Transition operators in acoustic-wave diffraction theory. II. Short-wavelength behavior, dominant singularities of $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$

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This second paper of a series deals with special aspects of transition- (T-) operator theory for diffraction of time-harmonic, acoustic scalar waves from an impenetrable obstacle with surface $\partial\Omega$. It was shown in the first paper [G. E. Hahne, preceding paper, Phys. Rev. A **43**, 976 (1991)] that the computation of the T operator and complete Green's function for the case of "sound-hard" (Neumann-type) and "sound-soft" (Dirichlet-type) boundary conditions on $\partial\Omega$ reduces to the determination of the "radiation impedance" operator $\check{Z}_{k_0}$ and the "radiation admittance" operator $\check{Z}_{k_0}^{-1}$, respectively, characteristic of $\partial\Omega$ at wave number k_0. In this paper, the short-wavelength and the long-wavelength behavior of these operators as two-point kernels on a smooth $\partial\Omega$ are studied for pairs of points that are close together. First, an exact, closed-form expression is obtained for both $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ for $\partial\Omega$=a plane, on the basis of which a "tangent-plane" approximation to $\check{Z}_{k_0}$ and to $\check{Z}_{k_0}^{-1}$ for diffraction from a general smooth-surfaced, convex obstacle is proposed. This approximation is shown to lead, by means of the method of stationary phase, to the familiar "physical optics" approximation and to the geometrical acoustics limit for diffraction with Neumann-type and with Dirichlet-type boundary conditions. Second, the dominant singularities in $\check{Z}_{k_0}$ and in $\check{Z}_{k_0}^{-1}$ are obtained for smooth $\partial\Omega$, and the results are compared to analogous results for $\partial\Omega$ as a sphere inferred from the spherical harmonic expansions of the two operators.

I. INTRODUCTION

This paper is the second of a series. It is concerned with applications of the general theory formulated in Ref. 1 for the transition- (T-) operator approach to the diffraction of time-harmonic acoustic scalar waves from fixed, impenetrable obstacles, with given boundary conditions for the wave function on the obstacle's surface. In paper I the theory was derived for general homogeneous impedance, or Robin (for the mathematician Gustave Robin), surface boundary conditions (SBC's). In this paper, we shall treat only the two important special cases of homogeneous Neumann (N) and homogeneous Dirichlet (D) SBC's. These are also known as "sound-hard" and "sound-soft" SBC's, respectively. In paper I, the detailed geometry of the obstacle was left general; in this paper, we specialize to obstacles whose bounding surface is smooth, that is, has a continuous local curvature matrix. In that part of the paper dealing with short-wavelength approximations, we assume also that the obstacle is convex. Our objective here is to work out the theory of paper I given these simple geometries and boundary conditions, partly in order to show that the formalism implies the known "physical optics" method and geometrical acoustics (also called ray acoustics) as limiting cases, and partly in order to exhibit explicitly the dominant singularities of certain operators that play an essential role in the formalism.

Let the interior Ω, the exterior Ω^{ex}, and the surface $\partial\Omega$ be, respectively, the subsets of Euclidean three-space E^3 occupied by the obstacle, by the uniform sound-transmitting fluid, and by the boundary of both of these two open sets. We fix a rectangular Cartesian coordinate system, and associate points in E^3 with three-vectors written $\mathbf{r}$, $\mathbf{r}_1$, etc.; points in E^3 that are also in $\partial\Omega$ will be denoted by $\mathbf{r}_\partial$, $\mathbf{r}_{\partial 1}$, and so on. The differential of volume of E^3 is denoted d^3r, and that of area of $\partial\Omega$ by dA, possibly with subscripts. The vector $\hat{\mathbf{n}}(\mathbf{r}_\partial)$ is the unit outward-pointing (toward Ω^{ex}) normal vector to $\partial\Omega$ at $\mathbf{r}_\partial$.

We sometimes deal with functions on E^3 that are discontinuous across $\partial\Omega$. The limiting values of such functions on $\partial\Omega$ are normally taken from values in Ω^{ex}, and are signified explicitly by affixing a $+$ to the function's argument: $f(\mathbf{r})\rightarrow f(\mathbf{r}_\partial+)$ with $\mathbf{r}\in\Omega^{ex}$; $f(\mathbf{r}_\partial-)$ is the limit as $\mathbf{r}\in\Omega$ approaches $\mathbf{r}_\partial$.

Let c be the constant speed of sound in Ω^{ex}, and let k_0, $-\infty<k_0<+\infty$, be the wave number of a time-harmonic sound wave, with $\omega=k_0c$ as the associated angular frequency. The fundamental assumption in paper I was the existence of a T operator, called $T_{Nk_0}^{+}$ or $T_{Dk_0}^{+}$, such that Eq. (1) in paper I [labeled Eq. (I-1)] enabled the computation of the complete Green's function (Ref. 2, p. 806) for the diffracting system to be reduced to quadratures. The $+$ superscripts indicate that we are dealing with a causal diffraction problem, and, accordingly, with outgoing-wave boundary conditions [Ref. 3, Eq. (3.7)] on Green's functions as $r\rightarrow\infty$. The results of paper I for the complete Green's functions $G_{Nk_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)$ and $G_{Dk_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)$ in terms of the (obstacle-) free-space Green's function $G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)$ and the respective T operator are given in Eqs. (I-59) and (I-60). The only nonelementary operators involved in the latter formulas are $\check{Z}_{k_0}$, called the "radiation impedance" operator, in the N case and its inverse $\check{Z}_{k_0}^{-1}$, called the "radiation admittance" operator, in the D case. These nonlocal operators are defined in Eq. (I-28), and map the space of complex-valued functions, whose domain of definition is $\partial\Omega$, into itself linearly. Briefly, they are defined as follows: given outgoing-wave boundary conditions as $r \to \infty$ for a wave function satisfying the scalar Helmholtz equation in Ω^{ex}, the wave function is uniquely specified in Ω^{ex} by either the set of its exterior limiting values as $\mathbf{r} \to \mathbf{r}_\partial +$, or by the set of its exterior limiting normal gradients as $\mathbf{r} \to \mathbf{r}_\partial +$ (Ref. 3, Theorems 3.13, 3.21, and 3.25). Accordingly, the limiting-wave function on $\partial\Omega$ is uniquely determined by the limiting-normal-derivative function on $\partial\Omega$, and the converse is true as well; hence there must exist an invertible linear functional operator, which we designate by $\check{Z}_{k_0}$, that maps the latter function into the former for any allowable wave function.

The remainder of this paper is organized as follows. In Sec. II we shall determine closed-form expressions for $\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ and $\check{Z}_{k_0}^{-1}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ for the case that $\partial\Omega$ is a plane; these results in turn suggest straightforwardly what will be called the "tangent-plane" approximation for the respective operator, given that $\partial\Omega$ is smooth, and that $|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|$ is small compared to the magnitudes of the local radii of curvature of planar slices of $\partial\Omega$. In Sec. III we shall combine the method of stationary phase and the tangent-plane approximation to show that, when Ω is convex, the physical postulates that encompass the basis of the familiar physical optics method (cf. Ref. 4, Chap. I.2.13.4) are obtained asymptotically as $|k_0| \to \infty$. In Sec. IV, we shall apply the results of Sec. III to derive further consequences of the approximation scheme, that is, we verify that physical optics as formulated here, plus the method of stationary phase, yields the geometrical acoustics limit (Ref. 4, p. 30 and Chap. I.2.13.1) for $G_{Nk_0}^{+}$ and $G_{Dk_0}^{+}$, and yields asymptotically as $|k_0| \to \infty$ the so-called "extended boundary conditions" Eq. (I-16) for the complete Green's functions, for diffraction from smooth-surfaced, convex obstacles. In Sec. V we shall work out, in what amounts to a small $|k_0|$ approximation, corrections to the tangent-plane approximations for $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ that are linear in the local curvature matrix of $\partial\Omega$. These results are compared with corresponding results for spherical $\partial\Omega$ derived in the Appendix by a different method. Section VI contains a discussion of work accomplished and of directions for further investigation. Finally, in the Appendix we shall obtain infinite-sum expansions, in terms of spherical harmonics, for $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ for the case that $\partial\Omega$ is a sphere of radius a, called $S^2(a)$. A small-$|k_0|$ expansion of these operators is used to derive closed-form expressions for enough terms that the remainder is a continuous kernel on $S^2(a)$.

