Multiparticle coherence calculations for synchrotron-radiation emission

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We have used Monte Carlo calculations to verify the classical electrodynamics result describing the multiparticle coherent enhancement at long wavelengths for synchrotron-radiation emission. In addition we show the derivation of the result that, for a bunch of electrons in a storage ring, the synchrotron-radiation emission has a multiparticle coherent enhancement that is given by the square of the Fourier transform of the longitudinal spatial distribution function of the electrons. The result has been verified for Gaussian and sinusoidal distributions. For a Gaussian distribution of particles in a bunch, this is seen to have an effect only at wavelengths close to the bunch length. For a sinusoidal distribution as in the transverse optical klystron, the enhancement can easily be seen to occur at one specific wavelength.

INTRODUCTION

This paper discusses and calculates multiparticle-coherent-enhancement effects in synchrotron-radiation emission. In particular, we extend the work of Nodvick and Saxon [1]. Although their paper is primarily concerned with the long-wavelength suppression of coherent effects due to the size of the vacuum vessel, they also present a formalism for calculating the coherence. We show that their analytical result using classical electrodynamics can be generalized, the coherent enhancement being given by the square of the Fourier transform of the particle distribution function. This treatment contains certain approximations and in order to check their validity, we have applied Monte Carlo methods to the same problem.

We have previously calculated the long-wavelength limit of the synchrotron-radiation spectrum [2]. This calculation, however, did not take into account possible enhancements due to the coherent superposition of the emissions from different electrons in the circulating bunch. In another paper, Michel [3] presented a calculation in which he predicted large coherent enhancements starting in the middle infrared. These arise from the argument that for wavelengths equal to the bunch length the N electrons in the bunch can be considered to be radiating coherently, as a kind of superparticle of charge Ne, and thus the output is proportional to N^2e^2 rather than Ne^2. For wavelengths less than the bunch length, he argues that the enhancement would be proportional to the number of electrons in a cube whose side is of the order of the wavelength. The coherent enhancement, then, would be around 10^{12} at wavelengths equal to the bunch length falling as λ^{-3} to shorter wavelengths. This issue of coherence is here examined in the light of recent measurements [4], which failed to show coherent enhancement where it would have been expected on the basis of the above arguments.

THEORY OF COHERENT EMISSION FROM N ELECTRONS

The following theory assumes that the electrons are contained in a bunch because of the radio-frequency cavity and the nature of electron storage rings. It is further assumed that the beam has no emittance, i.e., that all the electrons follow the same circular path. Finally we assume that the observer is a long distance from the source compared to the bending radius of the dipole magnet.

Using classical electrodynamics and the notation of Jackson [5], we can write that the radiation emitted from a particle (electron) of charge e, into a solid angle $d\Omega$, at a frequency ω, is given by

$$\frac{d^2I}{d\omega\, d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\beta})e^{i\omega[t-\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c]}dt\right|^2, \tag{1}$$

where ω is the angular frequency of the radiation, $\boldsymbol{\beta}$ is the ratio of the particle velocity to the velocity of light ($\mathbf{v}/c$), $\mathbf{r}(t)$ is the particle position, and $\hat{\mathbf{n}}$ is a unit-direction vector close to the photon emission direction for a distant observer.

We note that this has been solved numerically by several authors, a comprehensive treatment being given, using the same notation, by Krinsky, Perlman, and Watson [6]. The usual method for calculating the power radiated by N electrons is to multiply the result of Eq. (1) by N and by the revolution frequency, since one usually observes the electron bunch only over a few milliradians of orbit each revolution.

In a full calculation for N electrons the power is found by summing the electric field of each one, paying attention to the phase. We then obtain

$$\frac{d^2I}{d\omega\, d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\sum_{j=1}^{N}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\beta}_j)e^{i\omega[t-\hat{\mathbf{n}}\cdot\mathbf{r}_j(t)/c]}dt\right|^2, \tag{2}$$

where $\boldsymbol{\beta}_j$ is the ratio of the jth particle velocity to the velocity of light ($\mathbf{v}_j/c$) and $\mathbf{r}_j$ is the position of the jth particle. If we now assume $\beta_j=\beta$ for all particles, so that the relative positions of the particles do not change during the observation of the emission, and that the center of mass of the bunch is described by $\mathbf{r}(t)$, we can rewrite Eq. (2) to a good approximation as follows, where now the r_j terms refer to the positions of the particles (along the orbit) with respect to the center of mass of the bunch:

