\sum_j\left[\left\langle A_{\rm sp}^\dagger\left[z_j,t'-\frac{d-z_j}{c}\right]A_{\rm sp}\left[z_j,t''-\frac{d-z_j}{c}\right]\right\rangle G_R(d,z_j)+\left\langle A_{\rm sp}^\dagger\left[z_j,t'-\frac{d+z_j}{c}\right]A_{\rm sp}\left[z_j,t''-\frac{d+z_j}{c}\right]\right\rangle\times R_1G_RG_L(0,z_j)\right]\ ,$$

or

$$\left[\frac{c}{2d}\right]^2\langle A_{\rm sp}^\dagger(d_<,t')A_{\rm sp}(d_<,t'')\rangle=\frac{D^2}{2\beta^2}\sum_j\langle F_S^\dagger(z_j,t')F_S(z_j,t'')\rangle[G_R(d,z_j)+R_1G_RG_L(0,z_j)]$$
$$=\frac{D^2}{\beta}\sum_j[G_R(d,z_j)+R_1G_RG_L(0,z_j)]P_2(z_j)\delta(t'-t'')\ , \tag{30}$$

where we have used Eq. (A3) of Appendix A and written $P_2(z_j)$ for the steady-state upper-level probability for an atom at z_j. Similarly

$$\left[\frac{c}{2d}\right]^2\langle A_{\rm sp}(d_<,t')A_{\rm sp}^\dagger(d_<,t'')\rangle=\frac{D^2}{\beta}\sum_j[G_R(d,z_j)+R_1G_RG_L(0,z_j)]P_1(z_j)\delta(t'-t'')\ . \tag{31}$$

As in Appendix A we can replace D^2/β by $\frac{1}{2}\gamma_c/N(P_2-P_1)_t$, where now $(P_2-P_1)_t$ is the (z-independent) threshold population inversion and N is the number of atoms [17]. Then in the steady state ($\gamma t \gg 1$),

$$X_{\rm int}(\tau)=\frac{1}{8\gamma}\left[\frac{c}{2d}\right]\frac{\ln(R_1R_2)^{-1/2}}{(P_2-P_1)_t}\frac{1}{N}\sum_j[G_R(d,z_j)+R_1G_RG_L(0,z_j)]e^{-\gamma\tau}, \tag{32}$$

where we have used $P_1(z_j)+P_2(z_j)=1$ and $\gamma_c=-(c/2d)\ln R_1R_2$.

For a continuous and uniform distribution of atoms we can make the replacement

$$\frac{1}{N}\sum_j[G_R(d,z_j)+R_1G_RG_L(0,z_j)]\rightarrow\frac{1}{d}\int_0^d dz[G_R(d,z)+R_1G_RG_L(0,z)], \tag{33}$$

and

$$X_{\rm int}(\tau)=\frac{1}{8\gamma}\left[\frac{c}{2d}\right]\frac{\ln(R_1R_2)^{-1/2}}{(P_2-P_1)_t}e^{-\gamma\tau}\frac{1}{d}\int_0^d dz[G_R(d,z)+R_1G_RG_L(0,z)]. \tag{34}$$

We can write $G_R(d,z)=I_R(d)/I_R(z)$ and $G_L(0,z)=I_L(0)/I_L(z)$, where I_R and I_L are the right- and left-propagating intracavity laser intensities. Then

$$X_{\rm int}(\tau)=\frac{1}{8\gamma}\left[\frac{c}{2d}\right]\frac{\ln(R_1R_2)^{-1/2}}{(P_2-P_1)_t}e^{-\gamma\tau}\frac{1}{d}\int_0^d dz\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right], \tag{35}$$

and therefore

$$X(\tau)=\frac{1}{8\gamma}\left[\frac{c}{2d}\right]e^{-\gamma\tau}\left[(G_RT_1+G_LG_RR_1T_2)+\frac{\ln(R_1R_2)^{-1/2}}{(P_2-P_1)_t}\frac{1}{d}\int_0^d dz\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right]\right]. \tag{36}$$

We can now proceed as before to the calculation of the linewidth. The identification

$$X(0)=\langle A_R^\dagger A_R\rangle_{\rm ss}=n_{\rm ss}=\frac{d}{c}\frac{1}{T_2}\frac{P_{\rm out}(d)}{\hbar\omega}, \tag{37}$$

where $P_{\rm out}(d)$ is the output power from the mirror at $z=d$, implies the linewidth (FWHM) $\Delta\omega(d)=2\gamma$ given by

$$\Delta\omega(d)=\frac{1}{2}\left[\frac{c}{2d}\right]^2\frac{\hbar\omega}{P_{\rm out}(d)}\left[\left[G_RT_1T_2+\frac{T_2^2}{R_2}\right]+\frac{\ln(R_1R_2)^{-1/2}}{(P_2-P_1)_t}\frac{T_2}{d}\int_0^d dz\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right]\right]. \tag{38}$$

We have used the fact that $\gamma=(c/2d)[1-(G_LG_RR_1R_2)^{1/2}]$ is small to replace G_LG_R by $1/R_1R_2$ in this expression for the linewidth.

Although it is not at all obvious from this derivation that the linewidths of the fields emerging from the two ends of the cavity should be identical in the case of two-sided output coupling with spatially varying field amplitudes and phases, we shall prove that this is in fact the case.

IV. INTERNAL AND EXTERNAL CONTRIBUTIONS TO THE LINEWIDTH

We shall evaluate the linewidth (38) for the plane parallel resonator of Fig. 1 with arbitrary output coupling, taking into account the interference between counterpropagating fields. We again use the directional gains

$$G_R=\frac{I_R(d)}{I_R(0)},\quad G_L=\frac{I_L(0)}{I_L(d)}, \tag{39}$$

and the boundary conditions at the planes $z=0$ and d,

$$I_R(0)=R_1I_L(0),\quad I_L(d)=R_2I_R(d), \tag{40}$$

to obtain the steady-state oscillation condition in the absence of noise:

$$G_RG_L=\frac{1}{R_1R_2}. \tag{41}$$

The coupled differential equations for the field intensities are

$$\frac{dI_R}{dz}=g_R(z)I_R,\quad \frac{dI_L}{dz}=-g_L(z)I_L, \tag{42}$$

where the gain coefficients $g_{R,L}$ are, in general, functions of both I_R and I_L. When $g_R=g_L$, the product $I_R(z)I_L(z)$ is constant and we find

$$\frac{I_R(d)}{I_L(0)}=\frac{I_R(0)}{I_L(d)}=\left[\frac{R_1}{R_2}\right]^{1/2}, \tag{43}$$

and

$$G_R=G_L=G=\frac{1}{(R_1R_2)^{1/2}}. \tag{44}$$

As discussed in Sec. III, (38) corresponds to a choice of symmetric ordering of the field operators. In this equation there are two distinct contributions to the laser linewidth. The first term in the large parentheses represents the "external" contribution due to the amplification of vacuum fluctuations entering the cavity from outside. The second term gives the "internal" contribution due to atomic noise fields inside the cavity. We find it convenient to express (38) in the general form

$$\Delta\omega(d)=\frac{1}{2}\left[\frac{c}{2d}\right]^2\frac{\hbar\omega}{P_{\text{out}}(d)}\left[\Gamma_{\text{ext}}+\frac{1}{(P_2-P_1)_t}\Gamma_{\text{int}}\right] . \tag{45}$$

With (39) and (40), Γ_{int} may be expressed as

$$\Gamma_{\text{int}}=\ln(R_1R_2)^{-1/2}\frac{T_2I_R(d)}{d}\times\int_0^d dz\left[\frac{I_R(z)+I_L(z)}{I_R(z)I_L(z)}\right] . \tag{46}$$

We shall examine (46) for three cases of interest: (a) the linear (exponential gain) regime where (44) applies with $G(z)=e^{\bar{g}z}$; (b) the nonlinear (saturated gain) regime where (44) applies with $G(z)$ no longer exponential; and (c) the nonlinear regime with interference between counter propagating intracavity fields taken into account, so that (44) is no longer valid.

