PHYSICAL REVIEW A VOLUME 43, NUMBER 2 15 JANUARY 1991

# Transition operators in acoustic-wave diffraction theory. I. General theory

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(Received 27 February 1990; revised manuscript received 26 July 1990)

The objective of this paper is the establishment of a formal theory of the scattering of time-harmonic acoustic scalar waves from impenetrable, immobile obstacles; the time-independent formal scattering theory of nonrelativistic quantum mechanics, in particular the theory of the complete Green's function and the transition ($T$) operator, provides the model. The quantum-mechanical approach is modified to allow the treatment of acoustic-wave scattering with imposed boundary conditions of impedance type on the surface $\partial\Omega$ of an impenetrable obstacle. With $k_0$ as the free-space wave number of the signal, a simplified expression is obtained for the $k_0$-dependent $T$ operator for a general case of homogeneous impedance boundary conditions for the acoustic wave on $\partial\Omega$. All the nonelementary operators that enter the expression for the $T$ operator are formally simple, rational algebraic functions of a certain invertible linear operator $\check{Z}_{k_0}$, which is called the radiation impedance operator, and which maps any sufficiently well-behaved, complex-valued function on $\partial\Omega$ into another such function on $\partial\Omega$. The nonlocal operators $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ are defined only implicitly, in that $\check{Z}_{k_0}^{-1}$ is the operator that maps the limiting-value function on $\partial\Omega$ of an outgoing-wave solution to the scalar Helmholtz equation into the uniquely corresponding limiting-normal-derivative function, and $\check{Z}_{k_0}$ does the inverse operation. Previous appearances in the literature of these operators, or their analogs for other elliptic, linear partial-differential equation systems, are cited. An analytical study of the dominant singularities of the operators (considered as two-point kernels on a smooth $\partial\Omega$), and of their behavior in the geometrical acoustics limit, is the subject of a second paper [G. E. Hahne, following paper, Phys. Rev. A **43**, 990 (1991)].

## I. INTRODUCTION

We shall establish a formal theory of the diffraction of time-harmonic acoustic scalar waves, analogous to the time-independent formal scattering theory for the particle waves of nonrelativistic quantum mechanics (Ref. 1, Chap. 2.5; Ref. 2, Chap. 5; Ref. 3, Chap. 7; Ref. 4, Chap. 8; Ref. 5, Chaps. 1 and 2). The formalism will be described as the transition- ($T$-) operator formulation of diffraction theory, because of the central role played by this entity in the mathematical framework.

This paper is the first of a series, and is referred to as paper I. A companion paper,[6] which deals with subjects that derive from those of this paper, will be referred to herein as paper II.

We shall recapitulate the basics of the theory of scalar-wave diffraction in Sec. II. But first, in the following paragraphs, we shall state the fundamental formula [see Eq. (1)], outline the scope of this paper, and indicate the topics to be covered in paper II.

Acoustic-wave diffraction will be understood to be the scattering of time-harmonic scalar waves from an immobile, impenetrable obstacle that is embedded in a uniform, nondispersing, nonabsorbing propagation medium in which sound propagates with constant speed $c$, with specified boundary conditions for the acoustic wave on the obstacle's surface. We shall use the (obstacle-) free-space wave number $k_0$, where $-\infty < k_0 < +\infty$, to describe the (normally unstated) time dependence $\exp(-ik_0ct)$ of a signal. At least the full range of real $k_0$ needs to be accounted for, since the transformation of the Green's functions and the transition operator from the frequency domain to the time domain, and the reduction of the scattering ($S$) operator to an expression in terms of the $T_{Rk_0}^{+}$ operators [analogous to Ref. 1, p. 90, Eq. (5.32)], would require an integral over all real $k_0$.[7]

We consider waves in three-dimensional Euclidean space $E^3$; the points of $E^3$ are denoted by three-vectors $\mathbf{r}$ relative to some fixed origin of coordinates. Source-free waves are presumed to satisfy the scalar Helmholtz equation [Ref. 8, Eq. (7.2.3)]; complete Green's functions will be defined as in Ref. 8, Eq. (7.2.5)—albeit with a different normalization—with boundary conditions of the type described in Ref. 8, p. 806.

Let $G_{Xk_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)$ be the Green's function for a diffracting system, where the superscript + means outgoing waves at infinity, and the $X$ denotes a particular type of Green's function. $X$ will take the value (absent), $R$, $N$, or $D$ herein, where $X$ absent will denote a free-space Green's function, and $X=R$, or $N$, or $D$ denote that, respectively, some general homogeneous impedance (here called Robin, after the mathematician Gustave Robin), or homogeneous Neumann, or homogeneous Dirichlet boundary conditions, are satisfied by the Green's function at the obstacle's surface. The $N$- and $D$-type boundary conditions, and hence the associated complete Green's function, etc., are obtained as special cases of $R$-type boundary conditions. We shall make plausible the existence of, and work out simpler forms for, $T$ operators, called $T^+_{Rk_0}$, or, more explicitly, $T^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)$, associated with a given obstacle geometry, boundary conditions, and wave number. These $T$ operators are defined implicitly by the equation

$$G^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)=G^+_{k_0}(\mathbf{r}_1;\mathbf{r}_2)+(G^+_{k_0}T^+_{Rk_0}G^+_{k_0})(\mathbf{r}_1;\mathbf{r}_2)\ , \tag{1}$$

where the operator product on the right-hand side of Eq. (1) is defined as follows:[9]

$$(G^+_{k_0}T^+_{Rk_0}G^+_{k_0})(\mathbf{r}_1;\mathbf{r}_2)=\int_{E^3}d^3r_3\int_{E^3}d^3r_4\,G^+_{k_0}(\mathbf{r}_1;\mathbf{r}_3)T^+_{Rk_0}(\mathbf{r}_3;\mathbf{r}_4)G^+_{k_0}(\mathbf{r}_4;\mathbf{r}_2)\ . \tag{2}$$

The quantum-mechanical equivalent of Eq. (1) is Ref. 1, p. 89, Eq. (5.21b), or the un-numbered equation in Ref. 4 on p. 134 following Eq. (8.10), or Ref. 5, Eq. (1.4). Marcuvitz [Ref. 10, Eq. (15)] defined an operator that is analogous to $T$ for the scattering of classical electromagnetic waves from an obstacle.

The complete causal Green's function $G^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)$ is proportional to the acoustic velocity potential (Ref. 8, p. 308) at $\mathbf{r}_1$, given a spherically symmetric, point-source loudspeaker at $\mathbf{r}_2$. In Eq. (1), $G^+_{Rk_0}$ comprises the superposition of the signal $G^+_{k_0}$ that would be present if there were no obstacle, plus the complete scattered signal $G^+_{k_0}T^+_{Rk_0}G^+_{k_0}$. Note that the linear operator $T^+_{Rk_0}$ has the effect of creating the complete source distribution for the scattered wave from the initial free-space wave. If the $T^+_{Rk_0}$ operator satisfying Eq. (1) were known, it would yield full information on scattering phenomena at the specified wave number, for the given obstacle geometry and boundary conditions.

We anticipate some results of Sec. IV by saying that some of the advantages of the $T$-operator description of diffraction are that the entity $T^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)$ (i) is zero if either $\mathbf{r}_1$ or $\mathbf{r}_2$ is outside the scattering obstacle, (ii) has an elementary structure if either $\mathbf{r}_1$ or $\mathbf{r}_2$ is inside the obstacle, and (iii) has a nonelementary structure only if both $\mathbf{r}_1$ and $\mathbf{r}_2$ are on the surface of the obstacle. Thus the original diffraction problem in $E^3$ reduces to a certain mathematical problem in the two-dimensional surface of the scattering obstacle.

Perturbation theory in the manner of the sequence of Born approximations (Ref. 1, Chap. 2.2.4) is not available for the treatment of connected-obstacle diffraction, since the coupling of the wave to the scatterer is infinite within the obstacle. Nevertheless, if the $T$ operator for each of several disjoint and acoustically weakly coupled obstacles is known, the combined complete Green's function can be assembled by multiple-scattering theory (Ref. 1, Chap. 2.5.4). For example, suppose that there are just two obstacles $\Omega_1$ and $\Omega_2$, such that in the other's absence they would have the associated $T$ operators $T^+_{1,k_0}$ and $T^+_{2,k_0}$, respectively. Then the complete Green's function $G^+_{1\cup 2,k_0}$ when both obstacles are present has the multiple-scattering expansion

$$\begin{aligned}G^+_{1\cup 2,k_0}=&G^+_{k_0}+G^+_{k_0}T^+_{1,k_0}G^+_{k_0}+G^+_{k_0}T^+_{2,k_0}G^+_{k_0}+G^+_{k_0}T^+_{2,k_0}G^+_{k_0}T^+_{1,k_0}G^+_{k_0}+G^+_{k_0}T^+_{1,k_0}G^+_{k_0}T^+_{2,k_0}G^+_{k_0}\\&+G^+_{k_0}T^+_{1,k_0}G^+_{k_0}T^+_{2,k_0}G^+_{k_0}T^+_{1,k_0}G^+_{k_0}+G^+_{k_0}T^+_{2,k_0}G^+_{k_0}T^+_{1,k_0}G^+_{k_0}T^+_{2,k_0}G^+_{k_0}+\cdots\ .\end{aligned} \tag{3}$$

That is, the combined Green's function is an infinite sum of terms, such that each summand has an associated Feynman diagram with a cogent physical interpretation: the wave's free-space "propagation" from one place to another is represented by $G^+_{k_0}$, and each complete "bounce" of the wave from a given obstacle is represented by the corresponding $T$ operator.

