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Localization factor for multichannel disordered systems

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The localization factor (inverse of the localization length) is derived as a function of the transmission matrix $\underline{\tau}$ for multichannel disordered systems using a multiplicative ergodic theorem of Oseledets [Trans. Mosc. Math. Soc. **19**, 197 (1968)]. The wave-transfer-matrix formalism is used in which the transfer matrices are symplectic and elements of the special unitary group SU(d,d). Modeling the disordered multichannel system via a product of random-wave-transfer matrices and applying the theorem of Oseledets as a starting point, it is shown that the multichannel localization factor (the smallest Lyapunov exponent of the random matrix product) can be expressed asymptotically as a function of any one of three matrix norms of the transmission matrix $\underline{\tau}$. As a by-product of this analysis, we also develop expressions for the remaining Lyapunov exponents as a function of $\underline{\tau}$. The localization factor result is compared with two others appearing in the literature.

I. INTRODUCTION

In the study of one-dimensional localization, whether quantum mechanical or classical, the transfer-matrix formalism has proven to be a powerful tool in understanding important properties of the phenomenon. In this formalism, a single-channel system (random chain) or multichannel system (random wire), bordered on both ends by its undisordered counterpart, may be modeled via a product of random transfer matrices. For the single channel system, we can use 2×2 wave-transfer matrices, $\underline{W}_j$, so that the random matrix product appears as

$$\prod_{j=1}^{n} \underline{W}_j = \begin{bmatrix} \dfrac{1}{\tau_n} & -\dfrac{\rho_n}{\tau_n} \\ -\dfrac{\rho_n^*}{\tau_n^*} & \dfrac{1}{\tau_n^*} \end{bmatrix}$$

for the n-bay disordered system, where τ_n is the transmission coefficient, ρ_n is the reflection coefficient, $*$ is complex conjugate, and $|\tau_n|^2+|\rho_n|^2=1$. (Frequently, in the solid-state literature one finds this stated as $T+R=1$.)

As the single-channel system becomes very long, we can appeal to a theorem of Furstenberg [1] on products of random matrices to show that with probability 1,

$$\gamma = -\lim_{n\to\infty} \frac{1}{n}\ln|\tau_n|, \quad \gamma>0 \tag{1}$$

where γ is the localization factor or the inverse of the localization length on a per bay basis. This definition of localization factor is consistent with that found in [2–4]. This result tells us, for example, that

$$|\tau_n|^2 \sim (e^{-\gamma n})^2 , \tag{2}$$

the transmitted energy decays exponentially with the number of bays as that number becomes very large. Our goal in this paper is to find the multichannel analog to the single-channel result (1). In the multichannel case we will appeal to a theorem of Oseledets relevant to products of random matrices to derive our result. This mathematical approach allows us to present the multichannel localization factor in a general setting applicable to both classical (in acoustical and optical systems, e.g.) and quantum-mechanical localization.

Before turning to the details of the derivation, we want to recall briefly a few key papers that have influenced the study of localization from the transfer-matrix perspective. As mentioned above, the pioneering work of Furstenberg [1] on products of random matrices has provided rigorous results that have immediate applicability to the one-dimensional localization problem. McCoy and Wu [5] were apparently the first to recognize the importance of Furstenberg's theorem to disordered physical systems when they studied random Ising models of ferromagnetic materials. However, Matsuda and Ishii [6,7] were the first to bring Furstenberg's work to bear on the localization problem. They carefully related Furstenberg's results to eigenmode localization and wave propagation in disordered mass-spring chains and some simple quantum-mechanical models.

In 1968 the Russian mathematician Oseledets [8] proved a multiplicative ergodic theorem that has enhanced our understanding of the asymptotic behavior of products of random matrices. This theorem, as we will see shortly, has important applications to the study of localization in multichannel systems. More recently, Refs. [9–17] have examined solid-state localization from a transfer-matrix perspective, frequently exploiting the work of Oseledets. Mathematicians have taken renewed interest in the theory of products of random matrices as indicated by two recent publications [18,19]. In recent years classical localization has generated much interest with Refs. [2,20–22] advocating the transfer-matrix perspective. When thinking of classical localization, one would more naturally use the words single wave and multiwave instead of the words single channel and multichannel, respectively. Indeed, the results presented here were originally derived with classical localization in mind.

In Sec. II we state our assumptions about the wave-transfer-matrix properties we use in the multichannel-localization-factor derivation (the assumptions are stated in more detail in Appendix A). The multichannel localization factor is presented in Sec. III (the derivation is shown in Appendix B, where we also provide a formula for all d Lyapunov exponents as a function of the transmission matrix). Here we include a comparison with two results that have appeared in the literature. Concluding remarks are made in Sec. IV.