We first note that the linewidth of radiation emerging through the mirror at $z=0$ may be obtained by making the appropriate replacements in (38):

$$\Delta\omega(0)=\frac{1}{2}\left[\frac{c}{2d}\right]^2\frac{\hbar\omega}{P_{\text{out}}(0)}\times\left[\left[G_LT_1T_2+\frac{T_1^2}{R_1}\right]+\frac{\ln(R_1R_2)^{-1/2}}{(P_2-P_1)_t}\frac{T_1I_L(0)}{d}\times\int_0^d dz\left[\frac{I_R(z)+I_L(z)}{I_R(z)I_L(z)}\right]\right] . \tag{47}$$

Using the relation

$$\frac{P_{\text{out}}(d)}{P_{\text{out}}(0)}=\frac{T_2I_R(d)}{T_1I_L(0)}=\frac{T_2}{T_1}R_1G_R , \tag{48}$$

and the gain "clamping" condition (41), we deduce that

$$\Delta\omega(d)=\Delta\omega(0)\equiv\Delta\omega . \tag{49}$$

Even when the directional gains are unequal, therefore, measurements of the fundamental linewidth at either end of the laser will yield the same value.

A. Exponential gain regime

In this unsaturated regime $g_R=g_L=\bar{g}=(-1/2d)\ln R_1R_2$. Utilizing the differential equations (42), we may perform the integration in (46) to obtain

$$\Gamma_{\text{int}}=T_2I_R(d)\left[\frac{1}{I_L(z)}-\frac{1}{I_R(z)}\right]_0^d , \tag{50}$$

or, employing (39)–(41),

$$\Gamma_{\text{int}}=T_2\left[\frac{1}{R_2}-1-R_1G_R+G_R\right]=T_2\left[\frac{T_2}{R_2}+T_1G_R\right]=\Gamma_{\text{ext}} . \tag{51}$$

We see that in this regime the internal and external contributions to the linewidth are exactly equal. From (44) and (51), and using $P_1+P_2=(P_1+P_2)_t=1$, we find

$$\Delta\omega=\left[\frac{c}{2d}\right]^2\frac{\hbar\omega}{P_{\text{out}}(d)}\left[\frac{P_2}{P_2-P_1}\right]_t\times\left[\frac{[(R_1)^{1/2}+(R_2)^{1/2}][1-(R_1R_2)^{1/2}]}{R_1R_2}\right]\times T_2(R_1)^{1/2} . \tag{52}$$

In terms of the total output power P_{out} given by

$$\frac{P_{\text{out}}}{P_{\text{out}}(d)}=\frac{T_2I_R(d)+T_1I_L(0)}{T_2I_R(d)}=1+\frac{T_1}{T_2}\left[\frac{R_2}{R_1}\right]^{1/2} , \tag{53}$$

the linewidth is

$$\Delta\omega=\left[\frac{c}{2d}\right]^2\frac{\hbar\omega}{P_{\text{out}}}\left[\frac{P_2}{P_2-P_1}\right]_t\times\left[\frac{[(R_1)^{1/2}+(R_2)^{1/2}][1-(R_1R_2)^{1/2}]}{(R_1R_2)^{1/2}}\right]^2 . \tag{54}$$

We note that in this regime $G(z)=e^{\bar{g}z}$, and that if the approximation $e^{\bar{g}z}\approx 1+\bar{g}z$ is made, we recover $\Delta\omega_{\text{ST}}$ of Eq. (13) instead of (54).

B. Gain-saturation regime

The expression for the external contribution to the linewidth is obviously unchanged regardless of how strongly the intracavity field may saturate the gain. The amplification of the (right-going) vacuum field is determined by the single factor G_R of (38). As long as (44) is valid, this contribution will be unchanged. The internal contribution, however, will be modified by gain saturation. We will assume a homogeneously broadened gain medium, in which case, for zero atom-field detuning,

$$g_R=g_L=\frac{\bar{g}}{1+I_R+I_L} , \tag{55}$$

with the field intensities normalized to the saturation intensity. The differential equations (42) may be integrated along the length of the cavity using (55) and the relation $I_R(z)I_L(z)=\text{const}\equiv C$. We obtain for $I_R(z)$

$$\ln\frac{I_R(z)}{I_R(0)}+I_R(z)-I_R(0)-C\left[\frac{1}{I_R(z)}-\frac{1}{I_R(0)}\right]=\bar{g}z\ . \tag{56}$$

At $z=d$,

$$I_R(d)=\frac{(R_1)^{1/2}}{[(R_1)^{1/2}+(R_2)^{1/2}][1-(R_1R_2)^{1/2}]}\times[\bar{g}d+\ln(R_1R_2)^{1/2}]\ . \tag{57}$$

Equation (46) may be integrated using the above results. The resulting value of $\Gamma_{\rm int}$ will depend on the values of $\bar{g}d$ and the mirror reflectivities. In Figs. 2 and 3 we plot the ratio of external to internal contributions, for two values of $\bar{g}d$, as a function of mirror reflectivity R_2. In each case we show results for two values for R_1. These curves display the generic trend that gain saturation leads to an increase in $\Gamma_{\rm int}$. This increase is more pronounced for larger $\bar{g}d$ and when the product of the reflectivities is small or, equivalently, for larger values of G. Our numerical results also indicate that the two contributions become independent of $\bar{g}d$ and essentially equal for $G\leq 2$. This corresponds to a regime where either the external or internal contribution may be used to compute the linewidth.

C. Gain saturation with spatial hole burning

The saturated gain is modified when interference between the left- and right-propagating fields is properly taken into account. This interference produces spatial hole burning and a reduction in output power. For an asymmetric cavity the gain becomes directional, so that G_R and G_L are no longer equal, although steady-state gain clamping ensures that $G_RG_L=G^2=1/R_1R_2$. Both $\Gamma_{\rm ext}$ and $\Gamma_{\rm int}$ will be shown to increase due to spatial hole burning. In the paraxial and plane-wave approximations the directional gain coefficients are given by [18]

$$g_i=\frac{\bar{g}}{(a^2-b^2)^{1/2}}\left[1-\frac{a-(a^2-b^2)^{1/2}}{2I_i}\right],\quad i=R,L \tag{58}$$

where $a\equiv 1+I_R+I_L$ and $b\equiv 2(I_RI_L)^{1/2}$. As shown by Agrawal and Lax [18], the solution may be formulated in terms of shifted intensities

$$A=I_R-\lambda,\quad B=I_L-\lambda\ , \tag{59}$$

which satisfy the same set of equations as I_R and I_L do in the absence of interference effects. The constant shift λ is specified by the boundary conditions. With this change of variables the coupled differential equations (42) then become [18]

$$\frac{dA}{dz}=\frac{\bar{g}A}{1+A+B}\ , \tag{60}$$

$$\frac{dB}{dz}=\frac{-\bar{g}B}{1+A+B}\ , \tag{61}$$

with the product $AB=\lambda$. Using the notation $A_0\equiv A(z=0)$, $A_1\equiv A(z=d)$, the boundary conditions at $z=0$ and d are now

