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Quantization of second-order Lagrangians: Model problem

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Many aspects of a model problem, the Lagrangian of which contains a term depending quadratically on the acceleration, are examined in the regime where the classical solution consists of two independent normal modes. It is shown that the techniques of conversion to a problem of Lagrange, generalized mechanics, and Dirac's method for constrained systems all yield the same canonical form for the Hamiltonian. It is also seen that the resultant canonical equations of motion are equivalent to the Euler-Lagrange equations. In canonical form, all of the standard results apply, quantization follows in the usual way, and the interpretation of the results is straightforward. It is also demonstrated that perturbative methods fail, both classically and quantum mechanically, indicating the need for the nonperturbative techniques applied herein. Finally, it is noted that this result may have fundamental implications for certain relativistic theories.

I. INTRODUCTION

An extensive analysis, both classical and quantum mechanical, is presented for a particular model problem, introduced elsewhere [1] and described by a second-order Lagrangian, that is, one dependent upon accelerations in addition to positions and velocities. The purpose of this work is to examine the details of a completely solvable model problem in order to determine appropriate methods of treatment, in particular, the proper quantization, the correct interpretation of the results and the extraction of relevant information for such a problem. It is anticipated that the results of this analysis will also be valid for a class of second-order Lagrangians and be useful for certain physically interesting problems which cannot be solved exactly.

Although, in general, there are a number of reasons [2–4] for studying second-order Lagrangians, in our case, it is quite specific. It is the type of Lagrangian that arises from the conventional treatment of a particular relativistic action-at-a-distance (AAD) theory of particle dynamics, namely, the Fokker-Wheeler-Feynman (FWF) theory [5,6] of electrodynamics, its revisions and generalizations [7–12]. The theory is formulated in close analog to nonrelativistic particle dynamics, being written only in terms of the particles' positions and velocities as functions of the observer's time. However, in its exact form, it is multitimed, depending not only on the current time but also on all relative retarded times. At present, there is no known procedure to go from a multitimed Lagrangian to a Hamiltonian structure and, hence, to a quantum version. In order to circumvent this difficulty, the conventional approach [13–19] has been to make the $1/c$ power-series expansion of the interaction energy about the current time of the system. This results in an infinite-order Lagrangian, the $(1/c)^4$ term containing accelerations. Now, direct observation, calculations, and quantum electrodynamics have verified that the first two terms in this expansion contain considerable physical information. Since there is only a single interaction, the same should be true of the third term. All that is required are the techniques by which to extract this information. This work is intended as a first step in this direction.

The model problem is defined in the next section and represents the relative motion between two nonrelativistic particles whose interaction consists of the sum of a harmonic term plus a term quadratic in their relative accelerations. This acceleration dependence was chosen because it is the simplest that gives a meaningful problem, the resultant problem can be solved exactly, and the FWF theory yields a bilinear acceleration dependence. Now, in general, higher-order Lagrangians have been considered questionable [3] on the grounds that they may be nonlocal, that they may violate causality, and that their Hamiltonians may be unphysical in the sense that they do not properly represent the total energy of the system. Both the FWF theory and the model problem are local and satisfy causality, by their structure. We are expressly interested in such systems. The legitimate physical role that the Hamiltonian plays is seen from the analysis, at least to the extent that the model problem reflects the features of more complex problems. Of course, our model problem will have limitations and will lack characteristic aspects of more complex systems, such as a finite limit to a particle's speed and the distinction between the center-of-mass and center-of-momentum reference frames in the nonrelativistic and relativistic

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cases, respectively. Nevertheless, this investigation has been approached with the philosophy that it is essential for the first step to be a clear understanding of the specific second-order properties of certain Lagrangians, unobscured by other complexities. Such an understanding should provide a useful guide in the treatment of related problems. Finally, the Lagrangians of the FWF theory and our model problem are not parameter invariant, a feature generally assumed [2,3,4,20] for physical, i.e., relativistic, theories. Thus, although the formal structure is available for the non-parameter-invariant case, applications to physical systems appear to be nonexistent. This may be an unfounded omission.

Section II also contains the exact solution to the classical equation of motion plus an analysis of the results. The conversion to the canonical form is given in Sec. III. Here it is shown that the three methods, conversion to a problem of Lagrange [2], application of generalized mechanics [3], and Dirac's method [21] for constrained systems, yield the same canonical form. It is also shown in this section that canonical transformations can be used in the usual way and result in a precise interpretation of the Hamiltonian in this case. The resultant Hamiltonian can be quantized in the usual way and the quantum calculations are given in Sec. IV. In particular, the nonperturbative and perturbative calculations are shown to give different results. Further discussion is given in Sec. V, along with suggested implications of the results of this investigation.