II. WAVE-TRANSFER-MATRIX ASSUMPTIONS

Our assumptions about the wave-transfer matrix for multichannel systems will follow from two properties of the scattering matrix usually found in the solid-state literature [11,23,24]. These properties are discussed at length in Appendix A and we summarize them in this section. We assume that the $2d \times 2d$ scattering matrix of one disordered bay bordered on both sides by an undisordered system is both symmetric and unitary. This corresponds, respectively, to the assumptions of time-reversal symmetry and current conservation coupled with the exclusion of any evanescent states. In classical localization, symmetry of the scattering matrix implies symmetry of the corresponding impedance matrix, while the unitarity of the scattering matrix implies we have a dissipationless system and that we have excluded evanescent waves.

Our two assumptions about the scattering matrix $\underline{S}$ translate into two properties of the wave-transfer matrix $\underline{W}$. First,

$$\underline{S} \text{ symmetric} \Longrightarrow \underline{W} \text{ symplectic}$$

and, second,

$$\underline{S} \text{ unitary, } \det \underline{W} = 1 \Longleftrightarrow \underline{W}, \underline{W}^{\dagger} \in \mathrm{SU}(d,d) \ ,$$

where $\dagger$ is the complex conjugate transpose. The product of these $2d \times 2d$ random wave-transfer matrices is symplectic and an element of $\mathrm{SU}(d,d)$ and is written

$$\underline{V}_n = \prod_{j=1}^{n} \underline{W}_j = \begin{bmatrix} \underline{\tau}_n^{-1} & -\underline{\tau}_n^{-1}\underline{\rho}_n \\ -\underline{\tau}_n^{-1*}\underline{\rho}_n^* & \underline{\tau}_n^{-1*} \end{bmatrix} , \tag{3}$$

where now $\underline{\tau}_n$ is a transmission matrix and $\underline{\rho}_n$ is a reflection matrix. In Sec. III the subscript n will be suppressed in the transmission and reflection matrices.

III. THE MULTICHANNEL LOCALIZATION FACTOR

In this section we present Oseledets's theorem applied to products of random symplectic matrices and then the key result of the paper, the multichannel localization factor as a function of the transmission properties of the disordered system.

From a theorem of Oseledets [8,18,19] we let $\underline{W}_1, \underline{W}_2, \ldots, \underline{W}_n$ form a sequence of independent identically distributed random symplectic matrices of size $2d \times 2d$. Suppose also that

$$E(\sup\{\ln \sigma_{\max}(\underline{W}_1), 0\}) < +\infty \ .$$

If we set $\underline{V}_n = \underline{W}_n \cdots \underline{W}_1$, then the sequence of matrices $(\underline{V}_n^{\dagger}\underline{V}_n)^{1/2n}$ converges with probability 1 as $n \to \infty$ to a random matrix $\underline{B}$ with $2d$ nonrandom eigenvalues

$$e^{\gamma_1}, \ldots, e^{\gamma_d}, e^{-\gamma_d}, \ldots, e^{-\gamma_1} \ .$$

These γ_i's are the Lyapunov exponents of the random matrix product $\underline{W}_n \cdots \underline{W}_1$. We are, of course, interested in the instances when all the channels or waves are localized, so $\gamma_1 \geq \cdots \geq \gamma_d > 0$. In random dynamical systems, Lyapunov exponents are considered a measure of chaoticity [25].

The theorem of Furstenberg applied to $2d \times 2d$ matrices allows us to calculate γ_1, which is the uppermost Lyapunov exponent. However, in this multichannel case with $\gamma_d \leq \gamma_1$, γ_d represents the channel or wave with potentially the least amount of decay, and so it carries energy along the multichannel system farther than any other channel or wave. As a result, γ_d is the quantity of interest when calculating multichannel localization effects.

Note that we can also express the Lyapunov exponents of this random symplectic matrix product in terms of its singular values (see Appendix A) $\sigma_j = \sigma_j(\underline{V}_n)$. If we recall that the singular values of a symplectic matrix occur in reciprocal pairs,

$$\sigma_1, \ldots, \sigma_d, \sigma_d^{-1}, \ldots, \sigma_1^{-1} \ ,$$

where $\sigma_1 \geq \cdots \geq \sigma_d \geq 1$. Then, with probability 1,

$$\gamma_j = \lim_{n \to \infty} \frac{1}{n} \ln \sigma_j(\underline{V}_n), \quad 1 \leq j \leq d \ . \tag{4}$$

This result [18] will be very useful in Appendix B where we derive γ_d as a function of the transmission properties of the system.

