Soliton-collision problem in the nonlinear Schrödinger equation with a nonlinear damping term

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An exchange in the wave action (number of quanta) is analyzed by means of the perturbation theory for a soliton-soliton collision in the nonlinear Schrödinger equation with a nonlinear damping term that conserves the total wave action (it may account for the intrapulse Raman scattering in the model of a lossless optical fiber, or for the nonlinear Landau damping in a plasma). It is demonstrated that, if the colliding solitons have a sufficiently large relative velocity, the analytical results are in very good accordance with recently published numerical simulations [S. Chi and S. Wen, Opt. Lett. **14**, 1216 (1989)]. This warrants application of the present variant of the perturbative technique to other physical problems.

The present work is devoted to collisional dynamics of solitons governed by the nonlinear Schrödinger (NS) equation

$$iu_t+u_{xx}+2|u|^2u=\epsilon uR[u] , \tag{1}$$

with the nonlinear damping term on its right-hand side that conserves the total wave action (also called the number of quanta)

$$N=\int_{-\infty}^{+\infty}|u(x)|^2dx . \tag{2}$$

In physical applications, two terms of this kind are well known: the one with

$$R=(|u|^2)_x , \tag{3}$$

which furnishes the simplest description of the intrapulse Raman scattering[1,2] in the model of an optical fiber based on Eq. (1), and

$$R=\pi^{-1}\mathrm{P}\int_{-\infty}^{+\infty}(x-x')^{-1}dx|u(x')|^2 \tag{4}$$

($\mathrm{P}\int$ stands for the principal value of the integral), which takes account of the nonlinear Landau damping[3] of the Langmuir plasma waves described by Eq. (1).[4] In both cases, the coefficient ϵ in Eq. (1) is real. In the model of the optical fiber, the quantity N has the physical sense of energy, while in the plasma-physics problem it gives the total number of plasmons.

It is well known that a soliton, which has the form

$$u_{\text{sol}}(x,t)=2i\eta\,\text{sech}[2\eta(x-Vt)]\times\exp[(i/2)Vx+i(4\eta^2-V^2/4)t] \tag{5}$$

in the absence of perturbations ($\epsilon=0$), undergoes acceleration induced by nonlinear damping. In the lowest approximation, the perturbation-induced equations of motion for the soliton's amplitude and velocity V are

$$\dot{\eta}=0,\quad \dot{V}=C\epsilon\eta^n , \tag{6}$$

where $C=\frac{128}{15}$ and $n=4$ in the case of Eq. (3),[1,2] and $C\approx7.443$ and $n=3$ in the case of Eq. (4).[3] The acceleration also generates emission of radiation (the radiative decay of the soliton), but this effect is very weak.[5]

As the soliton's acceleration depends on its amplitude according to Eq. (6), the next natural step is to consider a collision between solitons with different amplitudes.[6,7] In Ref. 6, the collision problem was attacked numerically for the optical-fiber model with the perturbing term (3). The main finding of Ref. 6 is a collision-induced change of amplitudes of both solitons. This change may be interpreted as a transfer of energy from the slower soliton to the faster one, the sum of the amplitudes being conserved [for the soliton solution given by Eq. (5), the number of quanta bound in the soliton is $N_{\text{sol}}=4\eta$]. Independently, radiative losses in the soliton-soliton collision were studied in Ref. 7 by means of the perturbation theory for the case when the relative velocity of the solitons was much larger than their amplitudes. It has been demonstrated that the collision between the solitons with the amplitudes η_1 and η_2 and the relative velocity V ($V^2\gg\eta_1^2,\eta_2^2$) gives rise to the radiative losses which can be expressed as the changes $\delta^{(2)}\eta_n$ ($n=1,2$) of the amplitudes of the colliding solitons:

$$\delta^{(2)}\eta_{1,2}=-4\epsilon^2\eta_{1,2}\eta_{2,1}^2 \tag{7}$$

in the case of Eq. (3), and

$$\delta^{(2)}\eta_{1,2}=-16V^{-2}\epsilon^2\eta_{1,2}\eta_{2,1}^2 \tag{8}$$

in the case of Eq. (4). The superscript (2) in Eqs. (7) and (8) implies that these expressions have been obtained in second order in the perturbation parameter ϵ, which is typical for the radiative processes.

However, the change of the solitons' amplitudes generated by the adiabatic (nonradiative) exchange must be an effect of first order in ϵ,[8–10] i.e., more important. The main objective of the present Brief Report is to demonstrate that the fundamental numerical results of Ref. 6 can also be obtained analytically (and in a more general form) by means of the perturbation theory. Similarly, the same will be done for Eq. (4).

If the relative velocity V of the colliding solitons is large as compared to their amplitudes, the full wave field during the collision may be represented, in the lowest approximation, by the linear superposition of the unper-

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turbed solitons defined by Eq. (5):

$$u(x,t)=(u_{\rm sol})_1+(u_{\rm sol})_2 \ . \tag{9}$$

Inserting Eq. (9) into the right-hand side of Eq. (1), one notes that, for both Eqs. (3) and (4), it turns into a polynomial in $(u_{\rm sol})_1$ and $(u_{\rm sol})_2$. Straightforward analysis demonstrates that, in the case of Eq. (3), the dominating role in the evolution of the, e.g., first soliton is played by the term of the polynomial

$$P_1=\epsilon(u_{\rm sol})_1(u_{\rm sol})_2[(u_{\rm sol}^*)_2]_x \ , \tag{10}$$

the asterisk standing for the complex conjugation. Indeed, inserting the polynomial into Eq. (11) (see below), one notes that the dominating term must contain one power of $(u_{\rm sol})_2$ and one power of $(u^*_{\rm sol})_2$ (or their derivatives), lest the integral on the right-hand side of Eq. (11) should be exponentially small (the integrand will contain a rapidly oscillating exponent). Next, either $(u_{\rm sol})_2$ or $(u^*_{\rm sol})_2$ must be represented by its derivative, which gives an additional large multiplier $\sim V$. It is easy to see that this is only the term (10) which meets all these conditions.