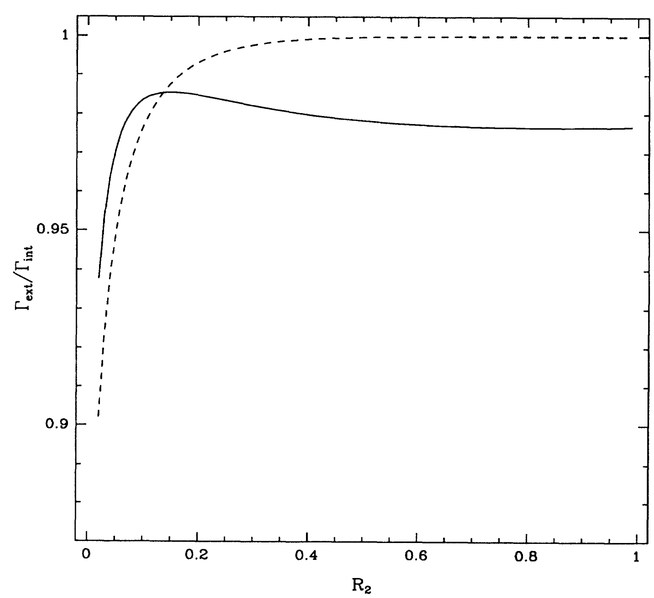

FIG. 2. Ratio of the external and internal contributions, in the gain-saturation regime, as a function of the mirror reflectivity R_2. $R_1=0.1$ (solid) and 0.9 (dashed) while $\bar{g}d=5$ in both cases.

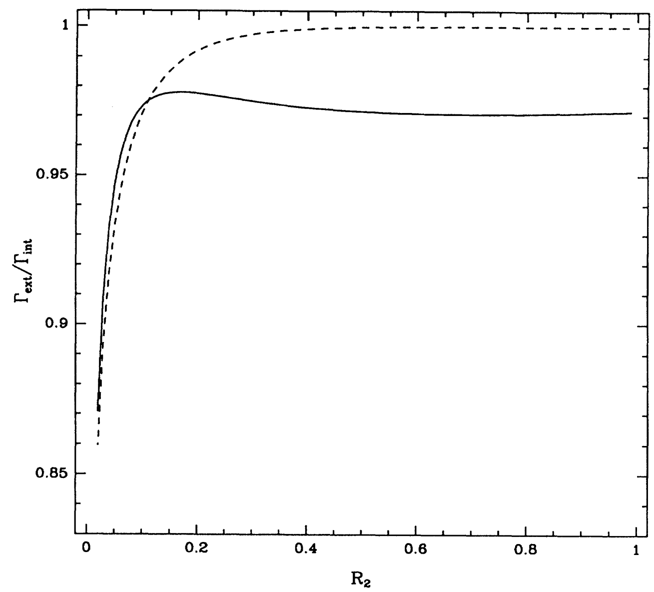

FIG. 3. Same as Fig. 3 but for $\bar{g}d=20.0$.

$$A_0+\lambda=R_1\left[\frac{\lambda}{A_0}+\lambda\right], \tag{62}$$

$$A_1+\lambda=\frac{1}{R_2}\left[\frac{\lambda}{A_1}+\lambda\right]. \tag{63}$$

We are interested in the solution of (60) for the general case of asymmetric resonators, subject to the above boundary conditions. The quadratic equation in A_0, resulting from (62), yields two solutions, both of which satisfy the requirement that the intensities be positive. However, the one given below is the only one consistent with the threshold gain condition, which implies that the intracavity laser field is zero at and below threshold. (It might be noted that this is the correct criterion to be applied to Eq. (4.3) in Ref. [18].) Consequently, A_0 is given by

$$A_0=\tfrac{1}{2}\{[\lambda^2(1-R_1)^2+4\lambda R_1]^{1/2}-\lambda(1-R_1)\} . \tag{64}$$

Integration of (60) leads to the following transcendental equation:

$$\ln\frac{A(z)}{A_0}+[A(z)-A_0]-\lambda\left[\frac{1}{A(z)}-\frac{1}{A_0}\right]=\bar{g}z , \tag{65}$$

satisfied by $A(z)$ at all points along the cavity. In particular, at $z=d$, we have

$$\ln\frac{A_1}{A_0}+(A_1-A_0)-\lambda\left[\frac{1}{A_1}-\frac{1}{A_0}\right]=\bar{g}d ,$$

which may be rewritten as

$$\ln[f(\lambda)]+[f(\lambda)-1]A_0+\frac{\lambda[f(\lambda)-1]}{f(\lambda)A_0}=\bar{g}d , \tag{66}$$

where

$$f(\lambda)=\frac{A_1}{A_0}=\frac{1}{R_2}\left[\frac{[\lambda^2(1-R_2)^2+4\lambda R_2]^{1/2}+\lambda(1-R_2)}{[\lambda^2(1-R_1)^2+4\lambda R_1]^{1/2}-\lambda(1-R_1)}\right]. \tag{67}$$

Numerical solution of (66) and (67) determines the parameter λ. Finally, having obtained A_0 and λ, the directional gain G_R is given by

$$G_R=\frac{A_1+\lambda}{A_0+\lambda}=\frac{1}{R_2}\left[\frac{[\lambda^2(1-R_2)^2+4\lambda R_2]^{1/2}+\lambda(1+R_2)}{[\lambda^2(1-R_1)^2+4\lambda R_1]^{1/2}+\lambda(1+R_1)}\right], \tag{68}$$

while G_L is determined by the gain clamping condition.

It is clear that for a symmetric resonator, with $R_1=R_2=R$,

$$G_R=G_L=\frac{1}{R} . \tag{69}$$

However, when the mirror reflectivities are unequal, the gain is strongly dependent on their ratio. This dependence is illustrated in Fig. 4, where G_R, G_L, and the gain obtained by neglecting interference effects $[G=1/(R_1R_2)^{1/2}]$ are displayed as functions of R_2 for a fixed value of R_1.

We now consider the effect of spatial hole burning on the linewidth. It is convenient to first rewrite (38) in terms of the total output power, using the general form of (53):

$$\frac{P_{\text{out}}}{P_{\text{out}}(d)}=1+\frac{T_1}{T_2}G_LR_2 . \tag{70}$$

Then

$$\begin{aligned}\Gamma_{\text{ext}}&=\frac{T_2^2}{R_2}+T_1G_RT_2+T_1G_LT_2+\frac{T_1^2}{R_1}\\&=\left[\frac{T_1}{(R_1)^{1/2}}+\frac{T_2}{(R_2)^{1/2}}\right]^2+T_1T_2\left[G_R+G_L-\frac{2}{(R_1R_2)^{1/2}}\right]\\&=\left[\frac{[(R_1)^{1/2}+(R_2)^{1/2}][1-(R_1R_2)^{1/2}]}{(R_1R_2)^{1/2}}\right]^2+\frac{T_1T_2}{R_1R_2}\frac{[(R_1R_2)^{1/2}G_R-1]^2}{G_R} .\end{aligned} \tag{71}$$

Spatial hole burning will always increase the laser linewidth because it reduces the output power of the laser, thereby increasing the relative noise power. We see from Eq. (71) that the external contribution is further increased for lasers with two-sided output coupling and asymmetric cavities. This additional increase vanishes for single-output resonators, or for those with mirrors of equal reflectivity, where $G_R=G_L=1/R$.