The multichannel localization factor (or the dth Lyapunov exponent of $\underline{V}_n$) is (see Appendix B)

$$\gamma_d = -\lim_{n \to \infty} \frac{1}{n} \sigma_{\max}(\underline{\tau}) \tag{5}$$

or

$$\gamma_d = -\lim_{n \to \infty} \frac{1}{n} \ln[\mathrm{tr}(\underline{\tau}\underline{\tau}^{\dagger})]^{1/2} \tag{6}$$

or

$$\gamma_d = -\lim_{n \to \infty} \frac{1}{n} \ln |\tau_{ij}|_{\max} \ , \tag{7}$$

where $\underline{\tau}$ is $d \times d$, and τ_{ij} is the ijth element of $\underline{\tau}$ and all the results hold with probability 1. In the limit (as is shown in Appendix B) all three results are equivalent and notice that all three agree with the single-channel relation of Eq. (1) where τ is a scalar transmission coefficient. Again, the reader should bear in mind that the three results above are only equivalent *asymptotically*. The easiest interpretation of the result can be placed on Eq. (7). This result says that the surviving channel which transmits energy the farthest is governed by the transmission coefficient with the largest absolute value, which makes perfect sense. As a byproduct of this analysis, we also develop in Appendix B expressions for the remaining

Lyapunov exponents as a function of $\underset{\sim}{\tau}$.

We are also able to compare our result with two others that have appeared in the literature. Imry [15] made exactly the same assumptions about the wave-transfer matrix as we have, and, through the work of Pichard, was aware of Oseledets's theorem. In his paper, Imry made some heuristic arguments concerning the conductance $2\,\mathrm{tr}(\underset{\sim}{\tau}\underset{\sim}{\tau}^\dagger)$, leading to the inverse localization length $1/\xi$ (the same as our multichannel localization factor, where each bay was measured in atomic units L), being

$$\frac{1}{\xi}=-\lim_{L\to\infty}\frac{1}{L}\ln\mathrm{tr}(\underset{\sim}{\tau}\underset{\sim}{\tau}^\dagger)\ .$$

The problem with this result is the missing square root over $\mathrm{tr}(\underset{\sim}{\tau}\underset{\sim}{\tau}^\dagger)$ [26].

Next we compare our result with Johnston and Kunz [11], who relied rigorously on theories of products of random matrices. In their paper, Johnston and Kunz used the work of Tutubalin and Virster [27,28], though they were aware of Pichard's studies. Arguing, as we have, that the smallest Lyapunov exponent of a random symplectic matrix product is the localization factor for long multichannel systems, they derived the localization factor as

$$\gamma_d=-\lim_{n\to\infty}\frac{1}{n}\ln|\tau_{ij}|\quad\text{for any }\tau_{ij}\ ;$$

in other words, the magnitudes of all elements of the transmission matrix behave asymptotically in the same way. This result differs from the one presented in Eq. (7) in that our γ_d involves the limit of $|\tau_{ij}|_{\max}$. Johnston and Kunz said their result was only proved for matrices of dimension 4×4 or smaller, while no such restriction holds for the Eq. (7) result. Even if the Johnston and Kunz result were found to hold for larger matrix dimensions, our results would not contradict. In addition, it is useful to evaluate whether the result of Ref. [11] makes sense for the undisordered or perfectly periodic system (though Johnston and Kunz made no explicit claim that their result held for such systems). For a perfectly periodic system with n bays, the transmission matrix $\underset{\sim}{\tau}$ would look like

$$\underset{\sim}{\tau}=\begin{bmatrix} e^{-ik_1 n} & & \\ & \ddots & \\ & & e^{-ik_d n}\end{bmatrix},$$

with all the off-diagonal terms zero. Using the results of Ref. [11], we are tempted to take any element of $\underset{\sim}{\tau}$ to get the proper localization factor. Yet if we choose any off-diagonal term, we get the following absurd result:

$$\gamma_d=-\lim_{n\to\infty}\frac{1}{n}\ln(0)$$
$$=-\lim_{n\to\infty}\frac{-\infty}{n}\ .$$

This is in contrast to Eq. (7) which takes the element of $\underset{\sim}{\tau}$ with the maximum absolute value, namely, $|e^{-ik_j n}|=1$, from which we find

$$\gamma_d=-\lim_{n\to\infty}\frac{1}{n}\ln(1)=0\ .$$

This is precisely the result for perfectly periodic systems, i.e., there is no localization.