Subsequent calculations are straightforward: One should find the quantity

$$\frac{d}{dt}(N_{\rm sol})_1\equiv\frac{d}{dt}\int_{-\infty}^{+\infty}|(u_{\rm sol})_1|^2dx = -i\int_{-\infty}^{+\infty}dx\, u_1^*(x)P_1(x)+\text{c.c.} \ , \tag{11}$$

corresponding to the perturbing term (10), and then, using the relation $\eta=\frac{1}{4}N_{\rm sol}$, it is necessary to calculate the quantity

$$\delta^{(1)}\eta_1=\frac{1}{4}\int_{-\infty}^{+\infty}dt\frac{d}{dt}(N_{\rm sol})_1 \ . \tag{12}$$

Substitution of Eqs. (5), (10), and (11) into Eq. (12) yields the eventual result

$$\delta^{(1)}\eta_{1,2}=4\epsilon\eta_1\eta_2\text{sgn}(V_{1,2}-V_{2,1}) \ . \tag{13}$$

The superscript (1) in Eq. (13) means that this result has been obtained in first order in ϵ, cf. Eq. (7). Evidently, Eq. (14) satisfies the conservation law $\delta^{(1)}\eta_1+\delta^{(1)}\eta_2=0$.

Now, it is relevant to compare the analytical result of Eq. (13) with the numerical findings of Ref. 6. In the notation adopted in the present work, the numerical simulations of Ref. 6 were performed for $\eta_1=\eta_2=\frac{1}{2}$ and $V_1=-V_2$ taking values from 2 to 200. The basic inference arrived at in Ref. 6 is that for $V_1\gtrsim 10$ and for ϵ not too large the quantity $\delta\eta/\eta$ is well described by the empiric formula

$$|\delta\eta|/\eta=2\epsilon \ . \tag{14}$$

Inserting $\eta_1=\eta_2=\frac{1}{2}$ into Eq. (13), one immediately realizes the origin of this "empiric" formula. The analytical expression of Eq. (13) applies as well to the case $\eta_1\neq\eta_2$, which was not simulated in Ref. 6.

The coincidence of the analytical expression with the numerical result of Eq. (14) gives a possibility for the direct check of the perturbative technique for the NS equation based on the simplest approximation of Eq. (9) in the case when the collision between the fast solitons is considered.

In Ref. 6, a contribution of the collision to the change of the solitons' velocities was singled out as well. The analytical approach relying upon Eq. (9) (based on the analysis of momentum balance instead of the balance of the number of quanta) yields the zero result for this effect, so that the next approximation in the small parameters $\eta_{1,2}/V$ is required. However, this effect, unlike the energy exchange and the radiative losses, is not of principal importance since the solitons' velocities change continuously between the collisions according to Eq. (6).

This technique has been checked for the present particular case; therefore it may be applied to other physical problems. For instance, for Eq. (4) the similar result is

$$\delta^{(1)}\eta_{1,2}=8\epsilon\eta_1\eta_2(V_{1,2}-V_{2,1})^{-1} \ , \tag{15}$$

cf. Eq. (8).

If one considers an ensemble of the solitons with different initial amplitudes and velocities, the ones with largest initial amplitudes will acquire largest velocities according to Eq. (6), and, according to Eqs. (13) and (15), they will further increase their amplitudes due to collisions with the slower solitons. As a matter of fact, a corresponding qualitative estimate has been given in Ref. 6 conformably to the solitons in the optical fiber. It can be relevant to note additionally that, if the mean distance between the solitons is L and a characteristic value of the initial amplitudes is η_0, the time T during which the solitons will separate into "large" and "small" ones scales as follows: $T\sim(L/\eta_0^5)^{1/2}\epsilon^{-1}$ for Eq. (3), and $T\sim(L/\eta_0)\epsilon^{-1}$ for Eq. (4).

The kinetics of the rarefied soliton gas was also analyzed in Ref. 7, but without the first-order exchange processes (only the radiative losses were taken into account). Therefore, the conclusion of Ref. 7 that the collisions result in complete decay of the solitons into radiation is wrong. However, the role of the radiative losses remains important because they put a limit on the collision-induced growth of the amplitudes of the fastest solitons. For instance, comparing Eqs. (13) and (7), one concludes that the amplitude cannot exceed (in the order of magnitude) the values $\sim\epsilon^{-1}$. In conclusion, it is pertinent to mention that similar results for this problem have been obtained by Yu.S. Kivshar (unpublished).

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[4] There are other physical problems that give rise to more complicated systems with damping, based on the NS equation and conserving the integral of Eq. (2). For instance, interaction of a dispersive shear wave on the surface of a nonlinearly elastic solid body covered by an elastic film with a nondispersive

damped acoustic surface wave is described by the modified Zakharov system:

$$iu_t + 2\lambda|u|^2u + 2nu + u_{xx} = 0 ,$$

$$n_{tt} - n_{xx} + \mu(|u|^2)_{xx} = \gamma n_{txx} .$$

This damped system conserves the total number of surface phonons defined by Eq. (2) [H. Hadouaj, B. A. Malomed, and G. A. Maugin (unpublished)].

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