The remainder of this paper is organized as follows. In Sec. II, we shall formulate the kinematics (geometry, functions, integrals) of the generic diffraction problem of scattering from a given, fixed obstacle, and introduce the dynamics, that is the equations of motion and boundary conditions for the Green's function, for $R$-type diffraction from an obstacle. In Sec. III A we shall, following Colton and Kress,[11] define four "primitive" operators that map the space of complex-valued functions, whose domain of definition is the boundary of the obstacle, into itself linearly. In Sec. III B, we shall define operators $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ that will be called the radiation impedance and admittance operator, respectively, on account of a partial analogy to the surface impedance and admittance of an obstacle (Ref. 8, p. 311); simple operator-algebraic expressions for $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ in terms of the primitive operators are derived. In Sec. III C we single out for mention some mathematical properties of the various operators that derive from the considerations of Secs. III A and III B. In Sec. III D we shall cite a number of previous investigations in which operators analogous to $\check{Z}_{k_0}$ or $\check{Z}^{-1}_{k_0}$ played a role. In Sec. IV we shall derive expressions for the operator $T^+_{Rk_0}$ in terms of the $\check{Z}_{k_0}$ operator (or simple algebraic functions of it) and of operators of elementary structure. Finally, in the Appendix we discuss mathematical sufficiency conditions that guarantee that an impedance boundary-value problem has a unique complete Green's function satisfying the principle of reciprocity, that is,

$$G^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)=G^+_{Rk_0}(\mathbf{r}_2;\mathbf{r}_1)\ . \tag{4}$$

In paper II we shall propose what will be called the "tangent-plane" approximation to the $T$ operator with either $N$-type or $D$-type boundary conditions being imposed. We shall then show, by means of applications of the method of stationary phase, that this approximate form yields the familiar "physical optics" method, and furthermore yields the correct geometrical acoustics (extreme short wavelength) limit for the corresponding complete Green's function for convex obstacles with smooth surfaces. Also in paper II, we shall investigate the dominant singularity structure of the operators $\check{Z}_{k_0}$ and $\check{Z}_{k_0}^{-1}$ for smooth $\partial\Omega$.

In the papers in this series, recourse will be had to suitably modified quantum-mechanical terminology and kinematics, such as "on the wave-number shell" or "on the frequency shell" (instead of "on the energy shell"), "wave-vector space" (in place of "momentum space"), position space, and representations of wave functions or matrices of operators in one or another (possibly continuous) basis of states. This parallelism is justified by the mathematical resemblance of the classical and quantum formalisms, and by the desirability of having Dirac's transformation theory available for mathematical applications in the context of classical wave theory. Some familiarity with the mathematical framework of quantum mechanics, as presented in, say, Levine (Ref. 1, Part I), or in Messiah (Ref. 12, Vol. I, Chap. VII), or in Dirac (Ref. 13, Chap. III), will be presumed on the part of the reader. We note in this connection that Deschamps[14] has advocated the use in classical electromagnetic field theory of a formalism for kinematics where certain inner products are denoted with symbols resembling Dirac's brackets, and of a dynamical approach akin to Feynman's, including a theory of wave creation, scattering, and detection events mediated by free-space propagation, each possible event sequence being describable by a diagram analogous to a Feynman diagram for a quantum scattering process.

## II. ELEMENTS OF ACOUSTIC DIFFRACTION PROBLEMS

In this section, we shall establish some geometrical, notational, and other kinematical conventions to describe a generic physical system of interest, and specify the dynamical problem, i.e., the equations of motion and boundary conditions, for the complete Green's function for the diffraction of acoustic waves from an obstacle.

We consider the scattering obstacle to occupy an open set $\Omega$, take $\Omega^{\rm ex}$ to be the connected, open subset of $E^3$ exterior to the obstacle that contains the uniform ambient fluid in which the acoustic wave propagates, and take $\partial\Omega$ to be the two-dimensional surface that is the boundary both of $\Omega$ and of $\Omega^{\rm ex}$. The open set $\Omega$ need not be connected, and can be empty, as in the case of diffraction from a thin plate, or from a thin screen with apertures.

All those, and normally only those, three-vectors $\mathbf{r}\in E^3$ with subscripts in the form $\mathbf{r}_{\partial\ldots}$ will represent points belonging to the given subset $\partial\Omega$ of $E^3$. We emphasize that $\mathbf{r}_\partial$ is not intended to be a tangent vector or normal vector to $\partial\Omega$, but is an ordinary three-vector that connects the origin with a point in the two-dimensional subset $\partial\Omega\subset E^3$. The unit vector $\hat{\mathbf{n}}(\mathbf{r}_\partial)$ is the outward-pointing (toward $\Omega^{\rm ex}$) normal to $\partial\Omega$ at $\mathbf{r}_\partial$, and is well defined if $\partial\Omega$ is smooth, as we shall assume is the case everywhere but on isolated points or curves in $\partial\Omega$. The direct product notation $(\Omega;\Omega)$, $(\partial\Omega;\Omega)$, $(\partial\Omega;\partial\Omega)$, etc., denotes subsets of $(E^3;E^3)=E^3\otimes E^3$; $(\mathbf{r}_1;\mathbf{r}_2)$ denotes a point in $(E^3;E^3)$, $(\mathbf{r}_{\partial 1};\mathbf{r}_2)$ a point in $(\partial\Omega;E^3)$, etc.

Let $\mathcal{F}(E^3)$ denote the linear space of complex-valued functions with domain $E^3$; such a function represents a *kinematically* allowable acoustic velocity-potential field (see, for example, Ref. 8, p. 308, or Ref. 11, Chap. 3.1) in obstacle-free space. We will deal with operators that map $\mathcal{F}(E^3)$ into itself linearly. The unit operator in this function space will be called $I$; $I$ has as its position-coordinate representatives the Dirac $\delta$ function on $E^3$, that is, $\delta^3(\mathbf{r}_1-\mathbf{r}_2)$.

We define the *bilinear* inner product $(\ ,\ )_{E^3}$ as follows:

$$(\Psi,\Phi)_{E^3}\equiv\int_{E^3}d^3r\,\Psi(\mathbf{r})\Phi(\mathbf{r})=(\Phi,\Psi)_{E^3} \tag{5}$$

for any pair $\Psi\in\mathcal{F}(E^3)$, $\Phi\in\mathcal{F}(E^3)$ for which the integral exists. If $T$ is a linear operator that maps $\mathcal{F}(E^3)$ into itself, we define the matrix element $(\Psi,T\Phi)_{E^3}$. The unique *transpose* $T^\tau$ of a linear operator $T$ is determined by the requirement that for all choices of suitable function pairs $\Psi,\Phi$, we have

$$(\Phi,T^\tau\Psi)_{E^3}=(\Psi,T\Phi)_{E^3}\ . \tag{6}$$

We call $T$ a *symmetric* operator if $T^\tau=T$.

Next, let $\Delta$ be any open set in $E^3$. Then we define the unit step function

$$\Theta_\Delta(\mathbf{r})=\begin{cases}+1 & \text{if } \mathbf{r}\in\Delta;\\ 0 & \text{otherwise}\ .\end{cases} \tag{7}$$

We can also consider $\Theta_\Delta$ to be a projection operator in $\mathcal{F}(E^3)$ such that $\Theta_\Delta$ has position-coordinate representatives $\Theta_\Delta(\mathbf{r}_1)\delta^3(\mathbf{r}_1-\mathbf{r}_2)$. Accordingly,

$$\Theta_\Omega+\Theta_{\Omega^{\rm ex}}=I\ , \tag{8}$$

except for operands having $\delta$-function-type singularities on $\partial\Omega$.

Integrals over $\partial\Omega$ will be defined with respect to the usual area measure, here called $dA$, that is "subduced" from the Euclidean metric on $E^3$ (Ref. 11, p. 33). When integrals over the product space $\partial\Omega\otimes\partial\Omega\otimes\partial\Omega\otimes\cdots$ of complex-valued functions, say $F$, occur, they are denoted as follows:

$$\int_{\partial\Omega}dA_1\int_{\partial\Omega}dA_2\int_{\partial\Omega}dA_3\cdots F(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2};\mathbf{r}_{\partial 3};\cdots)\ .$$

In this connection, we shall make extensive implicit use of a second kind of bilinear inner product $(\ ,\ )_{\partial\Omega}$, which is defined for function pairs $\psi(\mathbf{r}_\partial),\phi(\mathbf{r}_\partial)$ in the linear space $\mathcal{F}(\partial\Omega)$ of complex-valued functions with domain $\partial\Omega$:

$$(\psi,\phi)_{\partial\Omega}\equiv\int_{\partial\Omega}dA\ \psi(\mathbf{r}_\partial)\phi(\mathbf{r}_\partial)=(\phi,\psi)_{\partial\Omega}\ . \tag{9}$$

If $\partial\Omega$ does not have a finite area, we restrict ourselves to functions whose magnitude decreases rapidly outside a finite-area subset of $\partial\Omega$. If $Y$ is an operator that maps $\mathcal{F}(\partial\Omega)$ into itself linearly, the $\psi,\phi$ matrix element of $Y$ is defined as $(\psi,Y\phi)_{\partial\Omega}$. We define the transpose of an operator and symmetric operators in this function space as in Eq. (6) and its sequel. We call the unit operator acting in $\mathcal{F}(\partial\Omega)$ by $I_\partial$; $I_\partial$ is symmetric, and is represented by the two-dimensional Dirac δ function $\delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$.