Note that all three of our localization results, Eqs. (5), (6), and (7), only hold as $n\to\infty$. Indeed, all three must give equivalent answers in the limit. However, if we were to evaluate each of the three expressions for finite n, we would likely find three different answers. This is a consequence of the three matrix norms satisfying the following inequalities:

$$|\tau_{ij}|_{\max}\le\sigma_{\max}(\underset{\sim}{\tau})<[\mathrm{tr}(\underset{\sim}{\tau}\underset{\sim}{\tau}^\dagger)]^{1/2}\ .$$

Therefore, when doing numerical simulations of multichannel localization, averaging should be done on $\ln|\tau_{ij}|_{\max}$ so as not to mispredict the localization factor. Numerical averaging on $\ln|\tau_{ij}|_{\max}$ should be a computationally more efficient alternative to the numerical methods employed in [9,29,30] to find the dth Lyapunov exponent.

IV. CONCLUSION

In summary, we have derived the multichannel localization factor as a function of the transmission matrix of the disordered system. The transfer-matrix formalism, which has proven to be very powerful in the study of single-channel localization, was used here as well, in conjunction with a theorem of Oseledets dealing with products of random matrices. We expressed the multichannel localization factor (the dth Lyapunov exponent of the random matrix product) as

$$\gamma_d=-\lim_{n\to\infty}\frac{1}{n}\ln\sigma_{\max}(\underset{\sim}{\tau})\ ,$$

or

$$\gamma_d=-\lim_{n\to\infty}\frac{1}{n}\ln[\mathrm{tr}(\underset{\sim}{\tau}\underset{\sim}{\tau}^\dagger)]^{1/2}\ ,$$

or

$$\gamma_d=-\lim_{n\to\infty}\frac{1}{n}\ln|\tau_{ij}|_{\max}\ .$$

The result was compared with two others appearing in the literature. As a byproduct of this analysis, we also developed expressions for the remaining $d-1$ Lyapunov exponents as a function of $\underset{\sim}{\tau}$.

An analytical formula for γ_d would be desirable so as to avoid having to take a long matrix product as has been done in Refs. [9,29,30]. Recently, a number of attempts [31,32] have been made to derive Lyapunov exponents for random matrix products representing various multichannel systems. The work of Bougerol and Lacroix [18] may provide some additional guidance for such efforts. Indeed, an estimate of the dth Lyapunov exponent to first order in the variances of the random variables of the disordered multichannel system coupled with the results of this paper could provide powerful insights into the multichannel localization phenomenon.

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APPENDIX A: PROPERTIES OF THE SCATTERING AND WAVE-TRANSFER MATRICES

In this appendix we discuss some of the properties of the scattering matrices and wave-transfer matrices used in the paper. These matrices describe the propagation of traveling waves in the passbands of periodic or disordered periodic systems. We will state the scattering and wave-transfer matrices in their most general forms and then impose conditions on the scattering matrix and discuss what this implies for the wave-transfer matrix. The scattering and wave-transfer matrices are of dimension $2d\times 2d$. Note that we will suppress any subscripts on our transmission and reflection matrices. Scattering and wave-transfer matrices are discussed in [33] and for some specific disordered systems in [34].

The scattering matrix $\underline{S}$ in its most general form is

$$\begin{bmatrix} \overleftarrow{\mathbf{A}}_{j-1} \\ \overrightarrow{\mathbf{B}}_{j} \end{bmatrix} = \begin{bmatrix} \underline{r} & \underline{t} \\ \underline{\hat{t}} & \underline{\hat{r}} \end{bmatrix} \begin{bmatrix} \overrightarrow{\mathbf{B}}_{j-1} \\ \overleftarrow{\mathbf{A}}_{j} \end{bmatrix} , \tag{A1}$$

where $\overleftarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$ represent vectors of traveling-wave amplitudes in the indicated directions, $\underline{\hat{t}}$ and $\underline{r}$ are the transmission and reflection matrices, respectively, for the rightward traveling waves, and $\underline{t}$ and $\underline{\hat{r}}$ are the transmission and reflection matrices, respectively, for the leftward traveling waves.

The corresponding wave-transfer matrix $\underline{W}$ involves a rearrangement of the state vector, so that we relate waves on the right of a bay to those on the left of a bay:

$$\begin{bmatrix} \overleftarrow{\mathbf{A}}_{j} \\ \overrightarrow{\mathbf{B}}_{j} \end{bmatrix} = \begin{bmatrix} \underline{t}^{-1} & -\underline{t}^{-1}\underline{r} \\ \underline{\hat{r}}\underline{t}^{-1} & \underline{\hat{t}}-\underline{\hat{r}}\,\underline{t}^{-1}\underline{r} \end{bmatrix} \begin{bmatrix} \overleftarrow{\mathbf{A}}_{j-1} \\ \overrightarrow{\mathbf{B}}_{j-1} \end{bmatrix} . \tag{A2}$$