$$\frac{d^2I}{d\omega\, d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\sum_{j=1}^{N} e^{i\omega r_j/\beta c}\right|^2\left|\int_{-\infty}^{\infty}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\beta})e^{i\omega[t-\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c]}dt\right|^2 , \tag{3}$$

where we have now separated the time-dependent terms and retained them in the integral. Here we note that this is equivalent to Eq. (1), except for the summation term [5]. Since the particles are traveling very close to the speed of light, we can put $\beta=1$ and are left, then, with the problem of solving

$$T(\omega)=\left|\sum_{j=1}^{N} e^{i\omega r_j/c}\right|^2 . \tag{4}$$

Following Nodvick and Saxon [1], we rewrite the squared sum as the product

$$T(\omega)=\sum_{j=1}^{N} e^{i\omega r_j/c}\sum_{k=1}^{N} e^{-i\omega r_k/c} \tag{5}$$

$$=\sum_{j=1(j=k)}^{N} e^{i\omega(r_j-r_k)/c}+\sum_{j,k=1(j\neq k)}^{N} e^{i\omega(r_j-r_k)/c} \tag{6}$$

which in turn becomes

$$T(\omega)=N+\sum_{j,k=1(j\neq k)}^{N} e^{i\omega(r_j-r_k)/c} . \tag{7}$$

We now rewrite Eq. (7) as

$$T(\omega)=N+N(N-1)f(\omega) , \tag{8}$$

where $f(\omega)$ is defined by

$$f(\omega)=\frac{1}{N(N-1)}\sum_{j,k=1(j\neq k)}^{N} e^{i\omega(r_j-r_k)/c} . \tag{9}$$

At this stage we can see that we have an expression for the radiated power in the following form

$$\frac{d^2I}{d\omega\, d\Omega}=[N+N(N-1)f(\omega)]P(\omega) , \tag{10}$$

where $P(\omega)$ is the power radiated by a single electron and given by Eq. (1). The form of Eq. (10) allows us to see that $f(\omega)$ describes the coherence of the emitted radiation. $f(\omega)=0$ represents the incoherent limit where the total power from N electrons is simply N times the result from a single electron and $f(\omega)=1$ represents the coherent limit where the total power is N^2 times the result for a single electron.

Equation (10) was derived for a particular configuration of particles in the bunch and is formally correct for any N and for an arbitrary set of coordinates $\mathbf{r}_j$. However, when defined with respect to a particular particle configuration, $f(\omega)$ becomes very complicated due to its detailed dependence on the $\mathbf{r}_j$ terms; it can assume, for example, different values for particle configurations, which differ in detail but which nevertheless correspond to the same "mean" charge distribution. Equation (10) becomes useful primarily in the context of an ensemble average over particle configurations. The problem is similar to that encountered in disordered systems where configurational averaging must be performed.

We now define two normalized distribution functions. $S_1(\mathbf{r})$ describes a specific particle configuration:

$$S_1(\mathbf{r})=\frac{1}{N}\sum_{j=1}^{N}\delta(\mathbf{r}-\mathbf{r}_j) \tag{11}$$

and $S(\mathbf{r})$ is a continuous probability distribution function (e.g., Gaussian, sinusoidal, etc.) such that $NS(\mathbf{r})d^3r$ is the probability of finding a particle in the region d^3r about $\mathbf{r}$. $S(\mathbf{r})$ represents a particular ensemble average $\langle S_1(\mathbf{r})\rangle$ of $S_1(\mathbf{r})$. Specifically, if an infinite series of N-particle bunches is created under identical (accelerator) conditions, $S(\mathbf{r})$ is the average of the resulting $S_1(\mathbf{r})$. It can be thought of as a mean particle distribution function in which statistical fluctuations in the charge density due to the discrete nature of the charges have been averaged over. Figure 1 contains a schematic illustration of $S(\mathbf{r})$

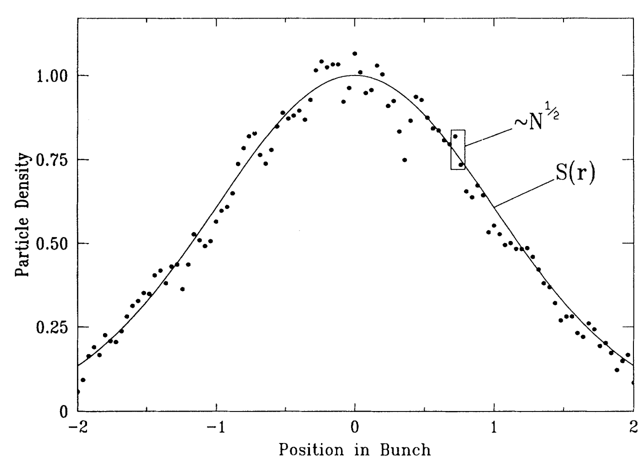

FIG. 1 Illustration of the difference between the smooth Gaussian function $S(r)$, shown by the solid line; and $S_1(r)$, a Gaussian distribution statistically filled with, in this case, 10 000 particles in 100 increments between -2σ and $+2\sigma$, shown as filled circles. The incremental change in electric field as a bunch of electrons passes an observer is proportional to $\sqrt{N}e$ (see box), and hence the intensity is proportional to Ne^2 or N times the value for a single electron.