We shall often encounter Green's functions, wave functions, etc., whose values are well defined for all points in $\Omega \bigcup \Omega^{\text{ex}}$, and whose values or gradients may be discontinuous across $\partial\Omega$. Appending a plus (respectively, minus) sign to the argument of a function on $\partial\Omega$ means that the limiting values from $\Omega^{\text{ex}}$ (respectively, $\Omega$) are to be taken, as in $\Psi(\mathbf{r}_\partial +)$ [respectively, $\Psi(\mathbf{r}_\partial -)$]; the $+$ limit is to be understood for such ambiguous functions if no sign is given explicitly. If a function $\Phi$ has two arguments in $\Omega^{\text{ex}}$, and if the limiting values on $(\partial\Omega;\partial\Omega)$ are order dependent, the argument that approaches $\partial\Omega$ *last* is given an extra $+$:

$$\lim_{\mathbf{r}_1 \to \mathbf{r}_{\partial 1}+} [ \lim_{\mathbf{r}_2 \to \mathbf{r}_{\partial 2}+} \Phi(\mathbf{r}_1;\mathbf{r}_2)] = \Phi(\mathbf{r}_{\partial 1}++;\mathbf{r}_{\partial 2}+) \ . \tag{10}$$

For the purposes of the derivations in Sec. IV it is convenient to define condensed notation for integrals over $\partial\Omega$. We let

$$\frac{\partial G^+_{Xk_0}}{\partial n_l}(\mathbf{r}_{\partial 1}+;\mathbf{r}_2), \quad \frac{\partial G^+_{Xk_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 2}+)$$

stand for the limiting exterior normal-derivative function on $\partial\Omega$ of the Green's function with respect to the left $(l)$ or right $(r)$ argument. We adopt the convention that whenever an operator of the type that maps $\mathcal{F}(\partial\Omega)$ into itself appears within an (operator)×(operator) product or (operator)×(function) product surrounded by brackets [ ], restriction of the *inner* variables to $\partial\Omega$ and an integration over $\partial\Omega$ is implied. For examples, let $\phi \in \mathcal{F}(\partial\Omega)$, and suppose $Y$ [with coordinate representatives $Y(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$] maps $\mathcal{F}(\partial\Omega)$ into itself; we have

$$[Y\phi](\mathbf{r}_{\partial 1}) \equiv \int_{\partial\Omega} dA_2 [Y(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial_2})\phi(\mathbf{r}_{\partial 2})] \ , \tag{11}$$

$$[YG^+_{k_0}](\mathbf{r}_{\partial 1};\mathbf{r}_2) \equiv \int_{\partial\Omega} dA_3 [Y(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 3}) G^+_{k_0}(\mathbf{r}_{\partial 3}+;\mathbf{r}_2)] \ , \tag{12}$$

$$\left[ \frac{\partial G^+_{k_0}}{\partial n_r} Y \frac{\partial G^+_{k_0}}{\partial n_l} \right](\mathbf{r}_1;\mathbf{r}_2) \equiv \int_{\partial\Omega} dA_3 \int_{\partial\Omega} dA_4 \left[ \frac{\partial G^+_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 3}+) Y(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 4}) \frac{\partial G^+_{k_0}}{\partial n_l}(\mathbf{r}_{\partial 4}+;\mathbf{r}_2) \right] . \tag{13}$$

Some further notational conventions will be given in Sec. IV.

Having dealt with needed kinematical preliminaries, we shall now proceed to formulate the dynamics of diffraction theory on the basis of Morse and Feshbach's (Ref. 8, Chap. 7.2) treatment of the complete Green's function $G^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)$ associated with a diffraction problem. Let $G^+_{k_0}(\mathbf{r}_1;\mathbf{r}_2)$ be the traveling-wave, free-space Green's function [Ref. 8, Eq. (7.2.17), with a different choice of normalization]:

$$G^+_{k_0}(\mathbf{r}_1;\mathbf{r}_2) = -\frac{1}{4\pi} \frac{\exp(ik_0|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} \ . \tag{14}$$

The superscript $+$ on the $G$ symbols indicates that outgoing-wave boundary conditions are satisfied at large distances from the source and from the scattering body (Ref. 11, Chap. 3.2, or Ref. 15, p. 189).

Let $A$ and $B$ be a pair of linear operators mapping $\mathcal{F}(\partial\Omega)$ into itself, and such that the physical dimension of $B$ is that of $A \times$(length); both $A$ and $B$ can be $k_0$ dependent, but we shall not make this fact explicit in the notation. These two operators will specify the particular impedance boundary conditions to be satisfied, in that the complete Green's function is required to satisfy

$$[AG^+_{Rk_0}](\mathbf{r}_{\partial 1};\mathbf{r}_2) + \left[ B \frac{\partial G^+_{Rk_0}}{\partial n_l} \right](\mathbf{r}_{\partial 1};\mathbf{r}_2) = 0 \ , \tag{15}$$

$$G^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2) = 0 \quad \text{if } \mathbf{r}_1 \in \Omega, \text{ or if } \mathbf{r}_2 \in \Omega, \text{ or both} \ . \tag{16}$$

We obtain the $N$ case, or the $D$ case, by specializing to $A = 0$ and $B = I_\partial$, or to $A = I_\partial$ and $B = 0$, respectively. The conditions Eq. (15) are the "surface" boundary conditions on $\partial\Omega$, and depend on the physical properties and the geometry of the obstacle; sufficient conditions on $A$ and $B$ so that there is a unique $G^+_{Rk_0}(\mathbf{r}_1;\mathbf{r}_2)$ satisfying reciprocity are given in the Appendix. The conditions Eq. (16) are the generalizations of so-called "extended" boundary conditions (see, for example, Ref. 16, p. 6) to complete Green's functions.

The Green's functions satisfy an inhomogeneous scalar Helmholtz equation in their left-hand arguments:

$$(\nabla^2_l + k_0^2) G^+_{Xk_0}(\mathbf{r}_1;\mathbf{r}_2) = \begin{cases} \delta^3(\mathbf{r}_1 - \mathbf{r}_2) \ \text{ if } X = R,\ N, \text{ or } D, \text{ for all } (\mathbf{r}_1;\mathbf{r}_2) \in (\Omega^{\text{ex}};\Omega^{\text{ex}}) \\ \text{or if } X = (\text{absent}), \text{ for all } (\mathbf{r}_1;\mathbf{r}_2) \in (E^3;E^3) \\ \text{undefined } \ \text{if } X = R,\ N, \text{ or } D, \text{ and } \mathbf{r}_1 \in \partial\Omega \\ 0 \ \text{ otherwise} \ . \end{cases} \tag{17}$$

We note that $G^{+}_{Xk_0}(\mathbf{r}_1;\mathbf{r}_2)-G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)$ is analytic (Ref. 11, p. 72) in $\mathbf{r}_1$ for any $(\mathbf{r}_1;\mathbf{r}_2)\in(\Omega^{\mathrm{ex}};\Omega^{\mathrm{ex}})$, $\mathbf{r}_2$ fixed, in particular for $\mathbf{r}_1$ in a neighborhood of $\mathbf{r}_2$.

More general types of nonlocal, linear, exterior boundary-value problems can be formulated (cf. p. 4 of Ref. 17, and other references cited therein), in which the speed of sound is position dependent or sound absorption takes place in $\Omega^{\mathrm{ex}}$, or in which the values of the wave function in $\Omega^{\mathrm{ex}}$ explicitly influence the limiting values on $\partial\Omega$ and reciprocally, at the level of Eqs. (15) and (17), by means of so-called "trace operator" terms and "Poisson operator" terms in the respective dynamical equations. We shall not consider these generalizations here.

## III. MATHEMATICAL PRELIMINARIES TO EXTRACTION OF THE $T$ OPERATORS

### A. Definition of the four primitive operators

Our goal now is to obtain an—insofar as is feasible explicit—expression for the operator $T^{+}_{Rk_0}$ such that Eq. (1) yields the Green's function $G^{+}_{Rk_0}$ defined in Sec. II. We shall in this subsection recapitulate the definitions of the four operators of Ref. 11, Chap. 2.7, and specify the relations between these operators and the limiting values and normal derivatives on $\partial\Omega$ of potentials derived from singlet and doublet layers on $\partial\Omega$.