II. DEFINITION AND LAGRANGIAN ANALYSIS OF THE MODEL PROBLEM

As noted in the Introduction, the Lagrangian considered in this work is taken to be [1]

$$L=\tfrac{1}{2}\mu\dot{x}^{\,2}-\tfrac{1}{2}kx^{2}-\tfrac{1}{2}\epsilon\ddot{x}^{\,2} \tag{2.1}$$

where the symbols have their usual meanings. The equations of motion can be obtained from a variational principle in the standard way [2]. First, form the fundamental integral

$$J=\int_{t_1}^{t_2}L\,dt\ . \tag{2.2}$$

Setting the variations of J with respect to changes in path equal to zero, subject to fixed end points, generalized in this case to variations in both x and $\dot{x}$ being zero at t_1 and t_2, yields

$$\mu\ddot{x}+\epsilon\dddot{x}+kx=0\ . \tag{2.3}$$

Equation (2.3) is a standard from and the solutions are found by letting

$$x=a\cos(\omega t+\delta)\ . \tag{2.4}$$

This yields

$$\epsilon\omega^4-\mu\omega^2+k=0 \tag{2.5}$$

with the solutions

$$\omega_1^2=\frac{1}{2\epsilon}[\mu-(\mu^2-4\epsilon k)^{1/2}],\quad \lim_{\epsilon\to 0}\omega_1^2=\frac{k}{\mu}=\omega_0^2\ , \tag{2.6}$$

$$\omega_2^2=\frac{1}{2\epsilon}[\mu+(\mu^2-4\epsilon k)^{1/2}],\quad \lim_{\epsilon\to 0}\omega_2^2=\frac{\mu}{\epsilon}=\omega_p^2\ . \tag{2.7}$$

The possible motions depend upon the value of ϵ. For $0<\epsilon<\mu^2/4k$ there are two independent harmonic modes. This is the range of ϵ that will be considered throughout the remainder of this work. Of course, one also has the cases $\epsilon>\mu^2/4k$ where only damped or runaway solutions occur and $\epsilon<0$ where one harmonic and one damped mode occur. Notice also that as ϵ tends to zero, although ω_1 goes to the normal limit, ω_2 diverges. For $0<\epsilon<\mu^2/4k$, the general solution is

$$x(t)=a_1\cos(\omega_1 t+\delta_1)+a_2\cos(\omega_2 t+\delta_2)\ . \tag{2.8}$$

The four arbitrary constants a_1, a_2, δ_1, and δ_2 are determined by four boundary conditions. If these are taken to be the time equal zero values of position, velocity, acceleration, and boost, that is, x_0, $\dot{x}_0$, $\ddot{x}_0$, and $\dddot{x}_0$, respectively, then one finds

$$a_1^2=\frac{1}{(\omega_2^2-\omega_1^2)^2}\left[(\omega_2^2x_0+\ddot{x}_0)^2+\frac{1}{\omega_1^2}(\omega_2^2\dot{x}_0+\dddot{x}_0)^2\right], \tag{2.9}$$

$$\tan\delta_1=-(\omega_2^2\dot{x}_0+\dddot{x}_0)/[\omega_1(\omega_2^2x_0+\ddot{x}_0)]\ , \tag{2.10}$$

$$a_2^2=\frac{1}{(\omega_2^2-\omega_1^2)^2}\left[(\omega_1^2x_0+\ddot{x}_0)^2+\frac{1}{\omega_2^2}(\omega_1^2\dot{x}_0+\dddot{x}_0)^2\right], \tag{2.11}$$

$$\tan\delta_2=-(\omega_1^2\dot{x}_0+\dddot{x}_0)/[\omega_2(\omega_1^2x_0+\ddot{x}_0)]\ . \tag{2.12}$$

The two modes are independent, because, by choice of boundary conditions, one can excite either one alone or any desired mixture. Notice also that the complete motion is given by the initial conditions and, hence, the equation of motion is exactly predictive. The only difference between this case and the usual ($\epsilon=0$) case is that two modes exist and more initial information is required.

A first integral, or constant, of the motion can be obtained in the usual way, that is, multiply Eq. (2.3) by $\dot{x}$ to obtain a total time derivative. The resultant constant is

$$h=\tfrac{1}{2}\mu\dot{x}^{\,2}+\tfrac{1}{2}kx^2+\epsilon\dot{x}\,\dddot{x}-\tfrac{1}{2}\epsilon\ddot{x}^{\,2}\ . \tag{2.13}$$

Substituting the solution, Eq. (2.8), into h yields

$$h=\tfrac{1}{2}\omega_1^2(\mu^2-4\epsilon k)^{1/2}a_1^2-\tfrac{1}{2}\omega_2^2(\mu^2-4\epsilon k)^{1/2}a_2^2\ . \tag{2.14}$$

Clearly h is a constant of the motion, as noted earlier. Also $h>0$ for $a_2=0$ but $h<0$ for $a_1=0$. Now, since we can excite the second mode alone, in this form it is not clear how one should interpret the two terms. Thus the problem needs to be examined in more detail.

Before continuing the analysis, we note that, for ϵ very small, one might be tempted to treat the terms proportional to ϵ perturbatively in Eq. (2.3). Although it turns out useful to know the precise consequences of such a calculation, it fails to give the complete solution. This failure can be understood as follows. In setting up the

perturbative equations, one makes a very explicit assumption about the mathematical character of the solution, namely, that it can be represented by its Taylor series in ϵ. x_1 satisfies this assumption and the method, not surprisingly, generates its Taylor series in ϵ exactly. x_2 does not satisfy this assumption, through ω_2, and, again not surprisingly, the method fails to indicate any trace of this mode. This establishes the necessity of having nonperturbative techniques for dealing with this type of problem. The next step is to determine a canonical formalism.