II. THE TANGENT-PLANE APPROXIMATION FOR $\check{Z}_{k_0}$ AND FOR $\check{Z}_{k_0}^{-1}$

According to Eqs. (I-32) and (I-34) we have

$$\check{Z}_{k_0} = -U_{k_0}(I_\partial - V_{k_0}^{\tau})^{-1} , \tag{1}$$

$$\check{Z}_{k_0}^{-1} = W_{k_0}(I_\partial + V_{k_0})^{-1} , \tag{2}$$

where the operators U_{k_0}, V_{k_0}, $V_{k_0}^{\tau}$, W_{k_0}, and I_∂ are defined in Eqs. (I-18)–(I-21) and following (I-9). If $\partial\Omega$ is a plane, then V_{k_0} and $V_{k_0}^{\tau}$ are both the zero operator, so that in this geometrical case the formulas

$$\check{Z}_{k_0} = -U_{k_0} , \tag{3}$$

$$\check{Z}_{k_0}^{-1} = W_{k_0} , \tag{4}$$

are exact. If $\partial\Omega$ is not planar but is smooth and convex, and if $|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|$ is small compared to the local curvature radii of $\partial\Omega$, then V_{k_0} and $V_{k_0}^{\tau}$ are in a sense small—see the discussion in Sec. V—and we are led to define the "tangent-plane" approximations $\tilde{Z}_{k_0}$ and $\tilde{Z}_{k_0}^{-1}$ as follows:

$$\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \approx \tilde{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \equiv -U_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) , \tag{5}$$

$$\check{Z}_{k_0}^{-1}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \approx \tilde{Z}_{k_0}^{-1}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \equiv W_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) . \tag{6}$$

III. DERIVATION OF THE PHYSICAL OPTICS METHOD

In this section we shall use the method of stationary phase to show that, when $\partial\Omega$ is smooth and convex, asymptotically as $|k_0| \to \infty$ the approximate radiation impedance and admittance operators of Eqs. (5) and (6) yield an equivalent of the physical optics method (Ref. 4, Chap. I.2.13.4) for diffraction. The results of this section will be applied in Sec. IV to show that the extended boundary conditions Eq. (I-16), and the respective geometrical acoustics limit, are obtained for the Green's functions $G_{Nk_0}^{+}$ and $G_{Dk_0}^{+}$ asymptotically as $|k_0| \to \infty$ when the approximate intermediate results are applied to Eqs. (I-59) and (I-60).

It proves to be the case that all the integrals needed for the present purpose and for Sec. IV are derivable from the following preliminary computation (we use the notational conventions of paper I, Sec. II), which is to be evaluated with the aid of the method of stationary phase (Ref. 5, Appendix III.3):

$$\Phi_{k_0}(\mathbf{r}_1;\mathbf{r}_2) \equiv \int_{\partial\Omega} G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_{\partial 3}) \frac{\partial G_{k_0}^{+}}{\partial n_l}(\mathbf{r}_{\partial 3};\mathbf{r}_2)\, dA_3 \tag{7}$$

$$= (16\pi^2)^{-1} \int_{\partial\Omega} |\mathbf{r}_1-\mathbf{r}_{\partial 3}|^{-1} |\mathbf{r}_{\partial 3}-\mathbf{r}_2|^{-1} [\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot(\mathbf{r}_2-\mathbf{r}_{\partial 3})/|\mathbf{r}_2-\mathbf{r}_{\partial 3}|] \times(-ik_0+1/|\mathbf{r}_2-\mathbf{r}_{\partial 3}|)\exp[ik_0 d(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)]\, dA_3 , \tag{8}$$

where we used the definition

$$d(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)\equiv|\mathbf{r}_1-\mathbf{r}_{\partial 3}|+|\mathbf{r}_{\partial 3}-\mathbf{r}_2| \;. \tag{9}$$

In what follows, we shall replace the factor $(-ik_0+1/|\mathbf{r}_2-\mathbf{r}_{\partial 3}|)$ in the integrand of Eq. (8) by $(-ik_0)$. This simplification is justified by the assumption that $|k_0|\to\infty$ for those applications that $\mathbf{r}_2$ remains at a nonzero distance from $\partial\Omega$. In another application, we will need the limiting normal derivative of Eq. (7) with respect to $\mathbf{r}_2$ as $\mathbf{r}_2\to\mathbf{r}_{\partial 2}+$. An estimate shows that in this case also the neglected terms are of order one higher power of $1/|k_0|$ than the terms retained.

Now fix $\mathbf{r}_1$ and $\mathbf{r}_2$ and let $\mathbf{r}_{\partial 3}$ range over $\partial\Omega$. The function $d(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ of Eq. (9) will have one or more stationary points. These can be of two types, either points of $\partial\Omega$ that are on the straight-line segment between $\mathbf{r}_1$ and $\mathbf{r}_2$, or points of tangency of $\partial\Omega$ with any of the one-parameter family of prolate ellipsoids of rotation having $\mathbf{r}_1$ and $\mathbf{r}_2$ as foci. Let $\mathbf{r}_{\partial a}$ be such a stationary point. For convenience, we define the following:

$$\begin{aligned}
&\hat{\mathbf{n}}(\mathbf{r}_{\partial a})=\hat{\mathbf{n}}_a \;,\\
&\mathbf{r}_1-\mathbf{r}_{\partial a}=r_{1a}\hat{\mathbf{r}}_{1a} \;,\\
&\mathbf{r}_2-\mathbf{r}_{\partial a}=r_{2a}\hat{\mathbf{r}}_{2a} \;.
\end{aligned} \tag{10}$$

Also, let $\hat{\mathbf{t}}_{a1}$ and $\hat{\mathbf{t}}_{a2}$ be tangent vectors to $\partial\Omega$ at $\mathbf{r}_{\partial a}$ such that $(\hat{\mathbf{t}}_{a1};\hat{\mathbf{t}}_{a2};\hat{\mathbf{n}}_a)$ is a right-handed triple of orthogonal unit vectors. Since $d(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ is stationary at $\mathbf{r}_{\partial 3}=\mathbf{r}_{\partial a}$, we must have

$$\begin{aligned}
&\hat{\mathbf{r}}_{1a}\cdot\hat{\mathbf{t}}_{a\alpha}=-\hat{\mathbf{r}}_{2a}\cdot\hat{\mathbf{t}}_{a\alpha} \quad \text{for } \alpha=1,2 \;,\\
&|\hat{\mathbf{r}}_{1a}\cdot\hat{\mathbf{n}}_a|=|\hat{\mathbf{r}}_{2a}\cdot\hat{\mathbf{n}}_a| \;.
\end{aligned} \tag{11}$$

We introduce local coordinates (u^1,u^2) into $\partial\Omega$ in a neighborhood of $\mathbf{r}_{\partial a}$ such that $\partial\Omega$ is given by

$$\begin{aligned}
\mathbf{r}_{\partial 3}&=\mathbf{R}_a(u^1,u^2)\\
&=\mathbf{r}_{\partial a}+\sum_{\alpha=1}^{2}u^\alpha\hat{\mathbf{t}}_{a\alpha}-\tfrac{1}{2}\sum_{\alpha,\beta=1}^{2}K_{a\alpha\beta}u^\alpha u^\beta\hat{\mathbf{n}}_a+O^3(u^1,u^2) \;,
\end{aligned} \tag{12}$$

where $K_{a\alpha\beta}$ is the 2×2 curvature matrix[6] of $\partial\Omega$ at $\mathbf{r}_{\partial a}$, and $O^3(u^1,u^2)$ is a term of third order in u^1,u^2. In what follows, we omit such third- and higher-order terms in u^1,u^2. In this quadratic approximation, we find for the local 2×2 metric tensor $g_{\alpha\beta}$, its inverse $g^{\alpha\beta}$, its determinant g, and the area element dA

$$\begin{aligned}
&g_{\alpha\beta}(u^1,u^2)=\delta_{\alpha\beta}+\sum_{\gamma,\zeta}K_{a\alpha\gamma}K_{a\beta\zeta}u^\gamma u^\zeta \;,\\
&g^{\alpha\beta}(u^1,u^2)=\delta_{\alpha\beta}-\sum_{\gamma,\zeta}K_{a\alpha\gamma}K_{a\beta\zeta}u^\gamma u^\zeta \;,\\
&g(u^1,u^2)=1+\sum_{\alpha,\beta,\gamma}K_{a\alpha\beta}K_{a\alpha\gamma}u^\beta u^\gamma \;,\\
&dA=[g(u^1,u^2)]^{1/2}du^1du^2 \;.
\end{aligned} \tag{13}$$

The local outward normal vector is

$$\hat{\mathbf{n}}(u^1,u^2)=[g(u^1,u^2)]^{-1/2}\left[\hat{\mathbf{n}}_a+\sum_{\alpha,\beta}K_{a\alpha\beta}u^\alpha\hat{\mathbf{t}}_{a\beta}\right] . \tag{14}$$