We define the four linear operators (called "primitive" herein) $U_{k_0}$, $V_{k_0}$, $V^{\tau}_{k_0}$, and $W_{k_0}$, each of which maps a suitably well-behaved, complex-valued function in $\mathcal{F}(\partial\Omega)$ into another such function, as follows:[18]

$$(U_{k_0}f)(\mathbf{r}_{\partial 1})\equiv -2\int_{\partial\Omega} dA_2 G^{+}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})f(\mathbf{r}_{\partial 2}) \,, \tag{18}$$

$$(V_{k_0}f)(\mathbf{r}_{\partial 1})\equiv -2\int_{\partial\Omega} dA_2 \frac{\partial G^{+}_{k_0}}{\partial n_r}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})f(\mathbf{r}_{\partial 2}) \,, \tag{19}$$

$$(V^{\tau}_{k_0}f)(\mathbf{r}_{\partial 1})\equiv -2\int_{\partial\Omega} dA_2 \frac{\partial G^{+}_{k_0}}{\partial n_l}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})f(\mathbf{r}_{\partial 2}) \,, \tag{20}$$

$$(W_{k_0}f)(\mathbf{r}_{\partial 1})\equiv -2\left.\left[\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\cdot\nabla_l \int_{\partial\Omega} dA_2 \frac{\partial G^{+}_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 2})f(\mathbf{r}_{\partial 2})\right]\right|_{\mathbf{r}_1=\mathbf{r}_{\partial 1}} . \tag{21}$$

The integrals on the right-hand sides of Eqs. (18)–(21) exist as (possibly improper) "principal-value" integrals[19] for any fixed $\mathbf{r}_{\partial 1}\in\partial\Omega$. The right-hand side of Eq. (21) can be evaluated in two ways, as $\mathbf{r}_1\rightarrow\mathbf{r}_{\partial 1}$ from either $\Omega$ or from $\Omega^{\mathrm{ex}}$. In view of the jump conditions Eq. (27), these limits coincide.

We note the important results (Ref. 11, pp. 61 and 62) that the operators $U_{k_0}$ and $W_{k_0}$ are symmetric, while the operators $V_{k_0}$ and $V^{\tau}_{k_0}$ are transposes of one another (and hence the notation is justified). We note also—see Sec. III B, Sec. III C, and Ref. 11, Eqs. (3.45) and (3.48)—that beyond these adjoint properties the primitive operators are not independent, that is, there exists a nonlinear operator-algebraic relation connecting them.

At least for $k_0\neq 0$, the four primitive operators admit of a simple physical interpretation in view of the relations giving a fluid's flow velocity field and pressure field in terms of a velocity potential field (Ref. 8, p. 308). In fact, let $\phi(\mathbf{r}_\partial)$ and $\psi(\mathbf{r}_\partial)$ be singlet- and doublet-layer source distributions, respectively, on $\partial\Omega$ such that the associated velocity potentials, for $\mathbf{r}_1\in\Omega\cup\Omega^{\mathrm{ex}}$, are [Ref. 11, Eqs. (2.31) and (2.33)]

$$u(\mathbf{r}_1)=-\int_{\partial\Omega} dA_2 G^{+}_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})\phi(\mathbf{r}_{\partial 2}) \,, \tag{22}$$

$$v(\mathbf{r}_1)=-\int_{\partial\Omega} dA_2 \frac{\partial G^{+}_{k_0}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 2})\psi(\mathbf{r}_{\partial 2}) \,. \tag{23}$$

Then the limiting values (which are proportional to the limiting surface pressures) and limiting normal gradients (which are the limiting normal fluid velocities) derived from $u$ and $v$ on $\partial\Omega$ can be expressed in terms of the operators $U_{k_0}$, $V_{k_0}$, $V^{\tau}_{k_0}$, and $W_{k_0}$ according to Table I (see Ref. 11, Theorems 2.12, 2.13, 2.19, and 2.21), where the + or − *subscripts* indicate whether $\partial\Omega$ is approached from $\Omega^{\mathrm{ex}}$ or $\Omega$, respectively. From the results given in Table I, the following jump conditions are obtained (Ref. 11, Theorems 2.12, 2.14, 2.18, and 2.21):

TABLE I. Effects of the operators $U_{k_0}$, $V_{k_0}$, $V^{\tau}_{k_0}$, and $W_{k_0}$.

| Result \ Operand | Singlet-layer density $\phi$ | Doublet-layer density $\psi$ |
|---|---|---|
| $u_{\pm}$, $v_{\pm}$ | $U_{k_0}/2$ | $(V_{k_0}\pm I_\partial)/2$ |
| $\dfrac{\partial u_{\pm}}{\partial n}$, $\dfrac{\partial v_{\pm}}{\partial n}$ | $(V^{\tau}_{k_0}\mp I_\partial)/2$ | $W_{k_0}/2$ |

$$u_+ - u_- = 0 \,, \tag{24}$$

$$v_+ - v_- = \psi \,, \tag{25}$$

$$\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} = -\phi \,, \tag{26}$$

$$\frac{\partial v_+}{\partial n} - \frac{\partial v_-}{\partial n} = 0 \,. \tag{27}$$

### B. The radiation impedance and admittance operators

Let $k_0$ be complex with $\mathrm{Im}(k_0) \geq 0$, and let $\Psi^+_{k_0}(\mathbf{r})$ be an outgoing-wave solution to the scalar Helmholtz equation in $\Omega^{\mathrm{ex}}$, for which the sources are all in $\Omega$ or are distributed on $\partial\Omega$. In view of the existence and uniqueness theorems for the exterior Dirichlet and exterior Neumann boundary-value problems (Ref. 11, Chaps. 3.3 and 3.4), the wave function $\Psi^+_{k_0}(\mathbf{r})$ is completely and uniquely determined either by its limiting-value function $\Psi^+_{k_0}(\mathbf{r}_\partial +)$, or by its limiting-normal-derivative function $(\partial\Psi^+_{k_0}/\partial n)(\mathbf{r}_\partial +)$ on $\partial\Omega$. Hence there is a one-to-one correspondence established, in that a limiting-value function on $\partial\Omega$ for some outgoing-wave solution uniquely determines a limiting-normal-derivative function on $\partial\Omega$, and conversely. Moreover, any reasonably well-behaved, complex-valued function on $\partial\Omega$ can serve in either role, and there is obviously a linear relationship between the limiting-value functions and the corresponding limiting-normal-derivative functions. There must, therefore, exist two mutually inverse operators—we shall suppress the $\partial\Omega$ dependence and call them $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$—which map $\mathcal{F}(\partial\Omega)$ into itself linearly, and which have the uniquely defining properties that for any outgoing-wave solution $\Psi^+_{k_0}(\mathbf{r})$,

$$\begin{aligned} \Psi^+_{k_0}(\mathbf{r}_\partial +) &= \left[\check{Z}_{k_0}\frac{\partial \Psi^+_{k_0}}{\partial n}\right](\mathbf{r}_\partial) \,, \\ \frac{\partial \Psi^+_{k_0}}{\partial n}(\mathbf{r}_\partial +) &= (\check{Z}^{-1}_{k_0}\Psi^+_{k_0})(\mathbf{r}_\partial) \,. \end{aligned} \tag{28}$$

We note that "mutually inverse" here means that

$$\check{Z}_{k_0}\check{Z}^{-1}_{k_0} = I_\partial = \check{Z}^{-1}_{k_0}\check{Z}_{k_0} \,. \tag{29}$$

The "háček" in $\check{Z}_{k_0}$ is intended to distinguish this entity from the conventional surface acoustic impedance of an obstacle (Ref. 8, p. 311); $\check{Z}_{k_0}$ depends on the geometry of the obstacle and the physical properties of the ambient, sound-transmitting fluid, but does not depend on the physical properties of the obstacle. In physical terms, $\check{Z}_{k_0}$ is proportional to the nonlocal operator that maps the local normal velocity field into the local pressure field just outside $\partial\Omega$, for a free-space acoustic wave satisfying outgoing-wave boundary conditions at infinity. Accordingly, we shall call the operators $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ the "radiation impedance" and "radiation admittance" operators for $\partial\Omega$, respectively. [As we are concerned primarily with diffraction in the frequency domain rather than in the time domain, we have incorporated into the operators defined by Eq. (28) certain simple factors that are normally not so incorporated—see Ref. 8, p. 311; note that the operators $\check{Z}_0$ and $\check{Z}^{-1}_0$ are well defined.] In this connection, Beranek (Ref. 20, Part XII) defined a complex number "radiation impedance" for an acoustic emitter in terms of the geometrical averages over the radiating surface of the normal velocity distribution and of the pressure distribution.

The operator $\check{Z}_{k_0}$ can also be construed from a different physical viewpoint: We suppose that $\Omega$ coincides with a fluid-filled cavity that is completely surrounded by a hypothetical material with surface acoustic impedance given by the nonlocal operator $\check{Z}_{k_0}$. Then any acoustic signal of frequency $k_0c$ that impinges on $\partial\Omega$ from the *inside* will be absorbed completely without reverberation by the material; the "room" $\Omega$ together with the given wall conditions on $\partial\Omega$ comprise an anechoic chamber, at least for signals with frequency $k_0c$. As we shall discuss further in Sec. III D, operators analogous to $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$, and the corresponding entities in the time domain, have a long history of appearance (sometimes implicit) in investigations of boundary-value problems.

The argument of Maccamy and Marin (Ref. 21, Appendix, proof of Lemma 1) for the two-dimensional case has an obvious three-dimensional analog; accordingly, both $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ must be symmetric operators, that is,

$$\begin{aligned} \check{Z}_{k_0} &= \check{Z}^{\tau}_{k_0} \,, \\ \check{Z}^{-1}_{k_0} &= \check{Z}^{-\tau}_{k_0} \,, \end{aligned} \tag{30}$$

where $-\tau$ means the inverse of the transpose (or, what proves to be the same, the transpose of the inverse) of an operator.