Now we require that the scattering matrix be symmetric. This means that

$$\underline{r}=\underline{r}^{T} ,$$

$$\underline{\hat{r}}=\underline{\hat{r}}^{T} ,$$

and

$$\underline{t}=\underline{\hat{t}}^{T} ,$$

where superscript T is matrix transpose. A little algebra shows that these are exactly the same conditions needed for the symplecticity of the wave-transfer matrix $\underline{W}$, namely, that

$$\underline{W}^{T}\underline{J}\,\underline{W}=\underline{J}$$

be satisfied, where

$$\underline{J}=\begin{bmatrix} \underline{0} & \underline{I} \\ -\underline{I} & \underline{0} \end{bmatrix} .$$

Thus,

$$\underline{S}\ \text{symmetric} \Longleftrightarrow \underline{W}\ \text{symplectic} .$$

Note also that a determinant identity for partitioned matrices applied to $\underline{W}$, and $\underline{t}=\underline{\hat{t}}^{T}$ prove that $\det\underline{W}=1$.

Now we impose the requirement that $\underline{S}$ be unitary, namely,

$$\underline{S}^{\dagger}\underline{S}=\underline{S}\,\underline{S}^{\dagger}=\underline{I} ,$$

where $\dagger$ is complex conjugate transpose. Now $\underline{S}^{\dagger}\underline{S}=\underline{I}$ tells us that

$$\begin{aligned} &\underline{r}^{\dagger}\underline{r}+\underline{\hat{t}}^{\dagger}\underline{\hat{t}}=\underline{I} ,\\ &\underline{t}^{\dagger}\underline{t}+\underline{\hat{r}}^{\dagger}\underline{\hat{r}}=\underline{I} ,\\ &\underline{r}^{\dagger}\underline{t}+\underline{\hat{t}}^{\dagger}\underline{\hat{r}}=\underline{0} , \end{aligned} \tag{A3}$$

These are precisely the same conditions, along with $\det\underline{W}=1$, that must hold when $\underline{W}$ is an element of the special unitary group $SU(d,d)$ or

$$\underline{W}^{\dagger}\underline{\Delta}\,\underline{W}=\Delta , \quad \det\underline{W}=1 ,$$

where

$$\underline{\Delta}=\begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & -\underline{I} \end{bmatrix} .$$

Similar conditions from $\underline{S}\,\underline{S}^{\dagger}=\underline{I}$, and $\det\underline{W}=1$, also tell us that $\underline{W}^{\dagger}$ is an element of $SU(d,d)$. We conclude that

$$\underline{S}\ \text{unitary},\ \det\underline{W}=1 \Longrightarrow \underline{W},\underline{W}^{\dagger}\in SU(d,d) .$$

Now imposing both symmetry and unitarity on the scattering matrix, we have

$$\underline{S}=\begin{bmatrix} \underline{r} & \underline{t} \\ \underline{t}^{T} & -\underline{t}^{-1*}\underline{r}^{*}\underline{t} \end{bmatrix} ,$$

where $\underline{r}=\underline{r}^{T}$ and

$$-\underline{t}^{-1*}\underline{r}^{*}\underline{t}=-\underline{t}^{T}\underline{r}^{*}\underline{t}^{-\dagger} .$$

Equivalently, when the wave-transfer matrix is symplectic and an element of $SU(d,d)$, we have

$$\underline{W}=\begin{bmatrix} \underline{t}^{-1} & -\underline{t}^{-1}\underline{r} \\ -\underline{t}^{-1*}\underline{r}^{*} & \underline{t}^{-1*} \end{bmatrix} .$$

We now derive a result that will be useful in Appendix B. From the condition $\underline{t}^{\dagger}\underline{t}+\underline{\hat{r}}^{\dagger}\underline{\hat{r}}=\underline{I}$ above, we can prove that

$$0<\mu_i[\underline{t}^{\dagger}\underline{t}]\le 1 ,$$

where $\mu_i[\ ]$ is the ith eigenvalue of the indicated argument. Also, note that

$$\mu_i[\underline{t}^{\dagger}\underline{t}]=\mu_i[\underline{t}\,\underline{t}^{\dagger}]$$

so that all the results stated below hold for $\underline{t}\,\underline{t}^{\dagger}$ as well as $\underline{t}^{\dagger}\underline{t}$. First we assume that $\underline{t}^{\dagger}\underline{t}$ is invertible so that it is positive definite:

$$\underline{t}^{\dagger}\underline{t}>\underline{0} .$$

We also have that $\underline{\hat{r}}^{\dagger}\underline{\hat{r}}$ is at least positive semidefinite:

$$\underline{\hat{r}}^{\dagger}\underline{\hat{r}}\ge\underline{0} .$$

From Eq. (A3) we have

$$\underline{t}^{\dagger}\underline{t}=\underline{I}-\underline{\hat{r}}^{\dagger}\underline{\hat{r}} .$$