We can expand $d(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ about $u^1=u^2=0$, retaining only terms up to second order in u^1,u^2:

$$d(\mathbf{r}_1;\mathbf{R}_a(u^1,u^2);\mathbf{r}_2)=r_{1a}+r_{2a}+\sum_{\alpha,\beta}\Lambda_{a\alpha\beta}u^\alpha u^\beta \;, \tag{15}$$

where

$$\begin{aligned}
\Lambda_{a\alpha\beta}\equiv\tfrac{1}{2}\{&(\hat{\mathbf{r}}_{1a}+\hat{\mathbf{r}}_{2a})\cdot\hat{\mathbf{n}}_aK_{a\alpha\beta}\\
&+(1/r_{1a})[\delta_{\alpha\beta}-(\hat{\mathbf{t}}_{a\alpha}\cdot\hat{\mathbf{r}}_{1a})(\hat{\mathbf{t}}_{a\beta}\cdot\hat{\mathbf{r}}_{1a})]\\
&+(1/r_{2a})[\delta_{\alpha\beta}-(\hat{\mathbf{t}}_{a\alpha}\cdot\hat{\mathbf{r}}_{2a})(\hat{\mathbf{t}}_{a\beta}\cdot\hat{\mathbf{r}}_{2a})]\} \;.
\end{aligned} \tag{16}$$

Let the eigenvalues of the symmetric, 2×2 matrix $\Lambda_{a\alpha\beta}$ be λ_{a1} and λ_{a2}, where both are assumed to be nonzero,[7] and define

$$\begin{aligned}
&\sigma_{a\alpha}\equiv k_0\lambda_{a\alpha}/|k_0\lambda_{a\alpha}| \quad \text{for } \alpha=1,2 \;,\\
&\Lambda_\alpha\equiv\det(\Lambda_{a\alpha\beta})=\lambda_{a1}\lambda_{a2} \;.
\end{aligned} \tag{17}$$

The method of stationary phase [Ref. 5, Appendix III, Eqs. (17)–(21)] now leads to the following approximate value for the integral of Eq. (8):

$$\begin{aligned}
\Phi_{k_0}(\mathbf{r}_1;\mathbf{r}_2)\underset{|k_0|\to\infty}{\sim}\sum_a\{&(-ik_0/|k_0|)(\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{2a})(16\pi r_{1a}r_{2a}|\Lambda_a|^{1/2})^{-1}\\
&\times\exp[i(\pi/4)(\sigma_{a1}+\sigma_{a2})+ik_0(r_{1a}+r_{2a})]\}+o(|k_0|^0) \;,
\end{aligned} \tag{18}$$

where the sum is over all stationary points $\mathbf{r}_{\partial 3}=\mathbf{r}_{\partial a}$ of $d(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ as $\mathbf{r}_{\partial 3}$ ranges over $\partial\Omega$.

We shall need to extract two limits from Eq. (18): (i) the limit with $\mathbf{r}_2$ fixed, as $\mathbf{r}_1\in\Omega^{\text{ex}}$ tends to $\mathbf{r}_{\partial 1}$; (ii) the limiting exterior normal derivative with respect to $\mathbf{r}_2\in\Omega^{\text{ex}}$ as $\mathbf{r}_2\to\mathbf{r}_{\partial 2}\in\partial\Omega$, that is, if $\mathbf{r}_1\in\Omega\cup\Omega^{\text{ex}}$ is fixed, we need the following derivative as ϵ approaches zero from positive values:

$$\lim_{\epsilon\to 0+}\frac{\partial}{\partial\epsilon}\Phi_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2}+\epsilon\hat{\mathbf{n}}(\mathbf{r}_{\partial 2})) . \tag{19}$$

In case (i), it is convenient to take the limit by allowing $\mathbf{r}_1$ to approach $\partial\Omega$ so that $\mathbf{r}_{\partial 1}=\mathbf{r}_{\partial a}$ is a stationary phase point that does not change in the limiting process. Then $\hat{\mathbf{r}}_{1a}$ is fixed, and we have

$$\Lambda_{a\alpha\beta}\underset{r_{1a}\to 0}{\sim}(2r_{1a})^{-1}[\delta_{\alpha\beta}-(\hat{\mathbf{t}}_{a\alpha}\cdot\hat{\mathbf{r}}_{2a})(\hat{\mathbf{t}}_{a\beta}\cdot\hat{\mathbf{r}}_{2a})] , \tag{20}$$

where we have used Eq. (11). Hence we have for $r_{1a}\to 0$

$$\begin{aligned}
&\lambda_{a1}\underset{r_{1a}\to 0}{\sim}1/(2r_{1a}) ,\\
&\lambda_{a2}\underset{r_{1a}\to 0}{\sim}\lambda_{a1}(\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{2a})^2 ,\\
&|\Lambda_a|^{1/2}\underset{r_{1a}\to 0}{\sim}|\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{2a}|/(2r_{1a}) ,\\
&\sigma_{a1}=\sigma_{a2}=k_0/|k_0| .
\end{aligned} \tag{21}$$

Thus the approximate limit of Eq. (18) is, in view of Eq. (I-14),

$$\Phi_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_2)\underset{|k_0|\to\infty}{\sim}-\tfrac{1}{2}[\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\cdot(\mathbf{r}_2-\mathbf{r}_{\partial 1})/|\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\cdot(\mathbf{r}_2-\mathbf{r}_{\partial 1})|]G^+_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_2) . \tag{22}$$

In case (ii), we retain only the dominant term in k_0, and hence will need only the derivative of $d(\mathbf{r}_1;\mathbf{r}_{\partial a};\mathbf{r}_2)$ as $\epsilon\to 0+$, where $\mathbf{r}_{\partial a}$ depends on $\mathbf{r}_2$ and hence on ϵ. We find that

$$\left.\frac{\partial d}{\partial\epsilon}(\mathbf{r}_1;\mathbf{r}_{\partial a};\mathbf{r}_2)\right|_{\epsilon=0+}=|\hat{\mathbf{n}}(\mathbf{r}_{\partial 2})\cdot(\mathbf{r}_1-\mathbf{r}_{\partial 2})|/|\mathbf{r}_1-\mathbf{r}_{\partial 2}| . \tag{23}$$

Hence we have

$$\frac{\partial\Phi_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 2}+)\underset{|k_0|\to\infty}{\sim}\tfrac{1}{2}[\hat{\mathbf{n}}(\mathbf{r}_{\partial 2})\cdot(\mathbf{r}_1-\mathbf{r}_{\partial 2})/|\hat{\mathbf{n}}(\mathbf{r}_{\partial 2})\cdot(\mathbf{r}_1-\mathbf{r}_{\partial 2})|]\frac{\partial G^+_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 2}) . \tag{24}$$

We define the step function $\Delta(\mathbf{r}_{\partial 1};\mathbf{r}_2)$ as

$$\Delta(\mathbf{r}_{\partial 1};\mathbf{r}_2)\equiv 1+\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\cdot(\mathbf{r}_2-\mathbf{r}_{\partial 1})/|\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\cdot(\mathbf{r}_2-\mathbf{r}_{\partial 1})| . \tag{25}$$

Note that when $\mathbf{r}_2\in\Omega^{\text{ex}}$ and $\mathbf{r}_{\partial 1}$ is in that part of $\partial\Omega$ that is "illuminated" by radial sound rays emanating from $\mathbf{r}_2$, then $\Delta(\mathbf{r}_{\partial 1};\mathbf{r}_2)=+2$, while if $\mathbf{r}_2\in\Omega$, or if $\mathbf{r}_{\partial 1}$ is on the shadow side of $\partial\Omega$ with respect to $\mathbf{r}_2\in\Omega^{\text{ex}}$, then $\Delta(\mathbf{r}_{\partial 1};\mathbf{r}_2)=0$.

We will generalize the statement of the physical optics method, from that of Ref. 4, Chap. I.2.13.4, to the following asymptotic approximations, which are suggested by the exact results for $\partial\Omega$ = a plane and by a physical intuition for diffraction at short wavelengths:

$$G^+_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_2)-\left[\tilde{Z}_{k_0}\frac{\partial G^+_{k_0}}{\partial n_l}\right](\mathbf{r}_{\partial 1};\mathbf{r}_2)\underset{|k_0|\to\infty}{\sim}\Delta(\mathbf{r}_{\partial 1};\mathbf{r}_2)G^+_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_2) , \tag{26}$$

$$\frac{\partial G^+_{k_0}}{\partial n_l}(\mathbf{r}_{\partial 1};\mathbf{r}_2)-[\tilde{Z}^{-1}_{k_0}G^+_{k_0}](\mathbf{r}_{\partial 1};\mathbf{r}_2)\underset{|k_0|\to\infty}{\sim}\Delta(\mathbf{r}_{\partial 1};\mathbf{r}_2)\frac{\partial G^+_{k_0}}{\partial n_l}(\mathbf{r}_{\partial 1};\mathbf{r}_2) . \tag{27}$$

That is, when $\mathbf{r}_2\in\Omega^{\text{ex}}$ is in the illuminated part (respectively, shadow part) of $\partial\Omega$ the left-hand side of Eqs. (26) and (27) are two times (respectively, zero times) the free-space function given by the first summand on the left-hand side.