and $S_1(\mathbf{r})$ for a Gaussian distribution of half-width σ.

In the ensemble-averaged version of Eq. (10), $f(\omega)$ is given by

$$f(\omega)=\frac{1}{N(N-1)}\left\langle \sum_{j,k=1(j\neq k)}^{N} e^{i\omega(r_j-r_k)/c}\right\rangle , \tag{12}$$

where the angle brackets explicitly denote the ensemble average. To evaluate Eq. (12) we shall make two assumptions. First we assume that for N large (i.e., of the order of the number of particles that in practice are contained in a bunch, $\sim 10^{12}$ at the National Synchrotron Light Source, Brookhaven), $f(\omega)$ is independent of N and is simply a frequency-dependent function, which in turn depends only on the mean particle distribution $S(\mathbf{r})$. This assumption is motivated by the behavior described by Eq. (10) in both incoherent [$f(\omega)=0$, power proportional to N], and coherent [$f(\omega)=1$, power proportional to N^2] limits. An equivalent statement is that the ensemble average of the restricted double sum in Eq. (7) is proportional to N^2-N for large N. We wish to express explicitly the dependence of $f(\omega)$ on $S(\mathbf{r})$. We have, from the ensemble average of Eq. (8)

$$\left\langle \sum_{j,k=1}^{N} e^{i\omega(r_j-r_k)/c}\right\rangle = N[1-f(\omega)]+N^2 f(\omega) . \tag{13}$$

Since $f(\omega)$ is independent of N by our first assumption, we can divide by N^2 and take the large-N limit to obtain

$$f(\omega)=\lim_{N\to\infty}\frac{1}{N^2}\left\langle \sum_{j,k=1}^{N} e^{i\omega(r_j-r_k)/c}\right\rangle . \tag{14}$$

Although the ensemble-averaged double sum in general contains terms proportional to both N and N^2 (the incoherent and coherent radiation terms, respectively) it is permissible to replace it with an expression that contains only the latter term, as the former will vanish in the limit. Equation (14) can be rewritten exactly as

$$f(\omega)=\lim_{N\to\infty}\left\langle \left| \int e^{i\omega r/c} S_1(\mathbf{r}) d^3 r \right| \right\rangle . \tag{15}$$

We now make our second assumption, that the ensemble average in Eq. (15) can be evaluated by replacing $S_1(\mathbf{r})$ with the (continuous) particle distribution function $S(\mathbf{r})$ to obtain finally

$$f(\omega)=\left| \int e^{i\omega r/c} S(\mathbf{r}) d^3 r \right|^2 . \tag{16}$$

Note that in replacing $S_1(\mathbf{r})$ with $S(\mathbf{r})$ we in effect threw away the incoherent radiation term in the ensemble-averaged double sum. Thus it is clear that one cannot evaluate Eq. (4) directly by converting the sums to integrals involving $S(\mathbf{r})$. Equation (16) yields the important general result that $f(\omega)$ is the magnitude squared of the Fourier transform of $S(\mathbf{r})$.

The above discussion is helpful in understanding the origin of coherent and incoherent emission of synchrotron radiation. One can think of coherent emission as being produced by the mean particle distribution $NS(\mathbf{r})$. It is not influenced by the statistical fluctuations about the mean (Fig. 1). Thus, if one were to imagine replacing the ensemble of discrete charge distributions $S_1(\mathbf{r})$ that actually exist in a storage ring with the corresponding continuous charge distribution $S(\mathbf{r})$, any incoherent radiation being generated would vanish but the coherent radiation intensity would remain unchanged. In contrast, any incoherent radiation results solely from the statistical fluctuations of the particle density, causing an electric field change of $\sqrt{N}e$—the intensity then being proportional to Ne^2. Note that the radiation intensity associated with a continuous charge distribution must automatically vary as N^2e^2 since the electric field that is generated must be proportional to the total charge (i.e., Ne).