Performing an eigenvector decomposition on the above Hermitian matrix, we get

$$\begin{aligned} \underline{t}^{\dagger}\underline{t}&=\underline{I}-\underline{\hat{r}}^{\dagger}\underline{\hat{r}}\\ &=\underline{U}\{\underline{I}-\mathrm{diag}(\mu_i[\underline{\hat{r}}^{\dagger}\underline{\hat{r}}])\}\underline{U}^{\dagger} . \end{aligned}$$

The positive definiteness of $\underline{t}^\dagger \underline{t}$ and the positive semidefiniteness of $\hat{\underline{r}}^\dagger \hat{\underline{r}}$ now imply

$$0 \le \mu_i[\hat{\underline{r}}^\dagger \hat{\underline{r}}] < 1$$

and

$$0 < \mu_i[\underline{t}^\dagger \underline{t}] \le 1 \ ,$$

which is the desired result.

Before we close this appendix, we mention the singular values of a matrix. Any reader not already familiar with singular values and the singular-value decomposition of a matrix is encouraged to consult [35]. The singular values σ_i of a complex $2d \times 2d$ matrix $\underline{W}$ are

$$\sigma_i(\underline{W}) = [\lambda_i(\underline{W}^\dagger \underline{W})]^{1/2}, \quad i = 1, \ldots, 2d$$

where we reserve $\lambda_i(\ )$ to indicate the ith eigenvalue of the transfer matrix $\underline{W}$ and products of transfer matrices and where we assume that the σ_i are ordered such that $\sigma_i \ge \sigma_{i+1}$. Note that the singular values of a symplectic matrix will occur in reciprocal pairs σ and $1/\sigma$. Note also that the maximum singular value σ_i of a matrix is usually denoted by $\sigma_{\max}$ and coincides with the spectral norm of a matrix:

$$\sigma_{\max}(\underline{W}) = \max_{z \ne 0} \frac{\|\underline{W}\underline{z}\|_2}{\|\underline{z}\|_2} = \|\underline{W}\|_2 \ ,$$

where $\|\underline{z}\|_2$ is the usual Euclidean length of the vector $\underline{z}$.

APPENDIX B: DERIVATION OF THE MULTICHANNEL LOCALIZATION FACTOR

The derivation of these results begins by recalling Eq. (4) for $j = d$,

$$\gamma_d = \lim_{n \to \infty} \frac{1}{n} \ln \sigma_d(\underline{V}_n) \ .$$

Recalling that the dth singular value of $\underline{V}_n$ is the positive square root of the dth eigenvalue of $\underline{V}_n^\dagger \underline{V}_n$, we have

$$\gamma_d = \lim_{n \to \infty} \frac{1}{2n} \ln \lambda_d(\underline{V}_n^\dagger \underline{V}_n) \ .$$

Recalling Eq. (3) and suppressing subscripts, consider the matrix

$$\underline{V}_n^\dagger \underline{V}_n = \begin{bmatrix} 2(\underline{\tau}\underline{\tau}^\dagger)^{-1} - \underline{I} & -\underline{\rho}^T(\underline{\tau}^*\underline{\tau}^T)^{-1} - (\underline{\tau}\underline{\tau}^\dagger)^{-1}\underline{\rho} \\ -\underline{\rho}^\dagger(\underline{\tau}\underline{\tau}^\dagger)^{-1} - (\underline{\tau}^*\underline{\tau}^T)^{-1}\underline{\rho}^* & 2(\underline{\tau}^*\underline{\tau}^T)^{-1} - \underline{I} \end{bmatrix} . \tag{B1}$$

Here $\underline{V}_n^\dagger \underline{V}_n$ is symplectic, so its eigenvalues will occur in reciprocal pairs $\lambda_1, \ldots, \lambda_d, 1/\lambda_d, \ldots, 1/\lambda_1$, where $\lambda_1 \ge \cdots \ge \lambda_d \ge 1$.

Our analysis will be simplified by recognizing the following [14]:

$$(\underline{V}_n^\dagger \underline{V}_n) + (\underline{V}_n^\dagger \underline{V}_n)^{-1} = \begin{bmatrix} 4(\underline{\tau}\underline{\tau}^\dagger)^{-1} - 2\underline{I} & \underline{0} \\ \underline{0} & 4(\underline{\tau}^*\underline{\tau}^T)^{-1} - 2\underline{I} \end{bmatrix} , \tag{B2}$$

where each block in the matrix is $d \times d$. The matrix has repeated eigenvalues $\lambda_1 + 1/\lambda_1, \ldots, \lambda_d + 1/\lambda_d$ for a total of $2d$ eigenvalues. However, we notice that these eigenvalues are the eigenvalues of the two diagonal blocks of this block diagonal matrix. The eigenvalues of each block are real because both blocks are Hermitian. In addition, each block is the complex conjugate of each other, and real eigenvalues being invariant with respect to complex conjugation, both blocks must have the same eigenvalues.