III. CANONICAL HAMILTONIAN FORMULATION

A. Conversion to a problem of Lagrange

The concept behind the conversion [2] to a problem of Lagrange is quite simple. One starts with a Lagrangian, e.g., Eq. (2.1), which one does not know how to handle, and converts it to an identical problem for which exact methods are known. In this instance, Eq. (2.1) is converted to a more general problem by setting $\dot{x}=y$ and $\ddot{x}=\dot{y}$. This more general problem is now made identical to the original problem by imposing the constraint

$$G=y-\dot{x}=0 \ . \tag{3.1}$$

A Lagrangian in the first-order form plus a constraint is known as a problem of Lagrange and it is known how treat such problems exactly [2,22]. The Lagrangian is replaced by

$$\Lambda=\tfrac{1}{2}\mu y^2-\tfrac{1}{2}kx^2-\tfrac{1}{2}\epsilon\dot{y}^2-\lambda_L(y-\dot{x}) \ . \tag{3.2}$$

Now, because of the inclusion of the constraint with the Lagrange multiplier, λ_L, x, and y play the role of independent variables and all the rules of first-order Lagrangians apply.

Of course, for the conversion to be useful, the Legendre transformation relating the velocities to the generalized momenta must be one to one. This requires [2] the determinant

$$D=\begin{vmatrix} \dfrac{\partial^2\Lambda}{\partial\dot{x}_i\partial\dot{x}_j} & \dfrac{\partial G}{\partial\dot{x}_i} \\ \dfrac{\partial G}{\partial\dot{x}_j} & 0 \end{vmatrix}\neq 0 \ . \tag{3.3}$$

Here, we use $x_1=x$ and $x_2=y$. This condition is satisfied for the model problem. Note that we have relaxed the condition on D used by Rund [2]. He assumes that Λ has a minimal value, whereas we cannot guarantee that *a priori* in our case. Thus

$$p_x=\frac{\partial\Lambda}{\partial\dot{x}}=\lambda_L \ , \tag{3.4}$$

$$p_y=\frac{\partial\Lambda}{\partial\dot{y}}=-\epsilon\dot{y} \ , \tag{3.5}$$

and

$$H=p_xy-\frac{1}{2\epsilon}p_y^2-\tfrac{1}{2}\mu y^2+\tfrac{1}{2}kx^2 \ . \tag{3.6}$$

H is now in canonical form, with the pairs (x,p_x) and (y,p_y) being the canonical variables satisfying the basic Poisson bracket relations. One can now apply canonical transformations and quantize in the usual way.

It should be noted that nothing is lost from the original problem. Hamilton's canonical equations are

$$\dot{x}=y \ , \tag{3.7}$$

$$\dot{y}=-\frac{1}{\epsilon}p_y \ , \tag{3.8}$$

$$\dot{p}_x=-kx \ , \tag{3.9}$$

$$\dot{p}_y=-p_x+\mu y \ . \tag{3.10}$$

Equation (3.7) is the constraint, Eq. (3.8) is the definition of p_y, and, when combined, these equations give the correct equation of motion, Eq. (2.3). Further, these relations can be used to write Eq. (2.13) in terms of the canonical variables. One finds h is identical to H. Thus understanding the role that H plays and its physical interpretation is necessary to complete the analysis of the model problem and is considered in Sec. III D.

B. Application of generalized mechanics

The formal structure of generalized mechanics can be discussed either from the point of view of the calculus of variations [2] or of jet theory [3]. Both approaches are quite formal and lengthy. In principle, the treatment requires an appropriate identification of a set of generalized canonical variables, the definition of their generalized momenta, and the specification of a theoretical structure that is identical to the canonical form of first-order problems. The steps are outlined below. Again, the starting point is Eq. (2.1). The fundamental momenta are the so-called Jacobi-Ostrogradsky momentum coordinates defined for our problem by

$$p_x=\frac{\partial L}{\partial\dot{x}}-\frac{d}{dt}\frac{\partial L}{\partial\ddot{x}}=\mu\dot{x}+\epsilon\dddot{x} \ , \tag{3.11}$$

$$p_y=\frac{\partial L}{\partial\ddot{x}}=-\epsilon\ddot{x} \tag{3.12}$$

where the notation has been chosen to match that of the preceding section. The equation of motion is

$$\dot{p}_x=\frac{\partial L}{\partial x} \ . \tag{3.13}$$

Note that Eq. (3.11) is equivalent to Eq. (3.10), Eq. (3.12) to Eq. (3.5), or Eq. (3.8) and Eq. (3.13) to Eq. (2.3). Thus there is a complete duality between the two methods. Notice also, that for p_y to be nontrivial, L must be at least quadratic in the acceleration. This causes p_x to contain $\dddot{x}$, which, in turn, in order to obtain a meaningful Legendre transformation between momenta and velocities, necessitates the definition of a canonical coordinate for p_y. The form of Eq. (3.12) implies $y=\dot{x}$ for p_y. Further, Eq. (3.13) implies x for p_x. Formally, for the methods of generalized mechanics to apply, one requires (see Grässer [20])

$$\det\begin{bmatrix} 0 & 0 & \delta_{hl} \\ 0 & \dfrac{\partial^2 L}{\partial \ddot{x}_i \ddot{x}_j} & 0 \\ \delta_{ks} & 0 & 0 \end{bmatrix} \neq 0 \ . \tag{3.14}$$

This is equivalent to Eq. (3.3). Although Grässer's work is intended for parameter-invariant problems, his derivation of this result is quite general.