Γ_{int} will also increase due to spatial hole burning. In terms of the variables defined in (59) and the total output power, (46) becomes

$$\Gamma_{\text{int}}=\ln(R_1R_2)^{-1/2}\left[T_2G_R+\frac{T_1}{R_1}\right]\frac{A_0+\lambda}{\lambda d}\times\int_0^d dz\left[\frac{A^2+2\lambda A+\lambda}{A^2+(\lambda+1)A+\lambda}\right]. \tag{72}$$

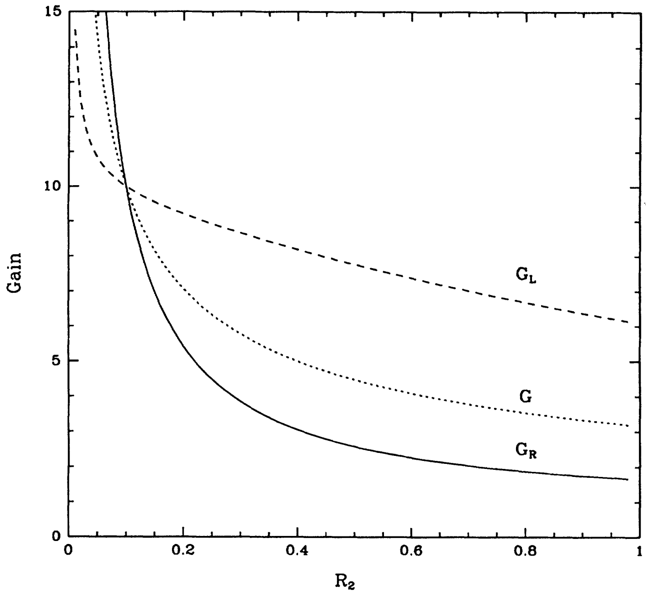

FIG. 4. Variation of the directional gains G_R, G_L as well as the gain G in the absence of hole burning, as a function of the mirror reflectivity R_2. R_1 is held fixed at 0.1 and $\bar{g}d$ is taken to be 20.0.

This equation may be integrated using (64)–(68). As before, the result will depend on the values of $\bar{g}d$ and the mirror reflectivities. In Figs. 5 and 6 we exhibit the ratio of external to internal contributions, as given, respectively, by Eqs. (71) and (72), for two values of $\bar{g}d$. We again consider two values of R_1 for each value of $\bar{g}d$. On contrasting these curves with those in Figs. 2 and 3, it is clear that the effects of spatial hole burning on the internal contribution are more pronounced than those of gain saturation alone. The two contributions are nearly equal only when both $R_1, R_2 \approx 1.0$. This corresponds to the regime where $\Delta\omega_{ST}$ is a good estimate for the laser linewidth.

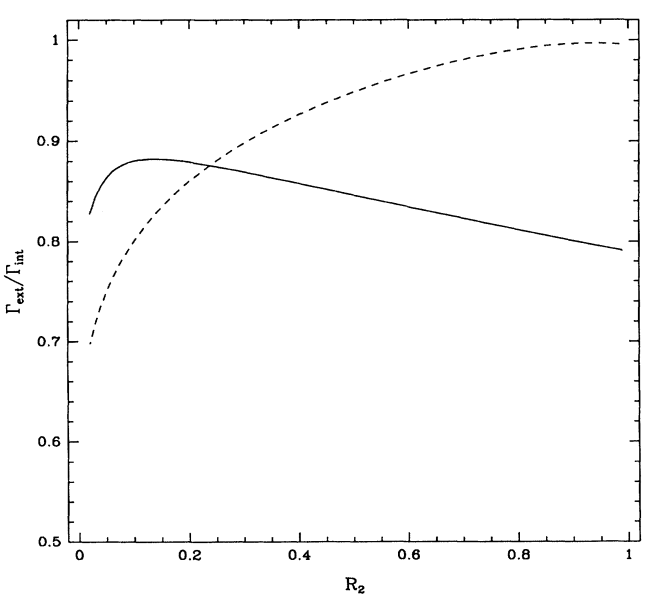

FIG. 5. Ratio of external and internal contributions, including effects of spatial hole burning, as a function of R_2. R_1 and $\bar{g}d$ have the same values as in Fig. 2.

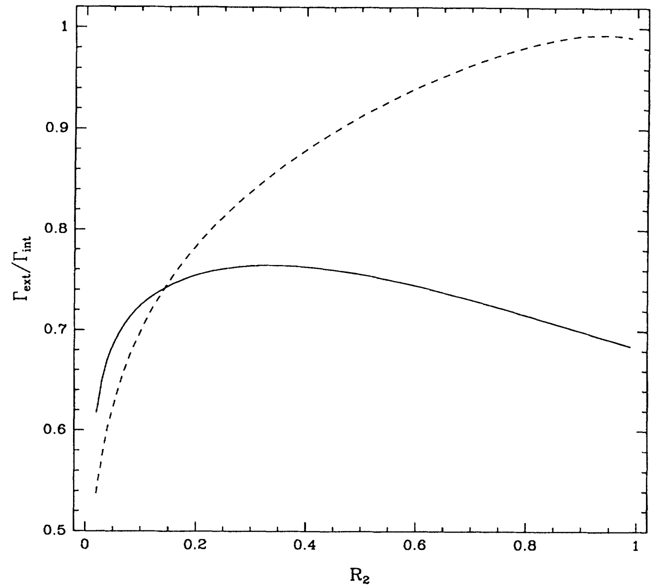

FIG. 6. Same as in Fig. 5 but for $\bar{g}d = 20.0$.

V. THE K FACTOR

The laser linewidth may be expressed in the form

$$\Delta\omega = K\Delta\omega_{ST} \ . \tag{73}$$

The factor K, usually referred to as the Petermann enhancement factor, or the "excess" spontaneous-emission factor [4,5], has been the subject of recent experimental studies [19].

In the unsaturated regime, the K factor is given by the ratio of (54) to (13). We find

$$K = \left[\frac{[(R_1)^{1/2} + (R_2)^{1/2}][1-(R_1R_2)^{1/2}]}{(R_1R_2)^{1/2}\ln(R_1R_2)} \right]^2 , \tag{74}$$

in agreement with previous calculations [20] in which the K factor was determined by integration over the explicit "adjoint modes" of the cavity. In our work, the K factor for this regime is obtained without reference to the adjoint modes and has a clear physical interpretation. It may be obtained almost trivially by calculating the amplification of vacuum fluctuations leaking into the cavity from the space external to the cavity. (See also Sec. VI.) This amplification depends only on the overall gain, and therefore provides the basis for the very simple derivation of the K factor (74) [9].