We shall now undertake some operator manipulations suggested by the argument of Ref. 11, Chap. 3.5, or Ref. 22, Sec. II. The results of Table I will be used to express the operators $\check{Z}_{k_0}$ and $\check{Z}^{-1}_{k_0}$ in terms of the operators defined in Eqs. (18)–(21). Let $u(\mathbf{r}_1)$ and $v(\mathbf{r}_1)$ be as defined in Eqs. (22) and (23), respectively. The limiting values and normal derivatives of $u(\mathbf{r}_1)$ as $\mathbf{r}_1 \rightarrow \mathbf{r}_{\partial 1}+$ are given in terms of the singlet-layer distribution $\phi(\mathbf{r}_{\partial 2})$ by the results of Table I; elimination of $\phi$ between these expressions yields

$$u_+ = U_{k_0}(V^{\tau}_{k_0} - I_\partial)^{-1}\frac{\partial u_+}{\partial n} \,, \tag{31}$$

and hence, comparing Eqs. (28) and (31), we must have

$$\begin{aligned} \check{Z}_{k_0} &= U_{k_0}(V^{\tau}_{k_0} - I_\partial)^{-1} \,, \\ \check{Z}^{-1}_{k_0} &= (V^{\tau}_{k_0} - I_\partial)U^{-1}_{k_0} \,. \end{aligned} \tag{32}$$

Similarly, applying the results of Table I to Eq. (23), and elimination of $\psi$, implies that

$$v_+ = (V_{k_0} + I_\partial)W^{-1}_{k_0}\frac{\partial v_+}{\partial n} \,. \tag{33}$$

We infer from Eqs. (28) and (33) that

$$\check{Z}_{k_0}=(V_{k_0}+I_\partial)W_{k_0}^{-1}\ ,$$
$$\check{Z}\,_{k_0}^{-1}=W_{k_0}(V_{k_0}+I_\partial)^{-1}\ . \qquad (34)$$

By taking adjoints of both sides of Eqs. (32) and (34) we find that

$$\check{Z}_{k_0}=(V_{k_0}-I_\partial)^{-1}U_{k_0}\ ,$$
$$\check{Z}\,_{k_0}^{-1}=U_{k_0}^{-1}(V_{k_0}-I_\partial)\ , \qquad (35)$$

$$\check{Z}_{k_0}=W_{k_0}^{-1}(V_{k_0}^{\tau}+I_\partial)\ ,$$
$$\check{Z}\,_{k_0}^{-1}=(V_{k_0}^{\tau}+I_\partial)^{-1}W_{k_0}\ . \qquad (36)$$

We also can obtain the adjoint forms Eqs. (35) and (36) by using the definitions Eq. (28) in Eqs. (3.79) and (3.80) of Ref. 11.

We shall conclude this subsection by establishing formal relationships between certain (two-sided) limiting values or limiting normal gradients of the $N$- and $D$-type complete Green's functions, and the operators $I_\partial$, $\check{Z}_{k_0}$, and $\check{Z}\,_{k_0}^{-1}$. The results, that is Eqs. (38)–(41), will be stated in a naïve mathematical form, which is meant to suggest a sequence of operations such as those that were used to obtain Table I.

We will use the following formula, which can be derived as sketched in Ref. 8, p. 808:

$$\Theta_{\Omega^{\rm ex}}(\mathbf{r}_1)G_{Xk_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)-G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)=\int_{\partial\Omega}dA_3\left[G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\frac{\partial G_{Xk_0}^{+}}{\partial n_l}(\mathbf{r}_{\partial 3}+;\mathbf{r}_2)-\frac{\partial G_{k_0}^{+}}{\partial n_r}(\mathbf{r}_1;\mathbf{r}_{\partial 3})G_{Xk_0}^{+}(\mathbf{r}_{\partial 3}+;\mathbf{r}_2)\right]\ , \qquad (37)$$

where for $X=R$, $N$, or $D$, the leading factor $\Theta_{\Omega^{\rm ex}}(\mathbf{r}_1)$ is redundant by Eq. (16).

We consider the $N$ case first. In Eq. (37), let $\mathbf{r}_1\to\mathbf{r}_{\partial 1}+$, followed by $\mathbf{r}_2\to\mathbf{r}_{\partial 2}++$. Using the results of Sec. III A, we obtain an operator equation which can be solved to yield the symmetric limit operator

$$G_{Nk_0}^{+}(\mathbf{r}_{\partial 1}+;\mathbf{r}_{\partial 2}++)=\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\ , \qquad (38)$$

where we have used Eq. (35). Physically speaking then, $\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ is proportional to the acoustic overpressure at the surface point $\mathbf{r}_{\partial 1}+$ with a time-harmonic point source at $\mathbf{r}_{\partial 2}+$, provided that the obstacle forces the sound wave to satisfy "sound-hard" boundary conditions on $\partial\Omega$. Next we let $\mathbf{r}_1\to\mathbf{r}_{\partial 1}+$, and then take the limiting normal derivative of both sides of Eq. (37) at $\mathbf{r}_{\partial 2}++$; an operator-algebraic equation is obtained, the solution of which is

$$\frac{\partial G_{Nk_0}^{+}}{\partial n_r}(\mathbf{r}_{\partial 1}+;\mathbf{r}_{\partial 2}++)=\delta_{\partial\Omega}^{2}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\ . \qquad (39)$$

For the $D$ case, we first take the limiting normal gradient of both sides of Eq. (37) as $\mathbf{r}_1\to\mathbf{r}_{\partial 1}+$, and then let $\mathbf{r}_2\to\mathbf{r}_{\partial 2}++$. The solution of the resulting operator equation is

$$\frac{\partial G_{Dk_0}^{+}}{\partial n_l}(\mathbf{r}_{\partial 1}+;\mathbf{r}_{\partial 2}++)=-\delta_{\partial\Omega}^{2}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\ . \qquad (40)$$

A fourth limiting form can be derived from Eq. (37) ($D$ case) by taking the limiting normal gradients of both sides first with respect to $\mathbf{r}_1$ as $\mathbf{r}_1\to\mathbf{r}_{\partial 1}+$, then with respect to $\mathbf{r}_2$ as $\mathbf{r}_2\to\mathbf{r}_{\partial 2}++$. The solutions of the resulting operator equation and Eq. (36) imply that we have the symmetric limiting operator

$$\frac{\partial^2 G_{Dk_0}^{+}}{\partial n_l\partial n_r}(\mathbf{r}_{\partial 1}+;\mathbf{r}_{\partial 2}++)=-\check{Z}\,_{k_0}^{-1}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})\ . \qquad (41)$$

### C. Mathematical observations

In this subsection, we shall note that the results of Secs. III A and III B give rise to certain mathematical properties of the operators discussed there; these properties do not seem to have been stated or investigated explicitly in the mathematical literature concerned with the diffraction problem.

We note first that if we combine Eq. (32) with Eq. (35), and Eq. (34) with Eq. (36), we find that the operator products $V_{k_0}U_{k_0}$ and $W_{k_0}V_{k_0}$ must be symmetric operators.

Suppose now that $\Omega$ is bounded and nonempty. The operators $U_{k_0}$ and $V_{k_0}^{\tau}-I_\partial$ each have as null space the linear span of the functions $(\partial u/\partial n)(\mathbf{r}_\partial-)$ corresponding to solutions $u(\mathbf{r})$ to the interior homogeneous Dirichlet boundary value problem for the scalar Helmholtz equation (Ref. 11, Theorems 3.30 and 3.22), while $V_{k_0}+I_\partial$ and $W_{k_0}$ have as null spaces the linear span of solution functions $v(\mathbf{r}_\partial-)$ corresponding to solutions $v(\mathbf{r})$ of the interior homogeneous Neumann boundary-value problem (Ref. 11, Theorems 3.32 and 3.17). If $\mathrm{Im}(k_0)\geq 0$, these null spaces are trivial, i.e., comprise just the zero function on $\partial\Omega$, except at countable, isolated sets of real $k_0$ values. But as a result of Ref. 11, Theorem 3.13, $\check{Z}_{k_0}$ and $\check{Z}\,_{k_0}^{-1}$ will exist whenever $\mathrm{Im}(k_0)\geq 0$, in particular, at the exceptional values of $k_0$. Hence, when $k_0$ tends to an eigenvalue of the interior homogeneous Dirichlet or interior homogeneous Neumann problem, the operator expressions of Eq. (32) and those of Eq. (34) represent operator quotients, the limits of which exist in a suitable sense and are the operator $\check{Z}_{k_0}$ or $\check{Z}\,_{k_0}^{-1}$ for the exceptional wave number. A further mathematical property of the operator $k_0\check{Z}_{k_0}$ is derived in the Appendix to this paper.