Now, when one defines the Jacobi-Ostrogradsky variables (x,y,p_x,p_y) as the canonical variables, the fundamental Poisson brackets in the usual way, and the Hamiltonian by

$$H = p_x\dot{x} + p_y\dot{y} - L \ , \tag{3.15}$$

one arrives exactly at Eq. (3.6) in the preceding section. The remainder of the discussion is then identical.

C. Application of Dirac's method for constrained systems

Dirac's method [21] for constrained systems can also be applied to this problem. Throughout, we closely follow the definitions and procedures outlined by Even [23], simply converting his results for field theories or Lagrangian densities to the case of point mechanics. Since one requires a constrained system, the starting point is Eq. (3.2) and is defined by the variables x, y, and λ_L and the velocities $\dot{x}$, $\dot{y}$, and $\dot{\lambda}_L$. Since its Hessian

$$\mathrm{Det}\left[\frac{\partial^2 \Lambda}{\partial \dot{x}_i \partial \dot{x}_j}\right] = 0 \ , \tag{3.16}$$

Λ is said to be degenerate and Dirac's method can be applied.

In this case, the canonical momenta are defined by

$$\begin{aligned} p_x &= \frac{\partial \Lambda}{\partial \dot{x}} = \lambda_L \ , \\ p_y &= \frac{\partial \Lambda}{\partial \dot{y}} = -\epsilon \dot{y} \ , \\ p_\lambda &= \frac{\partial V}{\partial \dot{\lambda}_L} = 0 \ . \end{aligned} \tag{3.17}$$

The constraints are defined by

$$\begin{aligned} \phi_1 &= p_x - \frac{\partial \Lambda}{\partial \dot{x}} = p_x - \lambda_L \ , \\ \phi_2 &= p_\lambda - \frac{\partial \Lambda}{\partial \dot{\lambda}_L} = p_\lambda \ . \end{aligned} \tag{3.18}$$

For the interim, ϕ_1 and ϕ_2 are not set to zero. Now p_y does not give a constraint because a constraint must be written in terms of the canonical variables and when one eliminates the $\dot{y}$ from $p_y - \partial\Lambda/\partial\dot{y}$, zero is obtained.

The Hamiltonian is, as usual,

$$H = p_x\dot{x} + p_y\dot{y} + p_\lambda\dot{\lambda}_L - \Lambda + X_1\phi_1 + X_2\phi_2 \ . \tag{3.19}$$

This can be written as

$$H = -\frac{1}{2\epsilon}p_y^2 - \tfrac{1}{2}\mu y^2 + \tfrac{1}{2}kx^2 + \lambda_L y + X_1'\phi_1 + X_2'\phi_2 \tag{3.20}$$

where $X_1' = X_1 + \dot{x}$ and $X_2' = X_2 + \dot{\lambda}_L$. Here X_1 and X_2 are undetermined Lagrange multipliers. Now, since the Poisson bracket relation $\{\phi_1,\phi_2\} = -1$, ϕ_1 and ϕ_2 are second class constraints. Thus one sets

$$\begin{aligned} \dot{\phi}_1 &= \{\phi_1, H\} = -X_2' - kx = 0 \ , \\ \dot{\phi}_2 &= \{\phi_2, H\} = -y + X_1' = 0 \end{aligned} \tag{3.21}$$

in order to obtain constraints on X_1' and X_2'.

Finally, evaluate all Poisson brackets between the variables. The only nonzero ones turn out to be

$$\{x,p_x\} = 1 = \{y,p_y\} \ . \tag{3.22}$$

Hence there are only two pairs of canonical variables. Thus one can now set

$$\phi_1 = 0 = \phi_2 \text{ and } p_x = \lambda_L \tag{3.23}$$

in H to retrieve exactly Eq. (3.6), in agreement with our previous procedure. The remaining arguments are again identical.

D. Canonical transformations

Since Eq. (3.6) has no semblance of the $\epsilon = 0$ Hamiltonian nor of two independent normal modes, we proceed by a series of canonical transformations to put it in a more recognizable and tractable form. Although this problem can be diagonalized by a single canonical transformation, a series of transformations is used because this is more illustrative, some of the intermediate results will be examined further and, finally, one wants to learn how to deal with more complicated cases.