When saturation and spatial-hole-burning effects are included, (74) is no longer valid and the K factor will be modified. The modified enhancement factor K' may be expressed as

$$K'=\frac{\Delta\omega}{\Delta\omega_{\rm ST}}=\frac{\Gamma_{\rm ext}+[1/(P_2-P_1)_t]\Gamma_{\rm int}}{2[P_2/(P_2-P_1)]_t(\ln R_1R_2)^2}\,, \tag{75}$$

or

$$K'=\frac{1}{2}\left[(\Gamma_{\rm int}+\Gamma_{\rm ext})+\left[\frac{P_1}{P_2}\right]_t(\Gamma_{\rm int}-\Gamma_{\rm ext})\right]\times\frac{1}{(\ln R_1R_2)^2}\,. \tag{76}$$

Values of K' may be obtained using (71) and (72) in Eq. (76). K' will reduce to K only when $\Gamma_{\rm int}=\Gamma_{\rm ext}$, i.e., only for the regime discussed in Sec. IV A. As acknowledged by the authors, the adjoint mode analyses of Refs. 5 and 20 are restricted to the same regime. The inclusion of saturation and interference effects necessary to obtain K' in terms of adjoint modes appears to us to be rather difficult. In contrast, Eqs. (71), (72), and (76) provide a straightforward way to include these effects, and to elucidate the physical origin of the various contributions to the enhancement factor.

In Figs. 7 and 8 both K and K' are shown as functions of R_2 for fixed values of R_1. The increase in the enhancement factor due to spatial hole burning is seen to be significant for small values of the product R_1R_2.

VI. PHYSICAL INTERPRETATIONS OF LASER LINEWIDTH

There is no question that the quantum linewidth of a laser may be attributed to spontaneous emission. In the

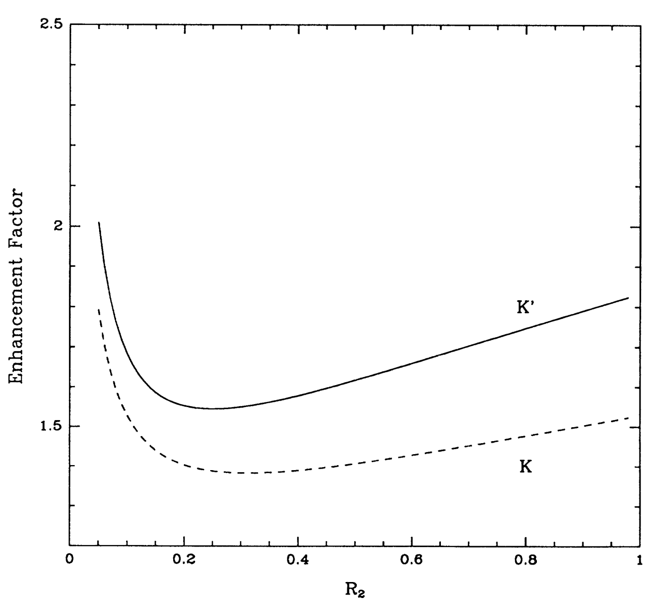

FIG. 7. K and K' as a function of mirror reflectivity R_2. R_1 is fixed at 0.1 and $\bar{g}d=5.0$, while the value $\frac{1}{2}$ is assumed for the ratio $(P_1/P_2)_t$.

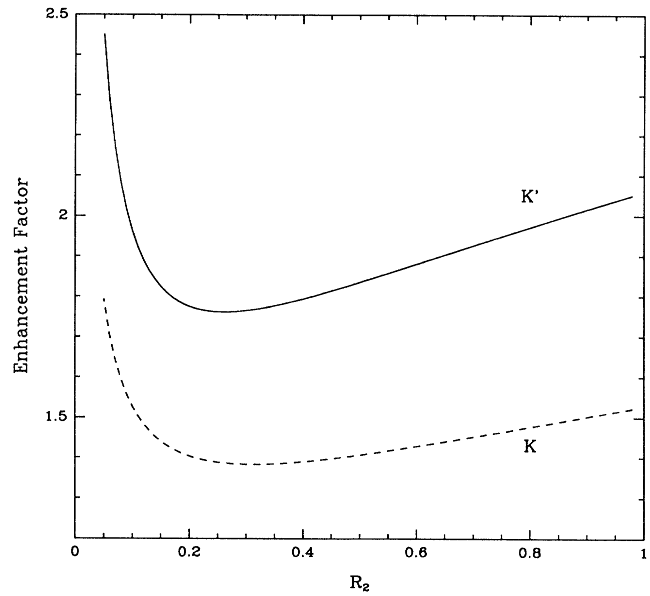

FIG. 8. Same as in Fig. 7 but for $\bar{g}d=20.0$.

Schawlow-Townes limit, for instance, the linewidth may be derived starting from the energy-time uncertainty relation, using for Δt the lifetime associated with spontaneous emission into the lasing mode [10].

We showed in Sec. II that this is not the *only* way to explain the finite laser linewidth. By considering non-normally ordered field correlation functions—which determine the *same* laser linewidth via a Fourier transform—we showed that the linewidth could also be attributed in part to the vacuum field leaking into the cavity. This is not surprising in view of the fact that spontaneous emission itself may be regarded in part as stimulated emission due to the vacuum field [14].

Such an alternative interpretation has a practical advantage. Namely, it leads to a very simple derivation of the K factor based on the amplification of the vacuum field propagating into the gain medium from the space external to the cavity. (See below.) In the "Schawlow-Townes limit" the amplification factor is close to unity, but for high-gain systems the amplification of vacuum fluctuations is appreciable, and its deviation from unity is precisely the origin of the K factor.

To appreciate the interplay of spontaneous emission and amplified vacuum fields in a heuristic, semiclassical framework, consider the equation

$$\frac{dI_N}{dz}=gI_N+R_{\rm sp}N_2\hbar\omega \tag{77}$$

describing the propagation of the "noise intensity" in an inverted medium. The gain coefficient g is independent of I_N if the noise intensity is too weak, compared with the coherent signal, to contribute substantially to gain saturation. $R_{\rm sp}$ is the rate of spontaneous emission into the single (signal) mode of frequency ω of interest, and N_2 is the number density of excited atoms. For simplicity we take g and N_2 to be independent of z. Then

$$I(d)-I(0)\cong\left[I_N(0)+\frac{R_{\rm sp}\hbar\omega N_2}{2g}\right]gd \tag{78}$$

for gd small compared with unity.

The rate of stimulated emission (or absorption) into a single mode of intensity I is $\sigma I/\hbar\omega$, where σ is the stimulated emission cross section. If we regard spontaneous emission as stimulated emission due to the vacuum field [14], and replace $R_{\rm sp}$ by $\sigma I_N/\hbar\omega$ in (78), then

$$\begin{aligned}I(d)-I(0)&\cong\left[I_N(0)+\frac{\sigma I_N(0)N_2}{g}\right]gd\\&=\left[1+\frac{N_2}{N_2-N_1}\right]I_N(0)gd\ ,\end{aligned} \tag{79}$$

since $g=\sigma(N_2-N_1)$. Thus, except for the factor $N_2/(N_2-N_1)$, the vacuum and spontaneous-emission contributions to the change in the noise intensity as it propagates from $z=0$ to d are exactly the same.