The third observation stems from scrutiny of an iterative process suggested by Born,[23] and mentioned by Stratton and Chu,[24] for the improvement of an approximate solution to a boundary-value problem for the scalar

Helmholtz equation. This process converges in one iteration, generally to an incorrect source distribution, and does not afford a means for the systematic improvement of an initial approximate solution, as was noted by Franz[25] and Schelkunoff.[26] This property of Born's process can be analyzed into the present operator language. In fact, consider two-row vectors $\mathbf{Q}$ such that the first and second rows consists of, respectively, any singlet layer of sources and any doublet layer of sources on $\partial\Omega$. Operators in this "direct sum" space are two-by-two matrices, each of the four elements being an operator of the type encountered in Secs. III A and III B. One such operator is the identity—let us call it $I_\partial \oplus I_\partial$—which has $I_\partial$ for its two diagonal elements and the zero operator for its off-diagonal elements. Another operator of this type we shall call $P$, which is defined as follows:

$$P \equiv \begin{bmatrix} -\frac{1}{2}(V_{k_0}^{\tau} - I_\partial) & -\frac{1}{2}W_{k_0} \\ \frac{1}{2}U_{k_0} & \frac{1}{2}(V_{k_0} + I_\partial) \end{bmatrix} . \tag{42}$$

If we define the operator $X$ as

$$X \equiv \begin{bmatrix} 0 & -I_\partial \\ I_\partial & 0 \end{bmatrix} , \tag{43}$$

then there is a simple relation between $P$ and its adjoint (using an obvious definition) $P^{\tau}$:

$$P^{\tau} = X^{\tau}(I_\partial \oplus I_\partial - P)X . \tag{44}$$

It is straightforward to show, with the aid of Eqs. (32) and (34)–(36), that the operator $P$ is an idempotent projection operator, that is,

$$P^2 = P \quad \text{or} \quad P(I_\partial \oplus I_\partial - P) = 0 . \tag{45}$$

Moreover, as is easily verified, the "unit space" of $P$, that is, vectors such that $P\mathbf{Q} = \mathbf{Q}$, consists of vectors $\mathbf{Q}$ such that the resulting superposition of singlet- and doublet-layer acoustic potentials, calculated according to Eqs. (22) and (23) and added, is identically zero in $\Omega$, while a vector in the "null space" of $P$ gives rise to an acoustic potential that is identically zero in $\Omega^{\text{ex}}$. In view of Eq. (44), there is a simple relationship between the unit space and the null space of $P$, and the null space and the unit space of $P^{\tau}$, respectively.

We note that the "method of Neumann," as described by Hadamard,[27] for the solution of an exterior Neumann problem for the Laplace equation by means of the solution of an interior Dirichlet problem, is conveniently formulated in terms of these properties of the corresponding operator $P$.

### D. Citation of previous work involving $\check{Z}_{k_0}$, $\check{Z}_{k_0}^{-1}$, or their analogs

Following is a sketch of previous (in some cases implicit) appearances of operators akin to $\check{Z}_{k_0}$ to $\check{Z}_{k_0}^{-1}$ that were found in a search of the relevant literature.

A certain approximation scheme for the numerical treatment of diffraction theory problems makes implicit use of the operators $\check{Z}_{k_0}$ or $\check{Z}_{k_0}^{-1}$. This method, which presumes that $|k_0| \times$(obstacle dimensions) is of the order of, or less than, 1, and which makes direct numerical use of explicit discrete sets of basis functions on the obstacle's boundary, is originally due to Waterman,[28] and is sometimes known as the $T$-matrix method;[16,29] see also Ref. 11, pp. 104–106, and Refs. 30 and 31 for more theory and applications. (Analogous to the imprecise occasional usage in quantum scattering theory, the $T$ matrix referred to here denotes a complete set of only fully on the wave-number shell matrix elements of the $T$ operator.[32]) Although Waterman's $T$-matrix method can be subsumed within the general $T$-operator formalism, we shall not explore this thoeretical relationship in detail. We remark only, first, that Waterman's method, in the form of Eqs. (3.92) and (3.93) of Ref. 11, is closely allied to the mathematical task of finding a finite-rank kernel, as defined in Ref. 33, to approximate the operator $\check{Z}_{k_0}^{-1}$ or $\check{Z}_{k_0}$ of Sec. III B herein; second, that the unique solvability of the null-field equations (as formulated in Ref. 11, Theorem 3.45) derives from the existence of the operators $\check{Z}_{k_0}^{-1}$ and $\check{Z}_{k_0}$ for all $k_0$ such that $\text{Im}(k_0) \geq 0$; and third, that both in principle and in practice [see paper II], realizations of the $T$-operator approach have the capability of treating short-wavelength, as well as moderate-to-long-wavelength, diffraction phenomena.

We note that a least-squares method for treating potential theory problems that is similar to Waterman's method was proposed earlier by Davis and Rabinowitz,[34,35] and was implemented in acoustic radiation problems by Schenck.[36] The operator $\check{Z}_{k_0}$, which apart from simple factors maps the surface normal velocity distribution into the surface pressure field on an acoustic radiator, is defined implicitly in Eq. (11) of Ref. 36. A finite matrix approximation to the kernel $\check{Z}_{k_0}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ for surfaces that are spheres deformed only along radial lines (called star-shaped surfaces) is obtained in Eq. (20) of Ref. 37.

The "method of Neumann" described by Hadamard[27] and a procedure described by Hörmander[38] entail the solution of an integral equation derived from potential theory that amounts to a particular application of Eq. (35) for $\check{Z}_0$ and $\check{Z}_0^{-1}$, respectively. Taylor[39] defines, and studies the properties of, a "Neumann operator" $N$, which in the present context corresponds to the full spectrum of operators $\check{Z}_{k_0}^{-1}$, $-\infty < k_0 < \infty$, transformed to the time domain.

In the context of boundary-value problems associated with Laplace's equation in $E^2$, Kang [Ref. 40, Eq. (9), and the unnumbered equation on line 9 of p. 1446] states the analog of Eqs. (41) and (38), respectively. Kang[40] attributes the second result to Hilbert,[41] and attributes to Birkhoff[42] the definition of a generalized analog to $\check{Z}_{k_0}$ which Birkhoff called the "albedo function" [Ref. 42, §12]. Birkhoff also in effect defines the operator analogous to $\check{Z}_{k_0}^{-1}$ in a special case (cf. Problem II′ on p. 174 of Ref. 42), and states (Ref. 42, p. 175) that this operator is unbounded and that its kernel is nonintegrable. We shall analyze the dominant singular behavior of $\check{Z}_{k_0}$ and of $\check{Z}_{k_0}^{-1}$ for smooth $\partial\Omega$ in paper II.

Lévy [cf. Ref. 43, p. 117, Eq. (15)] in effect states an analog of Eq. (41) as a means of obtaining an immediate solution to the Dirichlet boundary-value problem for Laplace's equation in $E^2$. In related work Levi[44] (working in $E^2$), Cisotti[45] (in $E^3$), and Lévy[46,47] (in $E^2$ and $E^3$) studied the behavior of either $G_{N0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})$ or $(\partial G_{D0}/\partial n_r)(\mathbf{r}_1;\mathbf{r}_{\partial 2})$ when $\mathbf{r}_1$ is close to $\mathbf{r}_{\partial 2}$ (see also paper II).

Several comparatively recent papers[21,22,40,48–63] have dealt explicitly with the task of obtaining and verifying approximate forms of $\check{Z}_{k_0}^{-1}$ or its analogs, in some cases in the time domain. One subset of papers[21,22,48–52] investigated nonlocal approximations that were intended to realize $\check{Z}_{k_0}^{-1}$ directly as a kernel of finite rank, the application being to enforce outgoing-wave boundary conditions (also called radiation, absorbing, or nonreflecting boundary conditions) on an artificial outer boundary. In another subset, a local condition, usually involving differential operations, was sought to approximate the effects of $\check{Z}_{k_0}^{-1}$, where the $\partial\Omega$ was either an artificial outer boundary,[40,53–56] or a generic boundary,[57] or the boundary of the scattering obstacle itself;[58–63] the latter are called "on-surface radiation conditions," or the like, in Refs. 58–63.

## IV. STRUCTURE OF THE $T$ OPERATORS

In this section, we shall use the mathematical results of Secs. III A and III B to obtain an expression, of the type of Eq. (1), for the Green's function $G_{Rk_0}^{+}$ defined by Eqs. (15)–(17).