First, since y is closely related to the $\epsilon = 0$ momentum, we want y to play the role of a momentum. This can be simply achieved by interchanging the roles of y and p_y. Now the new momenta are mixed linearly, and also the new coordinates, in such a way so as to restore the $\epsilon = 0$ Hamiltonian. Finally, the roles of those new variables that now primarily represent y and p_y are restored. The net canonical transformation achieving this result can be written as

$$\begin{aligned} p_x &= \alpha_1 p_1 + \mu\omega_0\beta_1 x_2, \quad x = \alpha_1' x_1 + \frac{\beta_1'}{\mu\omega_0}p_2 \ , \\ y &= \frac{\alpha_2}{\mu}p_1 + \omega_0\beta_2 x_2, \quad p_y = \mu\alpha_2' x_1 + \frac{1}{\omega_0}\beta_2' p_2 \ . \end{aligned} \tag{3.24}$$

The constants have been chosen simply to keep the α's and β's dimensionless and for the final coordinates and momenta to have the usual dimensions. This yields H in the form

$$\begin{aligned} H = {} & \frac{1}{2\mu}C_1 p_1^2 + \tfrac{1}{2}kC_2 x_1^2 - \frac{1}{2\mu}C_3 p_2^2 - \tfrac{1}{2}kC_4 x_2^2 \\ & + \omega_0 C_5 p_1 x_2 + \omega_0 C_6 x_1 p_2 \ . \end{aligned} \tag{3.25}$$

The for nontrivial Poisson bracket relations for Eq. (3.24)

and the four conditions

$$\alpha_1=1,\quad C_1=2\alpha_1\alpha_2-\alpha_2^2=1\ ,$$
$$C_2=\alpha_1'^2-\frac{1}{\lambda^2}\alpha_2'^2=1,\quad C_3=\frac{1}{\lambda^2}\beta_2'^2-\beta_1'^2=1 \tag{3.26}$$

where

$$\lambda^2=k\epsilon/\mu^2 \tag{3.27}$$

are sufficient to define the transformation uniquely. With this choice [see also Eqs. (2.6) and (2.7)],

$$C_4=\beta_2^2-2\beta_1\beta_2=\frac{(1+\lambda^2)(1-3\lambda^2)}{\lambda^2(1-\lambda^2)}\ ,$$
$$C_5=\alpha_1\beta_2+\beta_1\alpha_2-\alpha_2\beta_2=-\frac{2\lambda}{(1-\lambda^2)^{1/2}}\ , \tag{3.28}$$
$$C_6=\alpha_1'\beta_1'-\frac{1}{\lambda^2}\alpha_2'\beta_2'=\frac{\lambda}{(1-\lambda^2)^{1/2}}\ ,$$

and

$$H=\frac{1}{2\mu}p_1^2+\tfrac{1}{2}kx_1^2-\frac{1}{2\mu}p_2^2-\frac{\mu\omega_p^2(1+\lambda^2)(1-3\lambda^2)}{2(1-\lambda^2)}x_2^2$$
$$-2\omega_0\lambda/(1-\lambda^2)^{1/2}p_1x_2+\omega_0\lambda/(1-\lambda^2)^{1/2}x_1p_2\ . \tag{3.29}$$

H is now in a tractable form. Notice that the first term in Eq. (3.29) comes primarily from the first and third terms of Eq. (3.6) and represents most of the real kinetic energy of the system, while the third term of Eq. (3.29) comes primarily from the second term of Eq. (3.6) and is actually an interaction term, required to make H a constant of the motion and with a particular structure. In the limit $\epsilon\to 0$, i.e., $\lambda\to 0$, the mixed terms drop out and H becomes the difference between the zero-order Hamiltonians of the two modes in this limit.

In this particular model problem a fourth canonical transformation can be made that diagonalizes the Hamiltonian. It is sufficient to set

$$x_1=\alpha_1\xi_1+\beta_1\pi_2,\quad p_1=\alpha_1'\pi_1+\beta_1'\xi_2\ ,$$
$$x_2=\alpha_2\xi_2+\beta_2\pi_1,\quad p_2=\alpha_2'\pi_2+\beta_2'\xi_1\ . \tag{3.30}$$

In this case the four nontrivial Poisson bracket relations, the two coefficients of the $\pi_{1,2}^2$ terms being set to $1/2\mu$ and $-1/2\mu$, respectively, and the two coefficients of the mixed terms being set equal to zero define the canonical transformation uniquely. The coefficients of the ξ_1^2 and ξ_2^2 terms are fixed. One finds

$$H=\frac{1}{2\mu}\pi_1^2+\tfrac{1}{2}\mu\omega_1^2\xi_1^2-\frac{1}{2\mu}\pi_2^2-\tfrac{1}{2}\mu\omega_2^2\xi_2^2 \tag{3.31}$$

where ω_1^2 and ω_2^2 are given by Eqs. (2.6) and (2.7). This result is exact and the entire formalism is consistent.

The two equations of motion give exactly the two solutions,

$$\xi_i=\xi_{i0}\sin(\omega_i t+\delta_i')\quad (i=1,2)\ . \tag{3.32}$$

Tracing back through the canonical transformations, $x(t)$ is obtained in the form given by Eq. (2.5) with

$$a_1=\frac{\xi_{10}}{(1-4\lambda^2)^{1/4}},\quad a_2=-\frac{\xi_{20}}{(1-4\lambda^2)^{1/4}}\ . \tag{3.33}$$

Further, Eq. (2.13) reduces to

$$H=\tfrac{1}{2}\mu\omega_1^2\xi_{10}^2-\tfrac{1}{2}\mu\omega_2^2\xi_{20}^2\ . \tag{3.34}$$

Apart from the minus sign, this is exactly what one would expect for two fundamental normal modes.