This heuristic argument is in fact directly relevant to the laser linewidth, since we have shown that the noise field that randomly perturbs the laser output and produces a finite spectral linewidth can appear to have contributions from both the amplified vacuum field and amplified spontaneous emission. We have referred to these contributions as "external" and "internal," and have shown explicitly that within the linear regime they are the same, just as in the above semiclassical argument. The different "weights" given to these two contributions depend simply on how field operators are ordered in the calculation. We will now demonstrate explicitly that different orderings lead to the same linewidth. For simplicity we will neglect spatial hole burning, although the equivalence of different orderings may be shown to hold independently of whether there is spatial hole burning.

In Sec. III we arrived at the expression (38) for the linewidth using the symmetrically ordered correlation function $X(\tau)$ defined by Eq. (22). Let us now consider the normally ordered correlation function

$$X'(\tau)=\left[\frac{c}{2d}\right]^2\int_0^t dt'\int_0^{t+\tau}dt''\langle A_{\rm sp}^\dagger(d_<,t')A_{\rm sp}(d_<,t'')\rangle\times e^{\gamma(t'+t''-2t-\tau)}\ . \tag{80}$$

In this case the vacuum field makes no explicit contribution. Now from (30) and (32) we can write $X'(\tau)$ in the form

$$X'(\tau)=\frac{\gamma_c}{8\gamma}\frac{e^{-\gamma\tau}}{(P_2-P_1)_t}\frac{1}{d}\times\int_0^d dz\,P_2(z)\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right]\ , \tag{81}$$

where $P_2(z)$ is the upper-level probability for an atom at z:

$$\begin{aligned}P_2(z)&=\tfrac{1}{2}[P_2(z)+P_1(z)]+\tfrac{1}{2}[P_2(z)-P_1(z)]\\&=\tfrac{1}{2}+\tfrac{1}{2}[P_2(z)-P_1(z)]\ .\end{aligned} \tag{82}$$

The expression $g(z)=\sigma N[P_2(z)-P_1(z)]$ for the gain coefficient at z, together with the definition $g_t=\gamma_c/c=\sigma N(P_2-P_1)_t$ for the threshold gain, implies

$$P_2(z)-P_1(z)=\frac{c}{\gamma_c}g(z)(P_2-P_1)_t \tag{83}$$

and therefore

$$\begin{aligned}X'(\tau)&=\frac{\gamma_c}{16\gamma}\frac{e^{-\gamma\tau}}{(P_2-P_1)_t}\frac{1}{d}\int_0^d\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right]+\frac{c}{16\gamma}e^{-\gamma\tau}\frac{1}{d}\int_0^d dz\,g(z)\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right]\\&=X_{\rm int}(\tau)+\frac{c}{16\gamma}e^{-\gamma\tau}\frac{1}{d}\int_0^d dz\,g(z)\left[\frac{I_R(d)}{I_R(z)}+R_1G_R\frac{I_L(0)}{I_L(z)}\right]\ ,\end{aligned} \tag{84}$$

where $X_{\rm int}(\tau)$ is given by Eq. (35). To demonstrate the equivalence of the symmetric and normal orderings, we must show that the second term on the right-hand side is equal to the quantity $X_{\rm ext}(\tau)$ defined in (28). This equality may be demonstrated straightforwardly using the general relations (39)–(44).

At the risk of belaboring the point, we emphasize how simply this equivalence of different orderings allows us to calculate the K factor [9], for if the "internal" and "external" contributions are the same, we can calculate the K factor using one or the other. More to the point, we can use only the external part, which depends only on the overall gain factors and not at all on the generation and propagation of spontaneous emission from the excited atoms in the gain medium.

From the literature on the K factor one could easily be left with the impression that "excess spontaneous-emission noise" is a fundamental property of a *single atom* in a lossy cavity. However, this is not the case. Several years ago a solution to the problem of an atom in a lossy, multimode cavity was given [21]. The analysis was fully quantum mechanical, beginning with the Hamiltonian for the initially excited atom, the quantized radiation field, and all the atoms making up the two dielectric mirrors defining the (open) cavity. For simplicity only modes propagating in the two directions normal to the mirror surfaces were included in the analysis. One result of this theory is that the spontaneous-emission rate of a single atom inside a single-mode lossy cavity is enhanced by the *Q factor* of the cavity, not the K factor [22]. This is in agreement with the old argument of Purcell [23]. The K factor is therefore an enhancement of spontaneous-emission noise associated with an atom *inside a gain medium*. Such an interpretation is in fact entirely consistent with Siegman's analysis.

VII. SUMMARY

In this paper we have shown how the Schawlow-Townes formula for the laser linewidth is modified when arbitrarily large output couplings are taken into account, as well as saturation and spatial hole burning. The possibility of large output coupling leads to an enhancement of the linewidth as described by the Petermann K factor, and saturation and spatial hole burning produce further enhancement of the linewidth.

It is interesting that the experimental results of Hamel and Woerdman [19] do, in fact, show a larger K factor than that predicted without saturation and spatial hole burning. This difference can easily be accounted for by comparing their experimental results with our K' rather than the K factor without saturation and spatial hole burning. However, there is an important caveat: the difference between K and K', for the numbers appropriate to the Hamel-Woerdman experiments, is due predominantly to spatial hole burning. But, as we have noted, this effect will be reduced by carrier diffusion which is not accounted for in this paper. We defer further discussion of this point to a later paper that will include an analysis of the $1+\alpha^2$ enhancement as well as explore the modifications on including saturation and spatial hole burning.

The physical picture emerging from our theory attributes the laser linewidth to *either* spontaneous emission alone or to both spontaneous emission and vacuum field fluctuations, depending on how one chooses to order field operators in the correlation function determining the linewidth via a Fourier transform. We have shown that conventional theories, which attribute the linewidth to both atomic and field noise sources, are in this conceptual sense oversimplified.

The freedom to choose operator orderings leads to a practical as well as conceptual advantage: we have shown that it allows not only for a simple derivation of the K factor, but also for a straightforward generalization to include the effects of saturation and spatial hole burning on the linewidth. However, in view of the confusion that has previously surrounded the K factor, we feel that the conceptual clarity emerging from our approach should not be underestimated. In the simplest possible terms we can conclude that the fundamental laser linewidth arises from spontaneous emission and the fluctuations of the vacuum field leaking into the laser cavity from the outside world. When there is large output coupling, and therefore high gain, both the spontaneous emission generated inside the cavity and the vacuum field propagating into the cavity are amplified, and it is this amplification that is neglected in the standard theories and that gives rise to the K factor.

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APPENDIX A: NOISE CORRELATION FUNCTION FOR ATOMS

Consider first the familiar Heisenberg equation of motion for the slowly varying part of the two-level atomic lowering operator $\sigma(t)=S(t)e^{-i\omega t}$ in the case of a single atom coupled to a single field mode:

$$\dot{S}(t)=DA(t)\sigma_z-\beta S(t)+F_S(t) \ . \tag{A1}$$

Here D is the atom-field coupling constant, σ_z and A are, respectively, the operators corresponding to the population inversion and the slowly varying part of the photon annihilation operator, and β is a decay rate associated with the homogeneous broadening of the atomic transition. $F_S(t)$ is the Langevin noise operator resulting from the dissipation mechanism giving rise to the line broadening.