Let us now consider, for any fixed $\mathbf{r}_2$, the wave function in $\mathbf{r}_1$

$$G_{Rk_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)-G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)\ . \tag{46}$$

In view of Eqs. (16) and (17), this wave function is a source-free solution to the scalar Helmholtz equation for $\mathbf{r}_1\in\Omega^{\rm ex}$, which satisfies outgoing-wave boundary conditions at infinity, and which is identically zero if $\mathbf{r}_2\in\Omega$. The boundary values and normal derivatives of this exterior solution must satisfy Eq. (28), that is,

$$\begin{aligned}G_{Rk_0}^{+}(\mathbf{r}_{\partial 1}+;\mathbf{r}_2)-G_{k_0}^{+}(\mathbf{r}_{\partial 1}+;\mathbf{r}_2)\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)\\ =\left[\check{Z}_{k_0}\left[\frac{\partial G_{Rk_0}^{+}}{\partial n_l}-\frac{\partial G_{k_0}^{+}}{\partial n_l}\Theta_{\Omega^{\rm ex}}\right]\right](\mathbf{r}_{\partial 1}+;\mathbf{r}_2)\ .\end{aligned} \tag{47}$$

The exterior limiting values and normal gradients on $\partial\Omega$ of $G_{Rk_0}^{+}$ also satisfy Eq. (15), so that we now have two functional equations for two unknown functions. Following some algebra, we find that

$$\begin{aligned}G_{Rk_0}^{+}(\mathbf{r}_{\partial 1}+;\mathbf{r}_2)=&\left[[-(A+B\check{Z}_{k_0}^{-1})^{-1}A\check{Z}_{k_0}+\check{Z}_{k_0}]\right.\\ &\left.\times\left[\check{Z}_{k_0}^{-1}G_{k_0}^{+}-I_\partial\frac{\partial G_{k_0}^{+}}{\partial n_l}\right]\right](\mathbf{r}_{\partial 1}+;\mathbf{r}_2)\\ &\times\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)\ ,\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial G_{Rk_0}^{+}}{\partial n_l}(\mathbf{r}_{\partial 1}+;\mathbf{r}_2)=&-\left[[\check{Z}_{k_0}^{-1}(A+B\check{Z}_{k_0}^{-1})^{-1}A\check{Z}_{k_0}]\right.\\ &\left.\times\left[\check{Z}_{k_0}^{-1}G_{k_0}^{+}-I_\partial\frac{\partial G_{k_0}^{+}}{\partial n_l}\right]\right]\\ &\times(\mathbf{r}_{\partial 1}+;\mathbf{r}_2)\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)\ .\end{aligned} \tag{49}$$

We note that the factors $\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)$ on the right-hand sides of Eqs. (48) and (49) are redundant: if $\mathbf{r}_2\in\Omega$, $G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)$ is (in $\mathbf{r}_1$) an outgoing-wave solution with no sources in $\Omega^{\rm ex}$, so that its boundary values and normal derivatives satisfy Eq. (28), and accordingly,

$$\left[\check{Z}_{k_0}^{-1}G_{k_0}^{+}-I_\partial\frac{\partial G_{k_0}^{+}}{\partial n_l}\right](\mathbf{r}_{\partial 1}+;\mathbf{r}_2)=0,\quad \text{if } \mathbf{r}_2\in\Omega\ . \tag{50}$$

Reciprocity for $G_{k_0}^{+}$ and Eq. (30) now imply that

$$\left[G_{k_0}^{+}\check{Z}_{k_0}^{-1}-\frac{\partial G_{k_0}^{+}}{\partial n_r}I_\partial\right](\mathbf{r}_1;\mathbf{r}_{\partial 2}+)=0,\quad \text{if } \mathbf{r}_1\in\Omega\ . \tag{51}$$

Having obtained the exterior limiting values and normal gradients on $\partial\Omega$ of $G_{Rk_0}^{+}$ with respect to its left-hand argument, we can use Eq. (37) to reconstruct the entire exterior solution. Before stating the intermediate result, we define a quantity that will appear in the solution of the Robin boundary-value problem:

$$\begin{aligned}&[G_{k_0}^{+}J_{Rk_0}^{\partial\Omega}G_{k_0}^{+}](\mathbf{r}_1;\mathbf{r}_2)\\ &\equiv\left[\left[G_{k_0}^{+}\check{Z}_{k_0}^{-1}-\frac{\partial G_{k_0}^{+}}{\partial n_r}I_\partial\right][(A+B\check{Z}_{k_0}^{-1})^{-1}A\check{Z}_{k_0}]\right.\\ &\quad\left.\times\left[\check{Z}_{k_0}^{-1}G_{k_0}^{+}-I_\partial\frac{\partial G_{k_0}^{+}}{\partial n_l}\right]\right](\mathbf{r}_1;\mathbf{r}_2)\ .\end{aligned} \tag{52}$$

We combine Eqs. (37), (48), (49), and (52); after some algebra, we obtain

$$\begin{aligned}G_{Rk_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)=&G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)\\ &-\left[\frac{\partial G_{k_0}^{+}}{\partial n_r}\check{Z}_{k_0}\left[\check{Z}_{k_0}^{-1}G_{k_0}^{+}-I_\partial\frac{\partial G_{k_0}^{+}}{\partial n_l}\right]\right](\mathbf{r}_1;\mathbf{r}_2)\\ &-[G_{k_0}^{+}J_{Rk_0}^{\partial\Omega}G_{k_0}^{+}](\mathbf{r}_1;\mathbf{r}_2)\ .\end{aligned} \tag{53}$$

The right-hand side of Eq. (53) is not yet in the precise form of Eq. (1) needed to abstract the $T$ operator by "removing" the $G_{k_0}^{+}$'s. In order to achieve the desired form, we first obtain, by replacing $G_{Xk_0}^{+}$ with $G_{k_0}^{+}$ in Eq. (37),

$$\begin{aligned}0=&[\Theta_{\Omega^{\rm ex}}(\mathbf{r}_1)-\Theta_{\Omega^{\rm ex}}(\mathbf{r}_2)]G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)\\ &+\left[\frac{\partial G_{k_0}^{+}}{\partial n_r}I_\partial G_{k_0}^{+}-G_{k_0}^{+}I_\partial\frac{\partial G_{k_0}^{+}}{\partial n_l}\right](\mathbf{r}_1;\mathbf{r}_2)\ .\end{aligned} \tag{54}$$

It is also convenient first to make the following definitions:

$$[G_{k_0}^+ H_{\Omega k_0} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \equiv -\tfrac{1}{2}\int_\Omega d^3r_3\{[(\nabla_r^2+k_0^2)G_{k_0}^+(\mathbf{r}_1;\mathbf{r}_3)]G_{k_0}^+(\mathbf{r}_3;\mathbf{r}_2)+G_{k_0}^+(\mathbf{r}_1;\mathbf{r}_3)[(\nabla_l^2+k_0^2)G_{k_0}^+(\mathbf{r}_3;\mathbf{r}_2)]\} \tag{55}$$

$$= -\tfrac{1}{2}[\Theta_\Omega(\mathbf{r}_1)+\Theta_\Omega(\mathbf{r}_2)]G_{k_0}^+(\mathbf{r}_1;\mathbf{r}_2) , \tag{56}$$

$$[G_{k_0}^+ F_{\partial\Omega} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \equiv \frac{1}{2}\left[\frac{\partial G_{k_0}^+}{\partial n_r} I_\partial G_{k_0}^+ + G_{k_0}^+ I_\partial \frac{\partial G_{k_0}^+}{\partial n_l}\right](\mathbf{r}_1;\mathbf{r}_2) . \tag{57}$$

The term "operator" is used herein to describe certain entities, as the $J_{Rk_0}^{\partial\Omega}$ in Eq. (52) and the $F_{\partial\Omega}$ in Eq. (57), which differentiate functions standing on both sides of a matrix element expression. The name operator needs qualification therefore, insofar as the result of an operation on a function in $\mathcal{F}(E^3)$ is not another function either in $\mathcal{F}(E^3)$ or $\mathcal{F}(\partial\Omega)$, and such "operators" generally cannot be multiplied. These entities can be characterized as distributions[64] on the product space $\mathcal{F}(E^3)\otimes\mathcal{F}(E^3)$.

Let us now multiply both sides of Eq. (54) by the factor $\frac{1}{2}$, and add corresponding sides of the resulting equation to Eq. (53). Following the use of Eqs. (8) and (55)–(57), the desired expression for $G_{Rk_0}^+$ is obtained:

$$\begin{aligned} G_{Rk_0}^+(\mathbf{r}_1;\mathbf{r}_2) &= G_{k_0}^+(\mathbf{r}_1;\mathbf{r}_2) + [G_{k_0}^+ H_{\Omega k_0} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad - [G_{k_0}^+ F_{\partial\Omega} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad + \left[\frac{\partial G_{k_0}^+}{\partial n_r}\check{Z}_{k_0}\frac{\partial G_{k_0}^+}{\partial n_l}\right](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad - [G_{k_0}^+ J_{Rk_0}^{\partial\Omega} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) . \end{aligned} \tag{58}$$

The specializations to the $N$ and $D$ cases are obtained, respectively, by setting the operators $A=0$ and $B=0$ in Eq. (58), with the results

$$\begin{aligned} G_{Nk_0}^+(\mathbf{r}_1;\mathbf{r}_2) &= G_{k_0}^+(\mathbf{r}_1;\mathbf{r}_2) + [G_{k_0}^+ H_{\Omega k_0} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad - [G_{k_0}^+ F_{\partial\Omega} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad + \left[\frac{\partial G_{k_0}^+}{\partial n_r}\check{Z}_{k_0}\frac{\partial G_{k_0}^+}{\partial n_l}\right](\mathbf{r}_1;\mathbf{r}_2) , \end{aligned} \tag{59}$$

$$\begin{aligned} G_{Dk_0}^+(\mathbf{r}_1;\mathbf{r}_2) &= G_{k_0}^+(\mathbf{r}_1;\mathbf{r}_2) + [G_{k_0}^+ H_{\Omega k_0} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad + [G_{k_0}^+ F_{\partial\Omega} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) \\ &\quad - [G_{k_0}^+ \check{Z}_{k_0}^{-1} G_{k_0}^+](\mathbf{r}_1;\mathbf{r}_2) . \end{aligned} \tag{60}$$

It is now straightforward to infer from Eqs. (58)–(60) expressions for matrix elements of the $T$ operators with respect to pairs of functions in $\mathcal{F}(E^3)$, as in Eq. (6); we shall not do so here, however. A discussion in the Appendix deals with the subject of imposing restrictions on the operators $A$ and $B$ so that the existence of the operator $(A+B\check{Z}_{k_0}^{-1})^{-1}$, which appears in Eqs. (48), (49), (52), and (58), can be guaranteed. The Appendix also specifies conditions on $A$ and $B$ so that the $T_{Rk_0}^+$ operator is symmetric, which implies in turn that the associated complete Green's function satisfies the principle of reciprocity, Eq. (4).