So the eigenvalues μ_j of $4(\underline{\tau}\underline{\tau}^\dagger)^{-1} - 2\underline{I}$ are

$$\mu_1 = \lambda_1 + \frac{1}{\lambda_1}, \ldots, \mu_d = \lambda_d + \frac{1}{\lambda_d} \ ,$$

where

$$\mu_1 \ge \cdots \ge \mu_d \ .$$

Now let $\mu_j[\ ]$ be the jth eigenvalue of the indicated argument. So,

$$\begin{aligned} \lambda_d + \frac{1}{\lambda_d} &= \mu_d[4(\underline{\tau}\underline{\tau}^\dagger)^{-1} - 2\underline{I}] \\ &= \mu_{\min}[4(\underline{\tau}\underline{\tau}^\dagger)^{-1} - 2\underline{I}] \\ &= 4\mu_{\min}[(\underline{\tau}\underline{\tau}^\dagger)^{-1}] - 2 \ , \end{aligned}$$

where we have used a couple of determinant identities in the last equation. Now, taking the same limit on both sides,

$$\lim_{n \to \infty} \frac{1}{2n} \ln \left[\lambda_d + \frac{1}{\lambda_d} \right] = \lim_{n \to \infty} \frac{1}{2n} \ln \{ 4\mu_{\min}[(\underline{\tau}\underline{\tau}^\dagger)^{-1}] - 2 \} \ .$$

We notice that

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{2n} \ln \left[\lambda_d + \frac{1}{\lambda_d} \right] &= \lim_{n \to \infty} \frac{1}{2n} \ln(\lambda_d) \left[1 + \frac{1}{\lambda_d^2} \right] \\ &= \lim_{n \to \infty} \frac{1}{2n} \ln(\lambda_d) \\ &\quad + \lim_{n \to \infty} \frac{1}{2n} \ln \left[1 + \frac{1}{\lambda_d^2} \right] . \end{aligned}$$

Recalling that $\lambda_d \ge 1$, the second term above must vanish in the limit. So we are left with (recalling the definition of γ_d)

$$\gamma_d = \lim_{n\to\infty} \frac{1}{2n}\ln(\lambda_d)$$
$$= \lim_{n\to\infty} \frac{1}{2n}\ln\{4\mu_{\min}[(\underline{\tau}\,\underline{\tau}^\dagger)^{-1}]-2\} .$$

Note that

$$\mu_{\min}[(\underline{\tau}\,\underline{\tau}^\dagger)^{-1}] = \frac{1}{\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger]} .$$

So we can write

$$\gamma_d = \lim_{n\to\infty} \frac{1}{2n}\ln\left[\frac{4}{\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger]}-2\right]$$

or

$$\gamma_d = \lim_{n\to\infty} \frac{1}{2n}\ln\left[\left[\frac{1}{\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger]}\right](4-2\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger])\right]$$

or

$$\gamma_d = \lim_{n\to\infty} \frac{1}{2n}\ln\left[\frac{1}{\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger]}\right] + \lim_{n\to\infty} \frac{1}{2n}\ln(4-2\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger]) .$$

In Appendix A we showed that $0<\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger]\le 1$, so that the second term above must vanish in the limit.

We are left with

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{2n}\ln\mu_{\max}[\underline{\tau}\,\underline{\tau}^\dagger] ,$$

or recalling the definition of singular values, with probability 1,

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n}\ln\sigma_{\max}(\underline{\tau}) . \tag{B3}$$

As a byproduct of this analysis, we can find all d of the Lyapunov exponents of $\underline{V}_n$ in terms of the transmission matrix $\underline{\tau}$. First, recall that Eq. (4) implies

$$\gamma_j = \lim_{n\to\infty} \frac{1}{2n}\ln(\lambda_j), \quad 1\le j\le d$$

and from earlier in this appendix,

$$\lambda_j+\frac{1}{\lambda_j} = \mu_j[4(\underline{\tau}\,\underline{\tau}^\dagger)^{-1}-2\underline{I}]$$
$$=4\mu_j[(\underline{\tau}\,\underline{\tau}^\dagger)^{-1}]-2 .$$

Note here that

$$\mu_j[(\underline{\tau}\,\underline{\tau}^\dagger)^{-1}] = \frac{1}{\mu_{d-j+1}[\underline{\tau}\,\underline{\tau}^\dagger]}, \quad 1\le j\le d .$$