Nevertheless, the analysis and interpretation for this model problem are clear. An important point is that the Hamiltonian, Eq. (3.6), differs from those for the usual first-order Lagrangian case, being the difference between the energies contained in the two normal modes. This character is preserved by the above procedures. In general, this particular feature might be considered unacceptable if not properly understood and we come back to this point in the discussion.

IV. QUANTIZATION OF THE MODEL PROBLEM

A. Exact problem

Once Eq. (3.31) has been derived the quantization is straightforward and follows the usual rules of first-order problems. Thus one will set

$$\xi_1=\left[\frac{\hbar}{2\mu\omega_1}\right]^{1/2}(\hat{A}_-+\hat{A}_+)\ ,$$
$$\xi_2=\left[\frac{\hbar}{2\mu\omega_2}\right]^{1/2}(\hat{B}_-+\hat{B}_+)\ ,$$
$$\pi_1=-i\mu\omega_1\left[\frac{\hbar}{2\mu\omega_1}\right]^{1/2}(\hat{A}_--\hat{A}_+)\ , \tag{4.1}$$
$$\pi_2=-i\mu\omega_2\left[\frac{\hbar}{2\mu\omega_2}\right]^{1/2}(\hat{B}_--\hat{B}_+)$$

where the $\hat{A}_\pm$ and $\hat{B}_\pm$ are the usual raising and lowering operators. Therefore

$$\hat{H}=\tfrac{1}{2}\hbar\omega_1(\hat{A}_-\hat{A}_++\hat{A}_+\hat{A}_-)-\tfrac{1}{2}\hbar\omega_2(\hat{B}_-\hat{B}_++\hat{B}_+\hat{B}_-) \tag{4.2}$$

and the eigenvalue spectrum of $\hat{H}$ is

$$E_H(n,N)=\tfrac{1}{2}\hbar\omega_1(2n+1)-\tfrac{1}{2}\hbar\omega_2(2N+1)=E_1-E_2\ , \tag{4.3}$$

say. The eigenfunctions are simply the pairwise products of the individual eigenfunctions. Thus the exact solution is straightforward. Of course, when one measures the energy of the system in an experiment one must see the sum of the two energy spectra, i.e., at least for the excitations. This is implied by the exact solution, Eq. (2.8), of the equation of motion, being the sum of the two normal modes. As noted before, this point is examined further in the discussion.

B. Intermediate case

This case is considered because in most realistic problems exact solutions are not possible and one must resort to some approximation procedure. Typically, in similar problems, one starts with a small zero-order Hamiltonian plus a nonperturbative interaction. Customarily, a canonical transformation is applied that redefines the problem so that it separates into a large zero-order part plus a small perturbative interaction. This step is exact and the perturbative term is only a remnant of the total, thus allowing an acceptable solution using standard techniques. For our problem, there is no obvious separation at the outset. However, after the first canonical transformation the Hamiltonian as given by Eq. (3.29) does separate. We are now in a position to determine if, at this point, the mixed terms can be reasonably treated by perturbative techniques. This information could be most useful for more complicated problems.

Therefore, start with Eq. (3.29), proceed as in Sec. IV A, and write

$$\begin{aligned}\hat{H}=&\tfrac{1}{2}\hbar\omega_0(\hat{a}_-\hat{a}_+ +\hat{a}_+\hat{a}_-)-\tfrac{1}{2}\hbar\Omega_0(\hat{b}_-\hat{b}_+ +\hat{b}_+\hat{b}_-)\\&+2i\delta\hbar\omega_0(\hat{a}_- -\hat{a}_+)(\hat{b}_- +\hat{b}_+)\\&-i\delta\hbar\Omega_0(\hat{a}_- +\hat{a}_+)(\hat{b}_- -\hat{b}_+)\ .\end{aligned}\tag{4.4}$$

The $\hat{a}_\pm$ and the $\hat{b}_\pm$ refer to the (x_1,p_1) and (x_2,p_2), respectively, and [see Eq. (3.29)]

$$\omega_0^2=k/\mu\ ,\quad \Omega_0^2=\omega_p^2\frac{(1+\lambda^2)(1-3\lambda^2)}{(1-\lambda^2)}\ ,\tag{4.5}$$

$$\delta=\frac{\lambda}{2(1-\lambda^2)^{1/2}}\left[\frac{\omega_0}{\Omega_0}\right]^{1/2},\tag{4.6}$$

$$\frac{\omega_0^2}{\Omega_0^2}=\frac{\lambda^2(1-\lambda^2)}{(1+\lambda^2)(1-3\lambda^2)}\ .\tag{4.7}$$

The first two terms of Eq. (4.4) will be taken to be the zero-order part and the last two terms will be treated by perturbation theory, because of the presence of the δ, which is small.

The zero-order eigenvalue is

$$E_H^0=\tfrac{1}{2}\hbar\omega_0(2n+1)-\tfrac{1}{2}\hbar\Omega_0(2N+1)\tag{4.8}$$

and the power-series expansion in terms of λ agrees with Eq. (4.3) to order λ^0 for the first term and to order λ^2 for the second term.