We have, therefore, confirmed that the approximations Eqs. (5) and (6) entail the asymptotic validity of the *Ansätze* of physical optics as formulated in Eqs. (26) and (27). Note that if the exact $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ are used, the left-hand sides of Eqs. (26) and (27) should be exactly zero when $\mathbf{r}_2\in\Omega$, by Eq. (I-28). It is now plausible, but we shall not attempt here to prove, that the error terms $\tilde{Z}_{k_0}-\check{Z}_{k_0}$ and $\tilde{Z}^{-1}_{k_0}-\check{Z}^{-1}_{k_0}$ are asymptotically small as $|k_0|\to\infty$, so that Eqs. (26) and (27) would also hold asymptotically if $\mathbf{r}_2\in\Omega^{\text{ex}}$ and the exact $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ are used in the left-hand sides.

IV. EXTENDED BOUNDARY CONDITIONS AND GEOMETRICAL ACOUSTICS LIMITS

We shall verify, using the results of Sec. III, that the operator approximations Eqs. (5) and (6) applied to Eqs. (I-59) for $G^+_{Nk_0}$ and (I-60) for $G^+_{Dk_0}$ yield the correct limiting functions asymptotically as $|k_0|\to\infty$.

With the aid of Eqs. (I-56) and (I-57), Eqs. (I-59) and (I-60) can be recast in a form that makes it convenient to apply the results of Sec. III:

$$G^{+}_{Nk_0}(\mathbf{r}_1;\mathbf{r}_2)=[1-\tfrac{1}{2}\Theta_\Omega(\mathbf{r}_1)-\tfrac{1}{2}\Theta_\Omega(\mathbf{r}_2)]G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)-\frac{1}{2}\left[\frac{\partial G^{+}_{k_0}}{\partial n_r}\left[I_\partial G^{+}_{k_0}-\check{Z}_{k_0}\frac{\partial G^{+}_{k_0}}{\partial n_r}\right]\right](\mathbf{r}_1;\mathbf{r}_2)$$
$$-\frac{1}{2}\left[\left[G^{+}_{k_0}I_\partial-\frac{\partial G^{+}_{k_0}}{\partial n_r}\check{Z}_{k_0}\right]\frac{\partial G^{+}_{k_0}}{\partial n_l}\right](\mathbf{r}_1;\mathbf{r}_2)\ , \tag{28}$$

$$G^{+}_{Dk_0}(\mathbf{r}_1;\mathbf{r}_2)=[1-\tfrac{1}{2}\Theta_\Omega(\mathbf{r}_1)-\tfrac{1}{2}\Theta_\Omega(\mathbf{r}_2)]G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)+\frac{1}{2}\left[\left[\frac{\partial G^{+}_{k_0}}{\partial n_r}I_\partial-G^{+}_{k_0}\check{Z}^{-1}_{k_0}\right]G^{+}_{k_0}\right](\mathbf{r}_1;\mathbf{r}_2)$$
$$+\frac{1}{2}\left[G^{+}_{k_0}\left[I_\partial\frac{\partial G^{+}_{k_0}}{\partial n_l}-\check{Z}^{-1}_{k_0}G^{+}_{k_0}\right]\right](\mathbf{r}_1;\mathbf{r}_2)\ . \tag{29}$$

We now apply to Eqs. (28) and (29) the approximate results Eqs. (26) and (27), and the corresponding equations with the roles of the first and second arguments interchanged. It is evident that zero will be obtained [to order $o(|k_0|^0)$] on the right-hand side of Eqs. (28) and (29) when both $\mathbf{r}_1\in\Omega$ and $\mathbf{r}_2\in\Omega$.

We consider next the cases that either $\mathbf{r}_1\in\Omega$ and $\mathbf{r}_2\in\Omega^{\rm ex}$, or $\mathbf{r}_1\in\Omega^{\rm ex}$ and $\mathbf{r}_2\in\Omega$, or $\mathbf{r}_1\in\Omega^{\rm ex}$ and $\mathbf{r}_2\in\Omega^{\rm ex}$ with $\mathbf{r}_1$ and $\mathbf{r}_2$ mutually "invisible." For these geometrical cases, two types of ray trajectories can arise connecting $\mathbf{r}_1$ and $\mathbf{r}_2$: First are the broken line segments (of which more than one can occur, one for each point of stationary phase) that reflect from the interior of $\partial\Omega$; these do not contribute to the implied integrals on the right-hand sides of Eqs. (28) and (29) because of Eqs. (25)–(27). Second are the rays that are straight-line segments between $\mathbf{r}_1$ and $\mathbf{r}_2$, and which intersect $\partial\Omega$ in one or two points (each such intersection is a point of stationary phase). We need to work out Eq. (18) in more detail for stationary phase points of this type.

For a stationary phase point on a straight-line ray, we can augment Eq. (11) by

$$\hat{\mathbf{r}}_{1a}+\hat{\mathbf{r}}_{2a}=\mathbf{0}\ , \tag{30}$$

so that Eq. (16) becomes

$$\Lambda_{a\alpha\beta}=(r_{1a}+r_{2a})(2r_{1a}r_{2a})^{-1}[\delta_{\alpha\beta}-(\hat{\mathbf{t}}_{a\alpha}\cdot\hat{\mathbf{r}}_{2a})(\hat{\mathbf{t}}_{a\beta}\cdot\hat{\mathbf{r}}_{2a})]\ ; \tag{31}$$

hence

$$\begin{aligned}&\lambda_{a1}=(r_{1a}+r_{2a})/(2r_{1a}r_{2a})\ ,\\&\lambda_{a2}=\lambda_{a1}(\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{2a})^2\ ,\\&|\Lambda_a|^{1/2}=\lambda_{a1}|\hat{\mathbf{n}}_a\cdot\mathbf{r}_{2a}|\ ,\\&\sigma_{a1}=\sigma_{a2}=k_0/|k_0|\ .\end{aligned} \tag{32}$$

Note that in this geometrical case we have

$$r_{1a}+r_{2a}=|\mathbf{r}_1-\mathbf{r}_2|\ . \tag{33}$$

Accordingly, Eq. (18) reduces to

$$\Phi_{k_0}(\mathbf{r}_1;\mathbf{r}_2)=-\tfrac{1}{2}(\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{2a}/|\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{2a}|)G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)\ . \tag{34}$$

We can now compute, say, the right-hand side of Eq. (28) when $\mathbf{r}_1\in\Omega$ and $\mathbf{r}_2\in\Omega^{\rm ex}$. Making use of Eq. (26), we find that

$$G^{+}_{Nk_0}(\mathbf{r}_1;\mathbf{r}_2)=\tfrac{1}{2}G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)-\int_{\partial\Omega_2}\frac{\partial G^{+}_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 3})G^{+}_{k_0}(\mathbf{r}_{\partial 3};\mathbf{r}_2)dA_3\ , \tag{35}$$

where $\partial\Omega_2$ is that part of $\partial\Omega$ that is illuminated by a source at $\mathbf{r}_2$. Since the line segment between $\mathbf{r}_1$ and $\mathbf{r}_2$ intersects $\partial\Omega$ within $\partial\Omega_2$ the stationary phase point lies in $\partial\Omega_2$, and the integral on the right-hand side of Eq. (35) is effectively a transposed case of the integral in Eq. (7), and can be estimated by means of Eq. (34) with $\mathbf{r}_1$ and $\mathbf{r}_2$ interchanged; since $\hat{\mathbf{n}}_a\cdot\hat{\mathbf{r}}_{1a}<0$ in this case, we find that

$$G^{+}_{Nk_0}\underset{|k_0|\to\infty}{\sim}0+o(|k_0|^0)\ . \tag{36}$$

The approximate operators $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ are symmetrical, so that even if the approximate operators are applied, the right-hand side of Eq. (28) satisfies the reciprocity principle Eq. (I-4); hence in the present approximation scheme Eq. (36) must hold in the case $\mathbf{r}_1\in\Omega^{\rm ex}$ and $\mathbf{r}_2\in\Omega$ as well.

If $\mathbf{r}_1\in\Omega^{\rm ex}$ and $\mathbf{r}_2\in\Omega^{\rm ex}$ and are mutually invisible, Eq. (28) becomes, following the use of Eq. (26),

$$G^{+}_{Nk_0}(\mathbf{r}_1;\mathbf{r}_2)=G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)-\int_{\partial\Omega_2}\frac{\partial G^{+}_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 3})G^{+}_{k_0}(\mathbf{r}_{\partial 3};\mathbf{r}_2)dA_3-\int_{\partial\Omega_1}G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\frac{\partial G^{+}_{k_0}}{\partial n_l}(\mathbf{r}_{\partial 3};\mathbf{r}_2)dA_3\ , \tag{37}$$

where $\partial\Omega_{1,2}$ are defined as following Eq. (35). Again applying Eqs. (7) and (34), and using arguments similar to those used in the preceding paragraph, we find that Eq. (36) holds in this case, for which a nonzero exact result is expected for $|k_0|<\infty$.