The dissipation rate β in (A1) must be related to a fluctuation process represented by $F_S(t)$ in order to prevent the *operator* $S(t)$ from decaying to zero. The fluctuation-dissipation relation between β and $F_S(t)$ is independent of the A field in (A1), because both the fluctuation and the dissipation arise from the coupling to degrees of freedom (a "reservoir") independent of A. We can obtain the desired relation, therefore, by considering the formal solution of (A1) in the absence of any field, which yields

$$\langle S^\dagger(t)S(t)\rangle=\int_0^t dt'\int_0^t dt''\langle F_S^\dagger(t')F_S(t'')\rangle e^{\beta(t'+t''-2t)} \tag{A2}$$

in the long-time limit $\beta t\gg 1$, which for simplicity we focus on here. Langevin operators are typically δ correlated, and in order to satisfy the identity $\langle S^\dagger(t)S(t)\rangle=\frac{1}{2}\langle 1+\sigma_z(t)\rangle$ in the long-time limit we choose

$$\langle F_S^\dagger(t')F_S(t'')\rangle=2\beta P_2\delta(t'-t'') \ , \tag{A3}$$

where P_2 is the steady-state value of $\frac{1}{2}\langle 1+\sigma_z\rangle$, i.e., the steady-state upper-level probability. Similarly the identity $\langle S(t)S^\dagger(t)\rangle=\frac{1}{2}\langle 1-\sigma_z(t)\rangle$ is satisfied if

$$\langle F_S(t')F_S^\dagger(t'')\rangle=2\beta P_1\delta(t'-t'') \ , \tag{A4}$$

where P_1 is the steady-state lower-level probability.

Now in many, if not most lasers, β is large compared with other rates and consequently S may be assumed to adiabatically follow the inversion:

$$S\cong\frac{D}{\beta}A(t)\sigma_z(t)+\frac{1}{\beta}F_S(t) \ . \tag{A5}$$

This is essentially the standard rate-equation approximation. In this approximation the Heisenberg equation for the field operator,

$$\dot{A}(t)=DS(t)-\tfrac{1}{2}\gamma_c A(t)+F_A(t)\ , \quad \text{(A6)}$$

becomes

$$\dot{A}(t)\cong\frac{D^2}{\beta}\sigma_z(t)A(t)+\frac{D}{\beta}F_S(t)-\tfrac{1}{2}\gamma_c A(t)+F_A(t) \quad \text{(A7)}$$

with $F_A(t)$ the Langevin noise operator associated with cavity damping.

The first term on the right-hand side of (A7) accounts for stimulated emission and absorption. The second term corresponds to spontaneous emission and may be identified with the last term in Eq. (5) of the text:

$$\frac{c}{2d}A_{\rm sp}(t)\rightarrow\frac{D}{\beta}F_S(t) \quad \text{(A8)}$$

and

$$\langle A^\dagger_{\rm sp}(t')A_{\rm sp}(t'')\rangle\rightarrow\left[\frac{2d}{c}\right]^2\left[\frac{D}{\beta}\right]^2\langle F^\dagger_S(t')F_S(t'')\rangle\ . \quad \text{(A9)}$$

Equation (A7) applies when a single atom is coupled to the field. For N atoms the factor D^2/β is replaced by ND^2/β, which may be identified as $cg/2(P_2-P_1)$, where g is the gain coefficient. Then

$$\begin{aligned}\langle A^\dagger_{\rm sp}(t')A_{\rm sp}(t'')\rangle&\rightarrow\left[\frac{N}{2}\right]\left[\frac{2d}{c}\right]^2\frac{cg}{2\beta N}(P_2-P_1)^{-1}\\&\quad\times\langle F^\dagger_S(t')F_S(t'')\rangle\\&=\frac{1}{2}\left[\frac{2d}{c}\right]^2\frac{cg}{P_2-P_1}P_2\delta(t'-t'')\ .\end{aligned} \quad \text{(A10)}$$

The factor $N/2$ is introduced to obtain the contribution from N atoms, each of which is equally likely to spontaneously emit a photon to the right or left along the cavity axis. (In the text we require separately the spontaneous-emission contributions to left- and right-going fields.) Now $g(P_2-P_1)^{-1}$ is just the number density of atoms times the stimulated emission cross section, and consequently is time independent. Therefore we can replace $cg(P_2-P_1)^{-1}$ by its steady-state value, $cg_t(P_2-P_1)^{-1}\cong\gamma_c(P_2-P_1)^{-1}$. Thus

$$\langle A^\dagger_{\rm sp}(t')A_{\rm sp}(t'')\rangle=\frac{1}{2}\left[\frac{2d}{c}\right]^2\frac{\gamma_c P_2}{P_2-P_1}\delta(t'-t'')\ , \quad \text{(A11)}$$

which is the result (6) used in the text. Equation (7) follows similarly.

APPENDIX B: NOISE CORRELATION FUNCTION FOR THE FIELD

The positive-frequency and slowly varying part of the free-space, right-going vacuum electric field operator outside the laser cavity has the standard form

$$E_{R,\rm vac}(t)=i\sum_k\left[\frac{2\pi\hbar\omega_k}{V}\right]^{1/2}a_k(0)e^{-i(\omega_k-\omega_0)t}\ , \quad \text{(B1)}$$

where ω_0 is the atomic transition frequency, V is a quantization volume, and $a_k(0)$, is the source-free, photon annihilation operator for the mode k. The factor $e^{i\omega_0 t}$ makes $E_{R,\rm vac}(t)$ the part of the field that drives the slowly varying atomic dipole operator $S(t)$. The spatial dependence of the vacuum fields will be of no consequence in what follows and is therefore ignored here.

From (B1) we have the vacuum expectation value

$$\langle E_{R,\rm vac}(t')E^\dagger_{R,\rm vac}(t'')\rangle=\sum_k\left[\frac{2\pi\hbar\omega_k}{V}\right]\times e^{-i(\omega_k-\omega_0)(t'-t'')}\ , \quad \text{(B2)}$$

since $\langle a_k(0)a^\dagger_{k'}(0)\rangle=\delta_{kk'}$. In the one-dimensional mode continuum limit we take $V=A_0L$, where A_0 is some cross-sectional area and L is a length, and $\sum_k\rightarrow(L/2\pi)\int dk$. Then

$$\begin{aligned}\langle E_{R,\rm vac}(t')E^\dagger_{R,\rm vac}(t'')\rangle&\rightarrow\frac{L}{2\pi c}\left[\frac{2\pi\hbar}{A_0L}\right]\int_0^\infty d\omega\,\omega e^{-i(\omega-\omega_0)(t'-t'')}\\&\cong\frac{\hbar\omega_0}{A_0c}\int_{-\infty}^{\infty}d\omega\, e^{-i(\omega-\omega_0)(t'-t'')}\\&=\frac{2\pi\hbar\omega_0}{A_0c}\delta(t'-t'')=\frac{d}{c}\left[\frac{2\pi\hbar\omega_0}{A_0d}\right]\delta(t'-t'')\ .\end{aligned} \quad \text{(B3)}$$

In the approach of Sec. II we require only the effective slowly varying field annihilation operator $A_{R,\rm vac}(t)$, which may be defined by writing

$$E_{R,\rm vac}(t)=i\left[\frac{2\pi\hbar\omega_0}{A_0d}\right]^{1/2}A_{R,\rm vac}(t)\ . \quad \text{(B4)}$$

Comparison with (C8) indicates that

$$\langle A_{R,\rm vac}(t')A^\dagger_{R,\rm vac}(t'')\rangle=\frac{d}{c}\delta(t'-t'')\ , \quad \text{(B5)}$$

and of course $\langle A^\dagger_{R,\rm vac}(t')A_{R,\rm vac}(t'')\rangle=0$. Clearly $A_{L,\rm vac}(t)$ has the same correlation properties.