It is easy to see that all odd-order eigenvalue contributions are zero and hence the next contribution is

$$E_H^2=\langle\Phi_0|\hat{H}_1|\Phi_1\rangle\ .\tag{4.9}$$

It is straightforward to find Φ_1 from the first-order equation, namely,

$$\hat{H}_0\Phi_1+\hat{H}_1\Phi_0=E_0\Phi_1+E_1\Phi_0\ .\tag{4.10}$$

Setting

$$\Phi_1=\sum_{m,M}\alpha_{m,M}\Phi_0(m,M)\tag{4.11}$$

one finds the nonzero coefficients to be

$$\begin{aligned}\alpha_{n-1,N-1}&=i\delta\left[\frac{\Omega_0-2\omega_0}{\Omega_0-\omega_0}\right]\sqrt{n}\sqrt{N}\ ,\\ \alpha_{n-1,N-1}&=i\delta\left[\frac{\Omega_0+2\omega_0}{\Omega_0+\omega_0}\right]\sqrt{n}\sqrt{N+1}\ ,\\ \alpha_{n+1,N-1}&=i\delta\left[\frac{\Omega_0+2\omega_0}{\Omega_0+\omega_0}\right]\sqrt{n+1}\sqrt{N}\ ,\\ \alpha_{n+1,N+1}&=i\delta\left[\frac{\Omega_0-2\omega_0}{\Omega_0-\omega_0}\right]\sqrt{n+1}\sqrt{N+1}\ .\end{aligned}\tag{4.12}$$

Now, Eq. (4.9) yields

$$\begin{aligned}E_H^2=&\hbar\omega_0\frac{\lambda^2(2n+1)}{4(1-\lambda^2)(1-\omega_0^2/\Omega_0^2)}\\&-\hbar\Omega_0\frac{\lambda^2(\omega_0^2/\Omega_0^2)(3-4\omega_0^2/\Omega_0^2)(2N+1)}{4(1-\lambda^2)(1-\omega_0^2/\Omega_0^2)}\ .\end{aligned}\tag{4.13}$$

The power series in λ results in the first term of the exact eigenvalue, Eq. (4.3), being correct to order λ^2 and the second term being correct to order λ^4.

In order to get the first term correct to order λ^4 it is necessary to calculate the fourth-order contribution, that is, E_H^4, which can be written [24,25] as

$$E_H^4=\langle\Phi_2|E_H^0-\hat{H}_0|\Phi_2\rangle-E_H^2\langle\Phi_1|\Phi_1\rangle\ .\tag{4.14}$$

In this form one requires only Φ_2 in addition to the previous results. The algebra is straightforward but lengthy. It does give the correct result. Thus the exact eigenvalue spectrum of H can be determined using standard techniques when H has not been exactly diagonalized, provided a satisfactory reduction is implemented.

C. Perturbative treatment of the ϵ-dependent term

In the application of relativistic AAD theories, as indicated in the Introduction, and in quantum electrodynamics, at some point an expansion is made as a power series in $1/c$, or, equivalently, the fine structure constant. It is always assumed that these series are convergent and, moreover, that each term in the expansion can be treated perturbatively. Although a formal perturbative structure to treat acceleration-dependent terms in a Lagrangian has effectively existed [15,26] for some time, to date, the extent to which such a calculation is or is not valid is completely unknown, simply because exact methods have been nonexistent. The present analysis and model problem can be used to advantage, finally, to make a comparison between a perturbative and a nonperturbative quantum-mechanical solution to this type of problem.

In quantum form the procedure is to set

$$L_0=\tfrac{1}{2}\mu\dot{x}^{\,2}-\tfrac{1}{2}kx^2\tag{4.15}$$

and

$$p_0=\frac{\partial L_0}{\partial\dot{x}}=\mu\dot{x}\ .\tag{4.16}$$

Thus the Hamiltonian, Eq. (3.7), becomes

$$H=\frac{1}{2\mu}p_0^2+\tfrac{1}{2}kx^2-\frac{\epsilon}{2\mu^2}\dot{p}_0^2-\frac{\epsilon}{\mu^2}p_0\ddot{p}_0 \ . \tag{4.17}$$

At this point additional variables are not introduced. Rather, one sets

$$\begin{aligned}\dot{p}_0&=\frac{i}{\hbar}[H,p_0] \ ,\\ \ddot{p}_0&=\frac{i}{\hbar}\left[H,\frac{i}{\hbar}[H,p_0]\right] \ .\end{aligned} \tag{4.18}$$

Notice that it is essential to use the total H in order to generate the proper series, which will turn out to be in terms of λ, Eq. (3.27). One obtains

$$\begin{aligned}\dot{p}_0&=-kx+\lambda^2kx+\cdots \ ,\\ \ddot{p}_0&=-\frac{k}{\mu}p_0-\lambda^2\frac{k}{\mu}p_0+\cdots \ .\end{aligned} \tag{4.19}$$

Thus

$$\begin{aligned}H=&\frac{1}{2\mu}(1+2\lambda^2+2\lambda^4\cdots)p_0^2\\ &+\tfrac{1}{2}k(1-\lambda^2+2\lambda^4+\cdots)x^2 \ .\end{aligned} \tag{4.20}$$

It is sufficient to note that the effective frequency is given by

$$\begin{aligned}\omega^2&=\omega_0^2(1+2\lambda^2+2\lambda^4+\cdots)(1-\lambda^2+2\lambda^4+\cdots)\\ &=\omega_0^2(1+\lambda^2+2\lambda^4+\cdots) \ .\end{aligned} \tag{4.21}$$

This is exactly the expansion for ω_1^2. Thus the perturbative quantum solution fails in precisely the same way as the classical case. This again demonstrates the need for nonperturbative techniques.