Similar arguments, which make use of Eqs. (29), (27), (7), and (34), show that if $\mathbf{r}_1\in\Omega$ and $\mathbf{r}_2\in\Omega^{\rm ex}$, or if $\mathbf{r}_1\in\Omega^{\rm ex}$ and $\mathbf{r}_2\in\Omega$, or if $\mathbf{r}_1\in\Omega^{\rm ex}$ and $\mathbf{r}_2\in\Omega^{\rm ex}$ and $\mathbf{r}_1$ and $\mathbf{r}_2$ are mutually invisible, the present approximation scheme leads to

$$G^{+}_{Dk_0}(\mathbf{r};\mathbf{r}_2) \underset{|k_0|\to\infty}{\sim} 0+o(|k_0|^0) . \tag{38}$$

Finally, we combine Eqs. (26), (27), (28), and (29) when $\mathbf{r}_1$ and $\mathbf{r}_2$ are mutually visible. In this case just one stationary phase point contributes, and it lies in the nonempty set $\partial\Omega_1\cap\partial\Omega_2$. In reducing Eqs. (27) and (28), we can drop the portion of the integrals outside of $\partial\Omega_1\cap\partial\Omega_2$, with the results

$$G^{+}_{Xk_0}(\mathbf{r}_1;\mathbf{r}_2) \underset{|k_0|\to\infty}{\sim} G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)\mp\int_{\partial\Omega_1\cap\partial\Omega_2}\left[\frac{\partial G^{+}_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 3})G^{+}_{k_0}(\mathbf{r}_{\partial 3};\mathbf{r}_2)+G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\frac{\partial G^{+}_{k_0}}{\partial n_l}(\mathbf{r}_{\partial 3};\mathbf{r}_2)\right]dA_3 , \tag{39}$$

where the upper (respectively, lower) sign holds for the case $X=N$ (respectively, $X=D$). The integrals on the right-hand side of Eq. (39) will be recognized as symmetrized versions (i.e., symmetrized with respect to the arguments $\mathbf{r}_1$ and $\mathbf{r}_2$, including the region of integration) of the formulas that would be derived from the conventional physical optics method (Ref. 4, Chap. I.2.13.4). Both the latter and Eq. (39) yield the same asymptotic form, as $|k_0|\to\infty$, for $G^{+}_{Nk_0}$ and $G^{+}_{Dk_0}$ as that which can be obtained by geometrical acoustics methods [see Ref. 8; the explicit formula in Ref. 4, Eq. I.112, is valid only if the plane of incidence and reflection of the ray contains a principal axis of the curvature matrix $K_{\alpha\alpha\beta}$, as defined in Eq. (12), of $\partial\Omega$ at the point of reflection of the ray].

V. DOMINANT SINGULARITIES OF $\check{Z}_{k_0}$ AND $\check{Z}^{-1}_{k_0}$

In this section we consider a different kind of limit from that considered in Secs. III and IV, that is, we study the behavior of the radiation impedance and admittance operators when $|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|$ is small compared to the local curvature radii of $\partial\Omega$, and, moreover, $|k_0|$ is sufficiently small that $|k_0|\,|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|\ll 1$. For this limiting behavior the oscillatory nature of the signal is unimportant; to the level of accuracy maintained in what follows, the results are obtainable from the potential theory (Laplace equation) case $k_0=0$.

Perspective as to the structure of the operators for a general smooth $\partial\Omega$ can be inferred from the treatment of the sphere in the Appendix. If we denote the terms of the types appearing in Eqs. (A3)–(A7) as being singularities of successively weaker orders, the dominant and the next weaker singular terms in the expansion Eq. (A12) for $\check{Z}_{k_0}$ and Eq. (A13) for $\check{Z}^{-1}_{k_0}$ are independent of k_0. Still less singular terms involve k_0^2; no terms linear in k_0 appear, since these are associated with contributions to the operators that have weaker singularities than the terms studied or mentioned.

The operators V_{k_0}, $V^{\tau}_{k_0}$, and $\check{Z}_{k_0}$ can be expressed as two-point kernels with integrable singularities of order $|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^{-1}$ as $\mathbf{r}_{\partial 1}\to\mathbf{r}_{\partial 2}$. These operators are known to be "smoothing" operators, that is, they will, when convolved with another integrable kernel, make the product kernel less singular. For example, the kernel $(V_{k_0})^2(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ has a logarithmic singularity as $\mathbf{r}_{\partial 1}\to\mathbf{r}_{\partial 2}$ and the kernel $(V_{k_0})^n(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ is a continuous bounded kernel for $n\geq 3$: by way of proof, it is sufficient to verify these results for $k_0=0$, so that the arguments of Kang working in E^2 (Ref. 9, p. 1446), Kellogg (Ref. 10, Chap. XI, Sec. 9), Cisotti (Ref. 11, Sec. 3), and Lévy (Ref. 12, p. 223) apply. Thus it is plausible that when $|k_0|$ is small, it is meaningful to make a formal series expansion of Eqs. (1) and (2) in powers of $V^{\tau}_{k_0}$ and V_{k_0}; the approximations Eqs. (5) and (6) will exhibit the dominant singularities of the respective operators, while the first-order terms in $V^{\tau}_{k_0}$ and V_{k_0} will provide the sought-after correction term in each case (we shall not attempt to compute still higher-order corrections herein for nonspherical $\partial\Omega$).

An obstruction to the application of Eq. (2) to long-wavelength problems is the circumstance that for $k_0=0$, $\partial\Omega$ bounded, and Ω nonempty, the constant function is an eigenfunction of the homogeneous Neumann boundary-value problem within Ω, and hence the constant function on $\partial\Omega$ lies in the null space of both of the operators $I_\partial+V_0$ and W_0 (Ref. 3, Theorems 3.17 and 3.32). [The operator obtained by taking the limit of Eq. (2) as $k_0\to 0$ should exist nevertheless and agree with $\check{Z}^{-1}_0$—see the remarks in paper I, Sec. III C.] We offer two arguments to the effect that the procedure used here will yield acceptable results in spite of this difficulty: First, with the methods of the Appendix, we can carry out the computations for the case in which $\partial\Omega$ is a sphere, using the spherical harmonic expansions of $I_\partial+V_0$ and W_0. The expansion of $(I_\partial+V_0)^{-1}$ in powers of V_0 diverges for zero-order spherical harmonics, but the correct result for $\check{Z}^{-1}_0$ is obtained to within a higher-order correction if only the first two terms in the expansion are taken. And second, we are dealing with highly local properties of $\check{Z}^{-1}_0$, so that our procedure cannot distinguish between compact and unbounded $\partial\Omega$—indeed, we shall approximate $\partial\Omega$ by a paraboloid—so that the global structure of $\partial\Omega$ plays no significant role here.

We shall treat first the case of $\check{Z}_{k_0}$ by means of the leading two terms in an expansion of Eq. (1). Furthermore, as just argued, the dominant two singularities in $\check{Z}_{k_0}$ will be independent of k_0, so that within the prescribed limitations we have

$$\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\approx\check{Z}_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \tag{40}$$

$$\approx -U_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})-(U_0V^{\tau}_0)(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) . \tag{41}$$

Equations (I-18), (I-20), and (I-14) yield

$$-U_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 3}) = -(2\pi|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 3}|)^{-1}\,, \tag{42}$$

$$V_0^{\tau}(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 2}) = -\frac{\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot(\mathbf{r}_{\partial 3}-\mathbf{r}_{\partial 2})}{2\pi|\mathbf{r}_{\partial 3}-\mathbf{r}_{\partial 2}|^3}\,. \tag{43}$$

We need to calculate the correction term

$$-\int_{\partial\Omega} U_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 3})V_0^{\tau}(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 2})dA_3\,. \tag{44}$$

We approximate $\partial\Omega$ locally by a paraboloid, and introduce local coordinates as in Eq. (12), with the origin and orientation of the coordinates being such that

$$\mathbf{r}_{\partial 1}\leftrightarrow(\sigma,0)\,,\quad \mathbf{r}_{\partial 3}\leftrightarrow(u^1,u^2)\,,\quad \mathbf{r}_{\partial 2}\leftrightarrow(0,0)\,, \tag{45}$$

where $\sigma > 0$.