It is interesting to compare the "incoherent" radiation from a synchrotron with that from a conventional source such as a discharge lamp. The total field in the latter case is the sum of N wave trains with equal amplitude but with random relative phases. Since the phases are random they are averaged over in obtaining the total intensity. In the case of synchrotron radiation the relative phases of the wave trains emitted by different volume elements d^3r within the bunch are not random but are fixed by their relative positions within the bunch. However, the amplitudes of the wave trains vary in a random fashion about the mean due to the statistical fluctuations in the number of particles contained within each volume element d^3r. Thus incoherent radiation from a synchrotron results from averaging over random amplitude fluctuations rather than random phase fluctuations.

APPLICATION TO SPECIFIC PARTICLE DISTRIBUTIONS

We now apply these results to two particle distributions of relevance to synchrotron-radiation users. As a separate verification of the result of Eqs. (10) and (16), we also apply Monte Carlo methods to the direct solution of Eq. (4), for the same distributions. For the Monte Carlo calculation, 10^6 particles were chosen by first generating a random r_j value for each particle followed by a random number between 0 and 1. The latter was tested against the chosen $S(r)$, normalized to have a maximum value of 1. The "particle" and "position" were accepted if the random number was less than or equal to $S(r)$. Once the particles were selected, then for each wavelength λ, the quantity

$$\left[\sum_{1}^{n}\cos(2\pi\lambda/r_j)\right]^2+\left[\sum_{1}^{n}\sin(2\pi\lambda/r_j)\right]^2$$

was calculated. This is identical to the "intensity" plotted in Figs. 2 and 3.

Gaussian distribution

If the electrons have a Gaussian distribution, then the probability of finding a particle at a position r relative to the center of the bunch is given to within a factor N by

$$S(r)=\frac{1}{\sqrt{2\pi}\sigma}e^{-r^2/2\sigma^2} , \tag{17}$$

where σ is the standard deviation. For this case then we have to evaluate

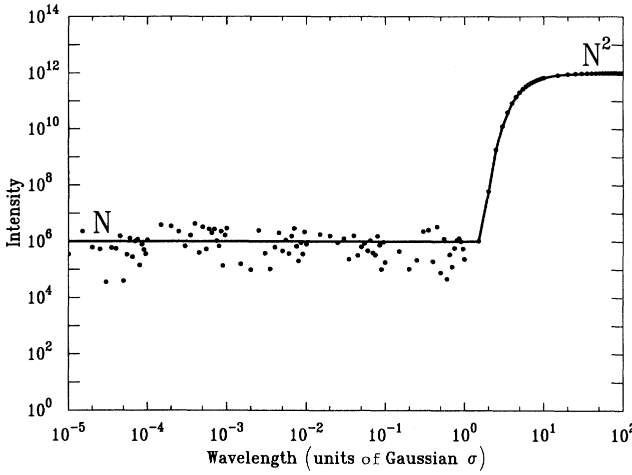

FIG. 2. Coherent enhancement factor for emission as a function of wavelength for a Gaussian distribution of particles within a bunch of an electron synchrotron. The solid line is the result of the calculation described in the text for $N+N(N-1)f(\omega)$ from Eqs. (10) and (19), while the filled circles are the result of a Monte Carlo calculation of Eq. (4) for the same distribution for 10^6 particles. These were distributed in a bunch extending from -4σ to $+4\sigma$. Both calculations are plotted as a function of the dimensionless quantity wavelength over σ, the latter describing the Gaussian bunch length.

$$f(\omega)=\left|\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{i\omega r/c}e^{-r^2/2\sigma^2}dr\right|^2 , \quad (18)$$

which is the Fourier transform of a Gaussian and yields the result

$$f(\omega)=e^{-4\pi^2\sigma^2/\lambda^2} . \quad (19)$$

In Fig. 2 we show as filled circles the result of a Monte Carlo calculation applied to Eq. (4) for 10^6 electrons for a Gaussian distribution. The particles were randomly distributed in a bunch extending from $+4\sigma$ to -4σ. We have plotted the function $N+N(N-1)f(\omega)$ versus wavelength (rather than frequency), as a function of the dimensionless quantity *wavelength*/σ. At short wavelengths (incoherent limit) the intensity can be seen to be proportional to N (10^6), while at longer wavelengths (coherent limit) it is equal to N^2(10^{12}). The solid line shows the result from Eqs. (10) and (19). The two calculations are seen to agree closely, confirming the validity of Eq. (16). Note that for this distribution, coherence only plays a role for wavelengths close to the bunch length. If the bunch length is long compared to the size of the vacuum vessel of the ring, considerable suppression is predicted [1].