APPENDIX C: REDUCTION OF LINEWIDTH ABOVE THRESHOLD

It is well known that the linewidth (13) is reduced by a factor of $\frac{1}{2}$ in the nonlinear, above-threshold regime of laser oscillation [1]. We shall present a simplified argument showing how this correction arises. As noted in Sec. II, the field and atomic noise sources may be treated as classical noise sources with correlations chosen to correspond to those of their quantum counterparts. Since it greatly simplifies the analysis compared with the full quantum treatment, and avoids any consideration of operator orderings, we take this approach here.

We write the classical counterpart of (3) in the notationally simplified form

$$\begin{aligned}\dot{A}(t)&=-\gamma A(t)+\frac{c}{2d}[V(t)+A_{\mathrm{sp}}(t)]\\&\equiv-\gamma A(t)+F_N(t)\ ,\end{aligned}\tag{C1}$$

where $F_N(t)$ is now a classical noise source of zero mean and $\gamma=\frac{1}{2}(\gamma_c-c\bar{g})$ as in Sec. II. $A(t)$ now represents a complex, *classical* field amplitude for the intracavity laser field. In the steady state,

$$\langle A^*(t)A(t+\tau)\rangle_{\mathrm{av}}=\int_0^t dt'\int_0^{t+\tau}dt''\langle F_N^*(t')F_N(t'')\rangle_{\mathrm{av}}\times e^{\gamma(t'+t''-2t-\tau)}\ ,\tag{C2}$$

where $\langle\ \rangle_{\mathrm{av}}$ denotes a classical ensemble average. Choosing

$$\begin{aligned}\langle F_N^*(t')F_N(t'')\rangle_{\mathrm{av}}&\equiv\frac{1}{2}\frac{P_2}{P_2-P_1}\gamma_c\delta(t'-t'')\ ,\\\langle F_N(t')F_N(t'')\rangle_{\mathrm{av}}&\equiv 0\ ,\end{aligned}\tag{C3}$$

we obtain

$$\langle A^*(t)A(t+\tau)\rangle_{\mathrm{av}}=\frac{1}{2}\frac{P_2}{P_2-P_1}\left[\frac{\gamma_c}{2\gamma}\right]e^{-\gamma\tau}\ ,\tag{C4}$$

which leads as in Sec. II to the linewidth $\Delta\omega=\Delta\omega_{\mathrm{ST}}$.

Now in Eq. (C2)— and in the quantum-mechanical equations (8) and (14) of the text—we have written the steady-state correlation function as if γ were a constant, independent of the field. In reality, of course, saturation of the gain medium implies a dependence of γ on the field intensity. Let us write

$$A(t)=F(t)e^{-i\phi(t)}\ ,\tag{C5}$$

where $F(t)$ and $\phi(t)$ are the real amplitude and phase, respectively, so that (C1) becomes

$$\dot{F}=-\gamma(F)F+\tfrac{1}{2}(F_Ne^{i\phi}+F_N^*e^{-i\phi})\ ,\tag{C6}$$

$$\dot{\phi}=\frac{i}{2F}(F_Ne^{i\phi}-F_N^*e^{-i\phi})\ ,\tag{C7}$$

where we indicate explicitly a dependence of γ on F, corresponding to saturation.

In the absence of noise the steady-state solution of (C6) is determined by the solution of the equation $\gamma(F)=0$. If we assume that the noise has only a negligible effect on the steady-state amplitude F_{ss} determined in this way, then

$$\begin{aligned}\langle A^*(t)A(t+\tau)\rangle_{\mathrm{av}}&=F_{\mathrm{ss}}^2\langle e^{-i[\phi(t+\tau)-\phi(t)]}\rangle_{\mathrm{av}}\\&=F_{\mathrm{ss}}^2\langle e^{-(1/2)[\phi(t+\tau)-\phi(t)]^2}\rangle_{\mathrm{av}}\ .\end{aligned}\tag{C8}$$

From (C7) we have

$$\phi(t+\tau)-\phi(t)=\frac{i}{2F}\int_t^{t+\tau}dt'[F_N(t')e^{i\phi(t')}-F_N^*(t')e^{-i\phi(t')}]\tag{C9}$$

and

$$\begin{aligned}\langle[\phi(t+\tau)-\phi(t)]^2\rangle_{\mathrm{av}}=\frac{1}{2F^2}\mathrm{Re}\int_t^{t+\tau}dt'\int_t^{t+\tau}dt''[&\langle F_N(t')F_N^*(t'')e^{i[\phi(t')-\phi(t'')]}\rangle_{\mathrm{av}}\\&-\langle F_N(t')F_N^*(t'')e^{i[\phi(t')+\phi(t'')]}\rangle_{\mathrm{av}}]\ .\end{aligned}\tag{C10}$$

We assume, as is usual in the theory of the laser linewidth, that the fluctuations in $\phi(t)$ are slow on a time scale over which the noise $F_N(t)$ has nonvanishing correlations. Then we can effectively assume the properties (C3) for the averages in the integrand of (C10) to obtain

$$\begin{aligned}\langle[\phi(t+\tau)-\phi(t)]^2\rangle_{\mathrm{av}}&=\frac{1}{2F^2}\int_t^{t+\tau}dt'\int_t^{t+\tau}dt''\frac{1}{2}\frac{P_2}{P_2-P_1}\gamma_c\delta(t'-t'')\\&=\frac{\tau}{4F^2}\frac{\gamma_cP_2}{P_2-P_1}\ ,\end{aligned}\tag{C11}$$

which implies the linewidth $\tau^{-1}\langle[\phi(t+\tau)-\phi(t)]^2\rangle_{\rm av}$ given by

$$\Delta\omega=\frac{P_2}{P_2-P_1}\left[\frac{\hbar\omega}{2P_{\rm out}}\right]\gamma_c^2=\tfrac{1}{2}\Delta\omega_{\rm ST}\ . \qquad \text{(C12)}$$

Therefore *phase fluctuations alone* produce the above-threshold linewidth $\frac{1}{2}\Delta\omega_{\rm ST}$. The effect of amplitude fluctuations in the linearized theory is to add an additional $\frac{1}{2}\Delta\omega_{\rm ST}$ to the linewidth, as is clear from our classical analysis. As emphasized by Lax [1], the stabilization of the amplitude in the nonlinear theory eliminates such a contribution to the linewidth. It should be emphasized that the effect of this nonlinear stabilization on $\Delta\omega$ is the same in our theory as in the conventional one pertaining to the "Schawlow-Townes limit." Therefore our expressions for $\Delta\omega$ should be multiplied by the factor $\frac{1}{2}$ in order to apply in the above-threshold regime. The K factor remains the same.

The formula (C7) leading to (C11) ignores any coupling between phase and amplitude fluctuations, and as such does not allow for the enhancement of the linewidth associated with the α parameter [6]. This enhancement is planned to be the subject of a future paper.

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