We can now compute the integrand in Eq. (44); omitting terms of higher order than the first in the curvature matrix $K_{\alpha\beta}$ of $\partial\Omega$ at $\mathbf{r}_{\partial 2}$, we find, using Eqs. (12), (14), (45), that Eq. (44) reduces to

$$(8\pi^2)^{-1}\int\int du^1du^2[(\sigma-u^1)^2+(u^2)^2]^{-1/2} \times\left[\sum_{\alpha,\beta}K_{\alpha\beta}u^{\alpha}u^{\beta}\right][(u^1)^2+(u^2)^2]^{-3/2}\,. \tag{46}$$

The integral equation (46) is to be taken over a disc, say, of fixed radius $A \gg \sigma > 0$, as we are interested only in the singular behavior of the integral as $\sigma\to 0$; contributions to the integral equation (46) from outside the disc change the result only by a continuous correction function of $(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$. The integral equation (46) can be evaluated in polar coordinates $(u^1,u^2)=(\rho\cos\phi,\rho\sin\phi)$, the integral over ρ being done first. If we drop the nonsingular contributions depending on the cutoff radius A, and use the result

$$\int_0^{2\pi}\cos 2\phi\,\ln(1-\cos\phi)d\phi = -\pi\,, \tag{47}$$

we find that Eq. (46) reduces to

$$(8\pi)^{-1}[(K_{11}+K_{22})\ln(1/\sigma)+\tfrac{1}{2}(K_{11}-K_{22})]\,. \tag{48}$$

We define the mean curvature $H(\mathbf{r}_{\partial 2})$ as

$$H(\mathbf{r}_{\partial 2})=\tfrac{1}{2}\mathrm{Tr}(K)\,; \tag{49}$$

our result for Eq. (44) can be expressed as follows for a more generally oriented coordinate system on $\partial\Omega$:

$$-(U_0V_0^{\tau})(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\approx(4\pi)^{-1}H(\mathbf{r}_{\partial 2})\ln(1/|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|) +(8\pi)^{-1}\sum_{\alpha,\beta}[K_{\alpha\beta}(\mathbf{r}_{\partial 2})-\delta_{\alpha\beta}H(\mathbf{r}_{\partial 2})]\{[\hat{\mathbf{t}}_{\alpha}\cdot(\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2})][\hat{\mathbf{t}}_{\beta}\cdot(\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2})]/|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^2\}\,. \tag{50}$$

Note that the second term on the right-hand side of Eq. (50) is bounded but discontinuous as $\mathbf{r}_{\partial 1}\to\mathbf{r}_{\partial 2}$, and vanishes identically when the curvature matrix $K_{\alpha\beta}$ is everywhere isotropic, that is, when $\partial\Omega$ is a sphere. The result Eq. (50) agrees with the corresponding part of the limiting values, for both source and field points on $\partial\Omega$, of what in the present notation is $-4\pi G_{N0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})$ obtained by Cisotti [Ref. 11, Eq. (77)] and Lévy (Ref. 12, p. 265, bottom line), in accordance with Eq. (I-38).

We note that once the singular terms of Eqs. (42) and (50) are subtracted from $\check{Z}_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$, the remainder should be a continuous kernel in $(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$, which kernel has a well-defined value as $\mathbf{r}_{\partial 2}\to\mathbf{r}_{\partial 1}$ for all $\mathbf{r}_{\partial 1}\in\partial\Omega$. This function on $\partial\Omega$ is not determined by Lévy's procedure,[13] and appears to be a function of the global, as well as the local, structure of $\partial\Omega$. In this connection, we note that the logarithmic term on the right-hand side of Eq. (50) does not scale properly with an overall change of the length scale of the system.

Now we consider $\check{Z}_{k_0}^{-1}$. As before, our objective is to obtain the next-most singular term beyond the zeroth-order approximation of Eq. (6). Thus we have from Eq. (2), given the stated limitations on k_0 and $|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|$,

$$\check{Z}_{k_0}^{-1}\approx\check{Z}_0^{-1} \tag{51}$$

$$\approx W_0-W_0V_0\,. \tag{52}$$

Let us first study the operator W_{k_0}. As defined in Eq. (I-21), this operator is not representable as a two-point kernel on $\partial\Omega$. For, suppose that Ω is the half-space $z<0$ of E^3; then we would have

$$W_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) = -2[\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\cdot\nabla_1][\hat{\mathbf{n}}(\mathbf{r}_{\partial 2})\cdot\nabla_2] \times G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)|_{\mathbf{r}_1=\mathbf{r}_{\partial 1},\mathbf{r}_2=\mathbf{r}_{\partial 2}} \tag{53}$$

$$=\frac{1}{2\pi}\left[\frac{-ik_0}{|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^2}+\frac{1}{|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^3}\right] \times\exp(ik_0|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|)\,. \tag{54}$$

Note that for $\mathbf{r}_{\partial 1}\neq\mathbf{r}_{\partial 2}$ the function on the right-hand side of Eq. (54) admits of a representation that is intrinsic to $\partial\Omega$, that is

$$[\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1})+k_0^2]U_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\,, \tag{55}$$

where $\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1})$ is the surface Laplace-Beltrami operator with respect to $\mathbf{r}_{\partial 1}$ coordinates. We can regard the right-hand side of Eq. (54) not as an ordinary function but as a generalized function, which is represented by Eq. (55) as a differential operator acting on a singular, but integrable, kernel. The application of the generalized kernel of Eq. (54) in a matrix element of the form described following Eq. (I-9), with smooth, rapidly decreasing functions $\psi(\mathbf{r}_\partial)$ and $\phi(\mathbf{r}_\partial)$, can be defined by Green's theorem:

$$\int_{\partial\Omega} dA_1 \int_{\partial\Omega} dA_2 \psi(\mathbf{r}_{\partial 1}) W_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\phi(\mathbf{r}_{\partial 2}) = \int_{\partial\Omega} dA_1 \int_{\partial\Omega} dA_2 \{[\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1}) + k_0^2]\psi(\mathbf{r}_{\partial 1})\} U_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\phi(\mathbf{r}_{\partial 2}) , \tag{56}$$

where the "boundary" terms vanish since either $\partial\Omega$ is compact or $|\psi(\mathbf{r}_{\partial 1}| \to 0$ rapidly as $|\mathbf{r}_{\partial 1}| \to \infty$. This procedure is an example of Hadamard's computation of the "finite part" of a divergent integral (cf. Ref. 14, Vol. II, pp. 785–788, and Ref. 9, p. 1445). It can be justified by performing the computation in the two-dimensional wave-vector space associated with the plane $\partial\Omega$, from which it can be inferred that no Dirac δ-function term of the type $Ck_0\delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$, where C is a dimensionless constant, as been omitted in passing from Eq. (54) to Eq. (55) and in turn to Eq. (56). Furthermore, a somewhat lengthy computation shows that for the case that $\partial\Omega$ is the paraboloid of Eq. (12), the symmetrized form of Eq. (55) has the property that

$$W_{k_0} \underset{\mathbf{r}_{\partial 1}\to\mathbf{r}_{\partial 2}}{\sim} \tfrac{1}{2}[\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1}) + \mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 2}) + 2k_0^2] \times U_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) + O(|\mathbf{r}_{\partial 1} - \mathbf{r}_{\partial 2}|^{-1}) . \tag{57}$$

[Note that in order to evaluate a matrix element expression of the operator on the right-hand side of Eq. (57), Green's theorem must be applied once for each $\mathcal{L}_{\partial\Omega}$.] Hence in this case also, and we presume for general smooth $\partial\Omega$, no singularity of the strength of a Dirac δ function is lost by the adoption of the approximation Eq. (57).

For the purpose of estimating the *second* term on the right-hand side of Eq. (52) the nonsymmetric form of Eq. (57) is sufficient, so that this term can be written as

$$-(W_0V_0)(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \approx -\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1}) \times \int_{\partial\Omega} U_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 3}) \times V_0(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 2}) dA_3 , \tag{58}$$

where the right-hand side of Eq. (58) is to be applied to a function on $\partial\Omega$ in the manner of Eq. (56). Computation of the integral on the right-hand side of Eq. (58) can be accomplished in a similar manner to, and to the same accuracy as, the integral of Eq. (44), with the result that our approximation for $\check{Z}_0^{-1}$ is

$$\begin{aligned}\check{Z}_0^{-1}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) = {} & \tfrac{1}{2}[\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1}) + \mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 2})] U_0(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \\ & + \mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1})\{(4\pi)^{-1}H(\mathbf{r}_{\partial 2})\ln(1/|\mathbf{r}_{\partial 1} - \mathbf{r}_{\partial 2}|) \\ & + (8\pi)^{-1}\sum_{\alpha,\beta}[K_{\alpha\beta}(\mathbf{r}_{\partial 2}) - \delta_{\alpha\beta}H(\mathbf{r}_{\partial 2})] \\ & \times [\hat{\mathbf{t}}_\alpha(\mathbf{r}_{\partial 1})\cdot(\mathbf{r}_{\partial 1} - \mathbf{r}_{\partial 2})][\hat{\mathbf{t}}_\beta(\mathbf{r}_{\partial 1})\cdot(\mathbf{r}_{\partial 1} - \mathbf{r}_{\partial 2})]/|\mathbf{r}_{\partial 1} - \mathbf{r}_{\partial 2}|^2\} .\end{aligned} \tag{59}$$

Since

$$\mathcal{L}_{\partial\Omega}(\mathbf{r}_{\partial 1})\ln(1/|\mathbf{r}_{\partial 1} - \mathbf{r}_{\partial 2}|) \approx -2\pi\delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) , \tag{60}$$

the correction term in Eq. (59) agrees with the corresponding term in Eq. (A12) for spherical $\partial\Omega$.