Sinusoidal distribution—free-electron lasers

In some devices a light beam of appropriate wavelength traveling parallel to the particles is superimposed on the fluctuating magnetic field of an undulator to "micro-bunch" the particles. An example is the transverse optical klystron [7] (TOK) at Brookhaven. This combination of electric and magnetic fields results in a "micro-bunching" of the particles within the Gaussian bunch. Although the degree of microbunching in practice is much less than 100%, for the purposes of the present calculation we will assume full modulation. The probability of finding a particle is, then, again to within a factor N, the product of the Gaussian function of Eq. (17) and the function

$$S''(r)=\sin\left[\frac{\omega' r}{c}\right] . \quad (20)$$

The Fourier transform of this probability distribution has in it a δ function, with a maximum at the frequency $2\omega'$. Due to the finite length of the bunch, there is also a maximum as before at long wavelengths. In Fig. 3 we show the result obtained from a Monte Carlo solution of Eq. (4) for a sinusoidal distribution of this kind. For this case we took 10^6 particles distributed in a Gaussian fashion from -4σ to $+4\sigma$, which was then further modulated by 200

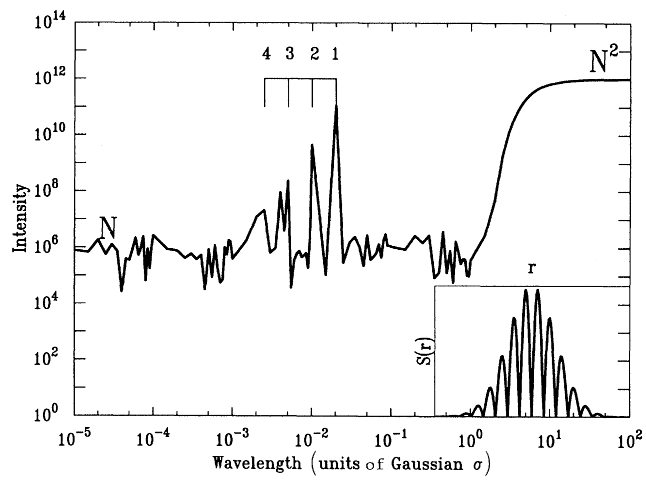

FIG. 3. Coherent enhancement factor $N+N(N-1)f(\omega)$ for emission as a function of wavelength for a sinusoidal distribution of particles within the Gaussian bunch of electrons in a synchrotron. [An example of the type of probability distribution S(r) is illustrated in the inset for 16 bunches for r lying between -4σ and $+4\sigma$ of the Gaussian bunch.] The data are the result of a Monte Carlo calculation of Eq. (4) for 10^6 particles, plotted as a function of the dimensionless quantity wavelength over σ, the latter describing the Gaussian bunch length. In this example there were 200 sinusoidal periods giving 400 smaller bunches between -4σ and $+4\sigma$ of the main bunch, or 50 bunches per σ. The overall behavior in the main figure is similar to Fig. 2, but an additional enhancement, predicted by Eq. (20), is observed. The numbered bars above the spectrum indicate the expected positions of the first through fourth harmonics of this enhancement. The first two harmonics are clearly seen in the Monte Carlo calculation, the third and fourth being somewhat lost in the noise.

sinusoidal periods. Thus the bunch was split into 400 smaller bunches, 50 per σ.

The solid line is the result of the Monte Carlo calculation for $N+N(N-1)f(\omega)$. The figure clearly shows the special enhancement at a wavelength/σ value of $\frac{1}{50}$ and in addition shows up to four harmonics, the third and fourth of which are somewhat buried in the statistical noise. At very short wavelengths (incoherent limit) the single-particle function has its intensity multiplied by N (10^6) and at very long wavelengths by $N^2(10^{12})$ coming from the overall Gaussian distribution as before.

DISCUSSION

The above calculations confirm that the form factor $f(\omega)$ giving the coherent enhancement for N particles emitting synchrotron radiation is the square of the Fourier transform of the electron probability distribution in the bunch. The range of coherent enhancement, then, for a Gaussian distribution of particles within a bunch is controlled by the bunch length, being more localized in frequency or wavelength space for longer bunches. This result has recently been confirmed by Nakazato *et al.* [8], using picosecond bunches of electrons from a linear accelerator. Experiments in the far-infrared region could benefit from such linear accelerators and even from new machines such as the advanced light source, which has shorter bunches. For a sinusoidal distribution the enhancement can readily be seen to occur at a single frequency. This result may assist in the understanding and hence development of coherent sources such as the transverse optical klystron.

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