So taking limits on both sides and discarding vanishing terms we find, with probability 1,

$$\gamma_j = -\lim_{n\to\infty} \frac{1}{n}\ln\sigma_{d-j+1}(\underline{\tau}), \quad 1\le j\le d .$$

This reproduces our result for γ_d, and also tells us that

$$\gamma_1 = -\lim_{n\to\infty} \frac{1}{n}\ln\sigma_d(\underline{\tau})$$
$$= -\lim_{n\to\infty} \frac{1}{n}\ln\sigma_{\min}(\underline{\tau}) .$$

Now we return to examining γ_d and proceed to show that, in addition to Eq. (B3),

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n}\ln[\mathrm{tr}(\underline{\tau}\,\underline{\tau}^\dagger)]^{1/2} .$$

First, examine

$$-\lim_{n\to\infty} \frac{1}{2n}\ln \mathrm{tr}(\underline{\tau}\,\underline{\tau}^\dagger) .$$

Take an eigenvector decomposition of the Hermitian matrix $\underline{\tau}\,\underline{\tau}^\dagger$ and rewrite this as

$$-\lim_{n\to\infty} \frac{1}{2n}\ln \mathrm{tr}[\underline{U}\,\mathrm{diag}(\mu_i)\underline{U}^\dagger] ,$$

where $\underline{U}$ is a unitary matrix. Recalling that $\mathrm{tr}(\underline{A}\,\underline{B}\,\underline{C})=\mathrm{tr}(\underline{B}\,\underline{C}\,\underline{A})$ for compatible matrices, we see that the above limit equals

$$-\lim_{n\to\infty} \frac{1}{2n}\ln \mathrm{tr}[\mathrm{diag}(\mu_i)]$$

or

$$-\lim_{n\to\infty} \frac{1}{2n}\ln(\mu_1+\cdots+\mu_d)$$

or

$$-\lim_{n\to\infty} \frac{1}{2n}\ln\left[\mu_1\left[1+\frac{\mu_2}{\mu_1}+\cdots+\frac{\mu_d}{\mu_1}\right]\right] .$$

Recalling that $\mu_1\ge\cdots\ge\mu_d>0$, we have that the term in the large parentheses is finite and bounded below by 1 and above by d, so when taking the limit, we are left with

$$-\lim_{n\to\infty} \frac{1}{2n}\ln\mu_1[\underline{\tau}\,\underline{\tau}^\dagger] ,$$

which is precisely equal to

$$-\lim_{n\to\infty} \frac{1}{n}\ln\sigma_{\max}(\underline{\tau}) = \gamma_d .$$

Thus we have indeed shown that, with probability 1,

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n}\ln[\mathrm{tr}(\underline{\tau}\,\underline{\tau}^\dagger)]^{1/2} . \tag{B4}$$

One final simplification in our result is now possible. Starting with

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{2n}\ln[\mathrm{tr}(\underline{\tau}\,\underline{\tau}^\dagger)] ,$$

let τ_{ij} be the ijth element of the matrix $\underline{\tau}$. Now (this is the square of the Frobenius norm of $\underline{\tau}$),

$$\mathrm{tr}(\underline{\tau}\,\underline{\tau}^\dagger) = \sum_{i=1}^{d}\sum_{j=1}^{d}|\tau_{ij}|^2$$
$$=|\tau_{11}|^2+|\tau_{12}|^2+\cdots+|\tau_{dd}|^2 .$$

We have that for one element of $\underline{\tau}$, $|\tau_{ij}|\ge|\tau_{kl}|$, $k\ne i$,

$l \neq j$, and we will denote it $|\tau_{ij}|_{\max}$. So,

$$\mathrm{tr}(\underline{\tau}\,\underline{\tau}^{\dagger}) = |\tau_{ij}|^2_{\max} \left[\sum_{i=1}^{d} \sum_{j=1}^{d} \frac{|\tau_{ij}|^2}{|\tau_{ij}|^2_{\max}} \right] .$$

So,

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{2n} \ln \left[|\tau_{ij}|^2_{\max} \left[\sum_{i=1}^{d} \sum_{j=1}^{d} \frac{|\tau_{ij}|^2}{|\tau_{ij}|^2_{\max}} \right] \right] ,$$

and because the term in the large parentheses is finite and bounded below by 1 and above by d^2, it vanishes after taking the limit, so we are left with, with probability 1,

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{2n} \ln |\tau_{ij}|^2_{\max}$$

or

$$\gamma_d = -\lim_{n\to\infty} \frac{1}{n} \ln |\tau_{ij}|_{\max} . \quad \text{(B5)}$$

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