The procedure used above is similar to that of Cisotti,[11] who studied the function analogous to $G_{N0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})$ for the interior region Ω; Lévy's[12] approach for both $G_{N0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})$ and $(\partial G_{D0}/\partial n_r)(\mathbf{r}_1;\mathbf{r}_{\partial 2})$ entailed the use of a preconstructed infinite family of solutions to Laplace's equation in E^3 to fit the boundary conditions on $\partial\Omega$ to desired accuracy. The present treatment and those of Refs. 11 and 12 are all local. Consequently, the results do not distinguish between Green's functions for the interior and exterior regions, and the results are comparable and agree to the order of terms computed here for $G_{N0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$. The results obtained herein for $\check{Z}_0^{-1}$ can also be extracted from Lévy's results (Ref. 12, bottom of p. 256) by the use of Eq. (I-41), augmented by manipulations of singular quantities analogous to those applied in Eqs. (53)–(58). Both Refs. 11 and 12 carried out the computation to one higher order (only G_{N0} was treated in Ref. 11), in that they included contributions that are not computed here and that are quadratic in $K_{a\alpha\beta}$ and linear in the unspecified cubic terms in Eq. (12). The results of Refs. 11 and 12 differ for these higher-order terms for G_{N0}; Lévy states (Ref. 12, p. 208, footnote 2, and p. 266, footnote 1) that Cisotti's results for these higher-order corrections are incorrect. The present work goes beyond that of Refs. 11 and 12 in the respect of establishing approximate local analytic forms for the operators W_0 and $\check{Z}_0^{-1}$ that are intrinsic to $\partial\Omega$; these operators are not representable as kernels on $\partial\Omega$, but, as is plausible from the results obtained here, are realizable as products of singular, but integrable, kernels and finite-order differential operators.

VI. SUMMARY AND DISCUSSION

The transition-operator formulation of diffraction theory holds promise of providing a unifying framework within which many disparate approaches to diffraction problems can be seen as different facets of the same underlying structure. The beginnings of this program have been carried out in paper I and the present work. In particular, it was noted in paper I, Sec. III D, that Waterman's null-field method (cf. Ref. 3, pp. 104–106), a medium-to-long-wavelength approximation for diffraction, takes a natural place in this scheme. Sections III and IV herein dealt with the other end of the wavelength range: "physical optics" and geometrical (ray) acoustics were derived from the starting point that Ω is taken convex, $\partial\Omega$ is smooth, sound-hard or sound-soft boundary conditions obtain, and the simple approximation to $\check{Z}_{k_0}$ or to $\check{Z}^{-1}_{k_0}$ proposed in Sec. II is applied as the essential ingredient in the respective T operator.

Section V was concerned with the improvement of the generalized (in the sense of incorporating finite-order differential operations) kernel approximations of Sec. II for $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$, for the long-wavelength regime and for nearby points on smooth (not necessarily convex) $\partial\Omega$. A higher-order approximation along the same lines, but for the special geometry that $\partial\Omega$ is a sphere, was derived in the Appendix.

The remaining paragraphs of this section are devoted to a discussion of conjectured elaborations and applications of the T-operator formalism.

An advantage of the T-operator approach to diffraction is that it can provide an alternative means of avoiding the nonuniqueness problems associated with integral equation methods based on single- or double-layer potentials. It is known that a suitable superposition of singlet and doublet layers yields uniquely solvable integral equations (Ref. 3, Chaps. 3.6 and 3.9; Ref. 15; and references given therein) for all k_0 such that $\mathrm{Im}(k_0)\geq 0$. The surface integral operators that arise in the T-operator formalism, given some restrictions for impedance boundary-value problems, exist for all k_0 such that $\mathrm{Im}(k_0)\geq 0$ (paper I, Sec. III C and Appendix). With trivial exceptions, however, analytical or numerical expressions for operators of the latter type are not known at the outset of a diffraction problem. But let us suppose the further development of the theory of operators as $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$, along the lines of obtaining suitable analytical approximations for their dominant singular terms, so that the remainders are continuous two-point kernels on $\partial\Omega$ for geometries and (long-to-medium) wavelengths of interest. Then these remainder terms could be obtained numerically and the given diffraction problem would be reduced to quadratures. We note in this connection that considerable analytical work has been done on an impedance boundary-value problem in E^2, with $\partial\Omega$ as a straight line—see Ref. 16, Chap. VI, §16.

Another artifice suggested by the T-operator approacn is the simulation of a complex (say, nonconvex) obstacle by numerically determined nonlocal impedance boundary conditions of the type of Eq. (I-15) on a simpler surface Σ, say, that circumscribes the actual obstacle boundary $\partial\Omega$. (For an example of such a geometry, and an approximation scheme that uses methods related to those discussed herein, see Ref. 17). The Helmholtz equation would have to be solved numerically in the domain between $\partial\Omega$ and Σ for a complete set (in a numerical sense) of conveniently chosen boundary conditions on that part of Σ which differs from $\partial\Omega$. Approximate operators $\check{Z}_{\Sigma k_0}$, $\check{Z}^{-1}_{\Sigma k_0}$, and $(A_\Sigma+B_\Sigma\check{Z}^{-1}_{\Sigma k_0})^{-1}A_\Sigma$ could then be determined so that in the region exterior to Σ the complete Green's function for the true obstacle is obtained by this substitute means. Several diffraction problems could be explored with reduced effort in this way provided that Σ and k_0 are kept fixed, for then only the operator quotient $A^{-1}_\Sigma B_\Sigma$, and not $\check{Z}_{\Sigma k_0}$ and $\check{Z}^{-1}_{\Sigma k_0}$, would change. The limitations of this procedure, in particular, the range of geometries and boundary conditions for which such a simulation is numerically stable or at least mathematically nonsingular, are unexplored.

A geometry that is involved in numerical applications is that of a rectangular parallepiped with edges a, b, c, say, for ranges of values of $|k_0a|$, $|k_0b|$, and $|k_0c|$. The eigenfrequencies and eigenfunctions for the interior Neumann and Dirichlet boundary-value problems are easily determined for these geometries; this circumstance, and presence of edges and corners, make these geometries of particular theoretical interest as test cases for schemes for obtaining approximations to $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$. Moreover, this class of $\partial\Omega$ can be useful in some numerical approximation schemes as an artificial boundary for a fine rectangular grid that covers the diffracting obstacle. For some theory and applications along these lines, primarily to wave propagation in two space dimensions, see Refs. 18 and 19, and further references given therein.

Concerning the short-wavelength regime, it is plausible, at least on physical grounds, that the transition-operator approach to diffraction is capable of subsuming the widely used approximation schemes known collectively as the geometrical theory of diffraction[20–24] (GTD). This theory is a collection of short-wavelength approximations that associate a quantitative scheme with an intuitively compelling ray-acoustical picture; the method comprises modifications and extensions of "classical" ray acoustics so as to incorporate certain types of diffraction phenomena that vanish in the classical geometrical acoustics limit. With respect to impenetrable obstacles, two generic types of nonclassical phenomena are treated: The first type is diffraction from convex obstacles with smooth surfaces; these phenomena are described in terms of so-called surface-diffracted rays (i.e., the ray manifestation of so called "creeping waves"), which propagate along geodesic curves in $\partial\Omega$ and continuously shed tangential "diffracted rays." The second type is diffraction caused by departures on $\partial\Omega$ from smoothness on curves or at points; characteristic examples are diffraction by the edge of a wedge or by a tip (apex of a half cone).

An initial step in correlating the first type of diffraction phenomena with the T-operator formalism would be to extend the present work by a study, with a combination of analysis and computation, of the behavior of $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ in the medium-to-short wavelength regime both for the sphere and for anisotropic convex surfaces as ellipsoids. The objectives would be to secure approximate useful forms for the operators that reproduce known results [as the extended boundary conditions of Eq. (I-16)] adequately, and to study the presumed approach of the operators to a short-wavelength approximation implied by the creeping wave picture as realized in the GTD for acoustic-wave scattering.[23] A recent, mathematically rigorous study[25] dealt with short-wavelength diffraction of D type, for strictly convex, smooth-surfaced obstacles, and for field points near the shadow boundary of the incoming signal; this paper did not, however, investigate the behavior as the source and field points approach the obstacle's boundary.

Comprehension in terms of T operators of the second class of diffraction phenomena treated by GTD would presumably entail a determination of the behavior of $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ for canonical cases of nonsmooth convex surfaces, as a wedge formed by two planes and with various wedge angles, a circular half cone with various aperture angles, and so on. It is plausible that the results, when properly formulated, could be generalized so as to apply at least locally to moderately deformed versions of these obstacles, in a manner analogous to the passage from Eqs. (3) and (4) to Eqs. (5) and (6). No published work is known to the writer that explicitly addresses the problem of approximating the behavior of $\check{Z}_{k_0}$ or of $\check{Z}_{k_0}^{-1}$, when one or both arguments of the associated kernel approach a line or point where $\partial\Omega$ is not smooth; various chapters of Ref. 4 provide some analytical starting points for investigations along these lines.