This particular calculation also illustrates that the introduction of any deviations from the exact canonical form of a Hamiltonian, such as those arising from the approximations leading to Eq. (4.20), can result in the loss of information. Nevertheless, such a calculation may yield essential information, as discussed in the next section.

V. DISCUSSION

In this work, many aspects of a model problem [1] with a second-order Lagrangian have been examined, both classically and quantum mechanically. In the classical case, the solutions of the Euler-Lagrange equations of motion and of Hamilton's canonical equations, obtained after converting to a canonical form, are identical. This conversion can be effected by either conversion to a problem of Lagrange [2], application of the methods of generalized mechanics [3], or the use of Dirac's method [21] for constrained systems. At this point, all of the usual methods of Hamiltonian mechanics apply. The generalized Hamiltonian can be studied by canonical transformations and can be put into diagonal form. In this case, it appears as the difference between two energy operators and its interpretation is clear. We elaborate on this point below. Once the classical problem has been reduced to standard first-order form, quantization is straightforward and the usual methods apply. The quantum results are consistent with the classical results.

The Hamiltonian arising in AAD theories for physical systems is normally identified as the total energy of the system and, hence, one imposes the requirement that its eigenvalue spectrum be lower bounded. This is in apparent direct conflict with the present case, e.g., Eqs. (3.34) and (4.3), and this situation needs clarification. There are a number of points that need to be recognized. First, the above requirement is entirely a Newtonian concept, arising from the structure of the theory and the fundamental assumption that nonrelativistic physical Lagrangians must be of first order. There would be no dispute or alternative if a proof existed for this assumption, however, in fact, just the opposite occurs. The no-interaction theorems [27–32] prove that the Lagrangians cannot be of first order for relativistic AAD theories. As indicated in the Introduction, one possibility is higher-order Lagrangians. Thus it is clearly prudent to be unprejudiced and objective on this point. Second, it is an elementary result of both classical mechanics [22,33] and quantum mechanics [34] that, like the Lagrangian, the Hamiltonian of a system is not unique. Consequently, the Hamiltonian need not represent the total energy, even if lower bounded, and need not be lower bounded. This in no way invalidates the formalism. Third, only when very specific *a priori* information is available [22], such as is the case for conservative nonrelativistic systems, can a particular Hamiltonian be identified as the total energy at the outset. When such information is not available, as is the case here, a different procedure must be found. Fourth, the formalism does not provide this information. It only guarantees that the Hamiltonian has the same units as the Lagrangian, energy in the present case, and that it is a constant of the motion. Fifth, our analysis demonstrates that, in the present case, the precise interpretation only comes after the complete solution is known and is unambiguous. The structure of Eqs. (3.31), (3.34), and (4.3) demonstrates that the Hamiltonian associated with the present second-order Lagrangian, although it does not represent the total energy of the system, does contain all of the information required to identify the total energy of the system. In this case, the eigenvalues of the Hamiltonian can only be interpreted as the difference in the actual energies contained in each mode. The total energy of the system must be the sum of these two energies and, hence, is properly lower bounded. It follows that no fundamental physical principles are violated by this example. Sixth, in principle, the Hamiltonian yields only the energy difference and, hence, in those problems where the identification might not be clear, one would require some additional information in order to make the correct interpretation. Our analysis indicates that, for those problems with a well-defined solution for ϵ going to zero, the perturbative solution will suffice for that purpose. Finally, it is clear the Newtonian concepts and results cannot be applied directly to high-order Lagrangians. In fact, they must be treated distinctively.

It is also interesting to make a few comments on the quantum results. In this formalism the phase space of the system is enlarged to consist of the position coordinate with its associated generalized momentum plus a velocity coordinate, or possibly a coordinate for the usual momentum, with its associated generalized momentum. The lower energy represents a resonance in the usual coordinate space, while the higher energy represents a resonance in this velocity space and is due entirely to the acceleration-dependent terms. If such terms arise in physical problems from high-order relativistic effects, they will be purely relativistic in nature, have no nonrelativistic limit, and will require distinctive treatment. Furthermore, we are, finally able to compare nonperturbative and perturbative quantum results. It is seen that the latter method fails in a way similar to that in the classical case, losing all trace of the higher mode. Now, although we may have lost some relativistic features by treating the kinetic energy nonrelativistically, one would have to be very optimistic to believe that this difference would be enough to make a perturbative treatment miraculously correct. One should expect it to be similarly inadequate. Thus these results raise some very fundamental physical questions, namely, are such higher resonances real, how are they manifested, how are they realized, and how can they be observed? This work lays the basis for further investigations into these questions.

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