Other possible generalizations are readily imagined that would increase the scope of applications of the transition-operator formalism, but we shall not attempt any further enumeration of these. The effectiveness of transition-operator theory as a unifying principle for a wide range of scattering phenomena is contingent on the derivation of a substantial body of physically cogent results and useful methods.

APPENDIX: $\check{Z}_{k_0}$ AND $\check{Z}_{k_0}^{-1}$ FOR SPHERICAL $\partial\Omega$

Let $\partial\Omega$ be a sphere $S^2(a)$ of radius a with its center at the origin of coordinates, so that $\mathbf{r}_{\partial 1}\in\partial\Omega$ is given by $\mathbf{r}_{\partial 1}=a\hat{\mathbf{r}}_1$. We use Eqs. (I-38) and (I-41), and the expansions of Green's functions in Ref. 4, Eqs. (10.68) and (10.5), to obtain the following expansions of the operators $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ in terms of the orthonormal spherical harmonics $Y_{lm}(\hat{\mathbf{r}})$ (as defined in Ref. 26), where $l=0,1,2,\ldots$, $m=-l,-l+1,\ldots,l-1,l$, and the argument $\hat{\mathbf{r}}$ is used to stand for spherical polar coordinates on $S^2(1)$:

$$\check{Z}_{k_0}(a\hat{\mathbf{r}}_1;a\hat{\mathbf{r}}_2)=(1/a)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{h_l^{(1)}(k_0a)}{k_0ah_l^{(1)\prime}(k_0a)}Y_{lm}(\hat{\mathbf{r}}_1)\times[Y_{lm}(\hat{\mathbf{r}}_2)]^* , \tag{A1}$$

$$\check{Z}_{k_0}^{-1}(a\hat{\mathbf{r}}_1;a\hat{\mathbf{r}}_2)=(1/a^3)\sum_{l,m}\frac{k_0ah_l^{(1)\prime}(k_0a)}{h_l^{(1)}(k_0a)}Y_{lm}(\hat{\mathbf{r}}_1)\times[Y_{lm}(\hat{\mathbf{r}}_2)]^* ; \tag{A2}$$

Eq. (A2) is given by Ref. 27, Eqs. (34) and (35). No closed-form expression in terms of known transcendental functions is presently available for either of these sums for general k_0. In this appendix, we shall obtain approximate forms for the sums in Eqs. (A1) and (A2) for the case that $|k_0|a|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|\ll 1$, which results become exact in the limit $k_0\to 0$. We shall give only very brief attention to the short-wavelength behavior of the sums.

We shall recapitulate some known or easily derivable results. Let $\mathcal{L}_{S^2(1)}(\hat{\mathbf{r}})$ be defined as following Eq. (55). Then we have

$$[\mathcal{L}_{S^2(1)}(\hat{\mathbf{r}})-\tfrac{1}{4}]Y_{lm}(\hat{\mathbf{r}})=-(l+\tfrac{1}{2})^2Y_{lm}(\hat{\mathbf{r}}) , \tag{A3}$$

$$\delta^2_{S^2(1)}(\hat{\mathbf{r}}_1;\hat{\mathbf{r}}_2)=\sum_{l,m}Y_{lm}(\hat{\mathbf{r}}_1)[Y_{lm}(\hat{\mathbf{r}}_2)]^* , \tag{A4}$$

$$(2\pi|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|)^{-1}=\sum_{l,m}(l+\tfrac{1}{2})^{-1}Y_{lm}(\hat{\mathbf{r}}_1)[Y_{lm}(\hat{\mathbf{r}}_2)]^* , \tag{A5}$$

$$(2\pi)^{-1}\ln(1+2/|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|)=\sum_{l,m}[(l+\tfrac{1}{2})(l+1)]^{-1}Y_{lm}(\hat{\mathbf{r}}_1)[Y_{lm}(\hat{\mathbf{r}}_2)]^* , \tag{A6}$$

$$(2\pi)^{-1}[1-|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|+\tfrac{1}{2}|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|^2\ln(1+2/|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|)]=\sum_{l,m}[(l+\tfrac{1}{2})(l+1)(l+2)]^{-1}Y_{lm}(\hat{\mathbf{r}}_1)[Y_{lm}(\hat{\mathbf{r}}_2)]^* ; \tag{A7}$$

Eq. (A4) is the closure property[28] for spherical harmonics, Eq. (A5) follows from the inverse distance formula[29] and the addition theorem for spherical harmonics (Ref. 30, Appendix IV), and Eqs. (A6) and (A7) can be derived by suitable manipulations of the inverse distance formula. Furthermore, we have the inequality[31]

$$|P_l(x)|\le 1 \quad \text{for } -1\le x\le 1,\ l=0,1,\ldots ; \tag{A8}$$

hence if A_l is any sequence of complex numbers bounded above in absolute value, then the sum

$$\sum_{l,m}A_l(l+\tfrac{1}{2})^{-3}Y_{lm}(\hat{\mathbf{r}}_1)[Y_{lm}(\hat{\mathbf{r}}_2)]^* \tag{A9}$$

converges absolutely and uniformly (in $\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2$), and thus represents a continuous kernel on $S^2(1)$.

We expand the coefficients in Eqs. (A1) and (A2) in what amounts to a descending series in powers of $l+\frac{1}{2}$, when $l \gg k_0 a$. Using Ref. 32, Eq. (10.1.3), we find

$$h_l^{(1)}(z)/[zh_l^{(1)\prime}(z)] = -(l+1)^{-1} - (z^2/2)[(l+\tfrac{1}{2})(l+1)(l+2)]^{-1} + O(l^{-4}) , \tag{A10}$$

$$zh_l^{(1)\prime}(z)/h_l^{(1)}(z) = -(l+1) + (z^2/2)[(l+\tfrac{1}{2})^{-1} + (l+\tfrac{1}{2})^{-1}(l+1)^{-1}] + [(3z^2/4)+(z^4/8)][(l+\tfrac{1}{2})(l+1)(l+2)]^{-1} + O(l^{-4}) . \tag{A11}$$

Combining the results Eqs. (A3)–(A11), we find that Eqs. (A1) and (A2) become

$$\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) = -(2\pi|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|)^{-1} + (4\pi a)^{-1}\ln(1+2a/|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|) - (k_0^2 a/4\pi)[1-|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|/a + (|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^2/2a^2)\ln(1+2a/|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|)] + R_1(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) , \tag{A12}$$

$$\check{Z}_{k_0}^{-1}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) = (2\pi|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|)^{-1}[\mathcal{L}_{S^2(a)}(\mathbf{r}_{\partial 2}) - 1/4a^2] - (2a)^{-1}\delta^2_{S^2(a)}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) + (k_0^2/4\pi)|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^{-1} + (k_0^2/4\pi a)\ln(1+2a/|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|) + (k_0^2/16\pi a)(6+k_0^2 a^2)[1-|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|/a + (|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|^2/2a^2)\ln(1+2a/|\mathbf{r}_{\partial 1}-\mathbf{r}_{\partial 2}|)] + R_2(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) , \tag{A13}$$

where the remainder terms $R_1(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ and $R_2(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ are continuous kernels on $S^2(a)$, both of which vanish identically as $k_0 \to 0$. It is not difficult to verify that the leading singular terms on the right-hand sides of Eqs. (A12) and (A13) agree with the results obtained by specializing Eqs. (42), (50), and (59) to spherical $\partial\Omega$.

For the purpose of confining the domain of a numerical integration of the Helmholtz differential equation to an agreeably small volume of E^3, it is desirable to have a localized approximation to the operator of Eq. (A2) or (A1), in order to effect radiation boundary conditions on an artificial spherical boundary $S^2(a)$ surrounding the obstacle, such that a is as small as feasible, but also so that $|k_0 a| \gg 1$. A first-order differential equation of Riccati type[33] is easily obtained for the logarithmic derivative of $h_l^{(1)}(z)$ or for the reciprocal function; such an equation provides an asymptotic series for the coefficient in Eq. (A2) or (A1) in descending powers of $k_0 a$, with the coefficients being polynomials of increasing order in $l(l+1)$. Correspondingly, an asymptotic expansion for the sum of Eq. (A2) or (A1) in terms of descending powers of $k_0 a$ with coefficients being polynomials in $\mathcal{L}_{S^2(a)}(\mathbf{r}_\partial)$ can be obtained. The leading terms of this expansion for Eq. (A2) agree with the leading terms in the sequence of approximations to $\check{Z}_{k_0}^{-1}$ derived by Jones[34] (see also Ref. 35). It is an open problem to obtain nonlocal asymptotic approximations to the sums of Eqs. (A1) and (A2) by analytic means, say along the lines of a variant of Watson's transformation (cf. Ref. 4, Chap. 10, and Ref. 36).

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