In Eqs. (58)–(60) we have achieved the principal aim of this paper, that is, we have found expressions, in terms of known operators, and of formally simple operators whose existence is at least plausible, such that Eq. (1) is satisfied. The only operators of the latter type that appear are $(A+B\check{Z}_{k_0}^{-1})A$, $\check{Z}_{k_0}$, $\check{Z}_{k_0}^{-1}$, or a product of these. Scalar-wave diffraction theory in the frequency domain, for given obstacle geometry and surface boundary conditions, thus reduces to finding suitable approximations for certain operators of this class for each frequency of interest, and quadratures.

## APPENDIX: MATHEMATICS OF THE IMPEDANCE BOUNDARY-VALUE PROBLEM

In this appendix we shall establish conditions on the operators $A$ and $B$ of Eq. (15) such that the existence, uniqueness, and reciprocity of the Green's function $G_{Rk_0}^+(\mathbf{r}_1;\mathbf{r}_2)$ are guaranteed. We will assume in this appendix that $\mathrm{Im}(k_0)\geq 0$ and that, unless otherwise stated, $k_0\neq 0$.

We sometimes use Dirac bra-ket notation for complex-valued functions on $\partial\Omega$: the value of a function $|\phi\rangle$ at $\mathbf{r}_\partial\in\partial\Omega$ is denoted $\langle\mathbf{r}_\partial|\phi\rangle$ and its complex conjugate is $\langle\phi|\mathbf{r}_\partial\rangle$. We define the sesquilinear (i.e., Hermitian) inner product for an ordered pair of such functions by

$$\langle\psi|\phi\rangle \equiv \int_{\partial\Omega} dA\,\langle\psi|\mathbf{r}_\partial\rangle\langle\mathbf{r}_\partial|\phi\rangle = \langle\phi|\psi\rangle^* . \tag{A1}$$

An operator $Y$ that maps this space of functions into itself has the "matrix element" $\langle\psi|Y|\phi\rangle$; the Hermitian adjoint $Y^\dagger$ of $Y$ is that operator which yields, for all choices of $|\psi\rangle$ and $|\phi\rangle$,

$$\langle\phi|Y^\dagger|\psi\rangle^* = \langle\psi|Y|\phi\rangle . \tag{A2}$$

If $Y^\dagger=Y$, the operator $Y$ is called *Hermitian*.

Let $\Psi_{k_0}^+(\mathbf{r})$ be an outgoing-wave solution in $\Omega^{\mathrm{ex}}$ to the scalar Helmholtz equation, such that $\langle\mathbf{r}_\partial|\Psi_{k_0}^+\rangle$ and $\langle\mathbf{r}_\partial|\partial_n\Psi_{k_0}^+\rangle$ are its limiting values and limiting normal derivatives, respectively, on $\partial\Omega$. Equation (28) implies that

$$\langle\mathbf{r}_\partial|\Psi_{k_0}^+\rangle = \langle\mathbf{r}_\partial|\check{Z}_{k_0}|\partial_n\Psi_{k_0}^+\rangle . \tag{A3}$$

Theorem 3.12 of Ref. 11 now reads as follows. With $|\partial_n\Psi_{k_0}^+\rangle$ as above, if $\mathrm{Im}(\langle\partial_n\Psi_{k_0}^+|k_0\check{Z}_{k_0}|\partial_n\Psi_{k_0}^+\rangle)\geq 0$, then necessarily the corresponding function $\Psi_{k_0}^+(\mathbf{r})\equiv 0$ in all $\Omega^{\mathrm{ex}}$, and accordingly $\langle\mathbf{r}_\partial|\Psi_{k_0}^+\rangle\equiv 0$ and $\langle\mathbf{r}_\partial|\partial_n\Psi_{k_0}^+\rangle\equiv 0$ everywhere on $\partial\Omega$. But, since $\langle\mathbf{r}_\partial|\partial_n\Psi_{k_0}^+\rangle$ can be any reasonable function, this is just the statement that the skew-Hermitian part $\mathcal{S}(k_0\check{Z}_{k_0})$ of the operator $k_0\check{Z}_{k_0}$, defined for any operator $Y$ as

$$\mathcal{S}(Y) \equiv (2i)^{-1}(Y-Y^\dagger) , \tag{A4}$$

is a negative-definite operator. Similarly, we can prove that $\mathscr{S}(k_0\check{Z}_{k_0}^{-1\dagger})$ is negative definite. This property of $k_0\check{Z}_{k_0}$ implies that, if $k_0$ is real, the integral over $\partial\Omega$ of the normal component of the time-averaged acoustic energy flux [Ref. 65, following Eq. (3)] is always positive for nonzero outgoing-wave solutions.

We shall now turn to the question of the invertibility of the operator $A+B\check{Z}_{k_0}^{-1}$, or, equivalently, of the operator $A\check{Z}_{k_0}+B$; this property has been taken for granted in Eqs. (48), (49), (52), and (58). The unqualified statement that an operator $X$ is invertible means that there is a unique operator $X^{-1}$ that is both a right and a left inverse operator for $X$.

We want to impose restrictions on the operators $A$ and $B$ which guarantee that the functional equation

$$(A+B\check{Z}_{k_0}^{-1})\phi=\psi \tag{A5}$$

has a unique solution $\phi\in\mathcal{F}(\partial\Omega)$ for every choice of $\psi\in\mathcal{F}(\partial\Omega)$. In order to encompass a wide range of possible applications, it is desirable that these restrictions be reasonably general and natural in form, and not involve relatively inacessible operators as $\check{Z}_{k_0}^{-1}$ and $\check{Z}_{k_0}$. As in the case of the theory of the Fredholm alternative (Ref. 11, Theorem 1.30), the analysis is conveniently divided into the establishment of uniqueness criteria and of general existence criteria for the solution of Eq. (A5). Let $0_{\partial\Omega}(\mathbf{r}_\partial)$ be the function in $\mathcal{F}(\partial\Omega)$ which has the value zero for all $\mathbf{r}_\partial\in\partial\Omega$. In order to prove that there is at most one solution to Eq. (A5), we want to restrict $A$ and $B$ so that the only solution function $\phi(\mathbf{r}_\partial)$ to the homogeneous equation

$$(A+B\check{Z}_{k_0}^{-1})\phi=0_{\partial\Omega} \tag{A6}$$

is itself $0_{\partial\Omega}$. Concerning the existence part of the argument, the right-hand side of the functional equation [say, for $G_{Rk_0}^{+}(\mathbf{r}_{\partial 1}+;\mathbf{r}_2)$] derived from Eqs. (15) and (47) appears to have no usable restrictions to a linear subspace of $\mathcal{F}(\partial\Omega)$. Accordingly, we want Eq. (A5) to have at least one solution for every $\psi\in\mathcal{F}(\partial\Omega)$, and therefore we need to restrict $A$ and $B$ so that the operator $A+B\check{Z}_{k_0}^{-1}$ maps $\mathcal{F}(\partial\Omega)$ *onto* itself; equivalently, we require that the homogeneous adjoint equation

$$(A^{\tau}+\check{Z}_{k_0}^{-\tau}B^{\tau})\chi=0_{\partial\Omega} \tag{A7}$$

has only the trivial solution $\chi=0_{\partial\Omega}$.

In what follows, a certain unified, and $\check{Z}_{k_0}$-independent, set of restrictions on $A$ and $B$ is proposed. The conditions Eqs. (A11) and (A12) are shown to be sufficient for the uniqueness of a solution to Eq. (A5), if the solution exists; we shall then prove that a solution to Eq. (A5) necessarily exists if $A$ and $B$ are also required to satisfy the conditions Eq. (A8) and (A16).

We note that, given the invertibility of $A+B\check{Z}_{k_0}^{-1}$, the operator expression $(A+B\check{Z}_{k_0}^{-1})^{-1}A\check{Z}_{k_0}$ appearing in Eqs. (52) and (58) should be symmetric in order that the Green's function satisfy Eq. (4). Accordingly, we restrict ourselves to operators $A$ and $B$ such that $BA^{\tau}$ is symmetric, that is,

$$BA^{\tau}=(BA^{\tau})^{\tau}=AB^{\tau}\ . \tag{A8}$$

Second, the impedance boundary conditions Eq. (15) are homogeneous; that is, if $X$ is any invertible linear operator mapping $\mathcal{F}(\partial\Omega)$ onto itself, then the simultaneous substitutions $A\to XA$ and $B\to XB$ will yield a physically equivalent boundary-value problem. [Note that Eq. (58) is invariant under this transformation.] In particular, if $B$ is invertible we choose $X=B^{-1}$, so that Eq. (A5) becomes

$$(\check{Z}_{k_0}^{-1}+B^{-1}A)\phi=\psi'\in\mathcal{F}(\partial\Omega)\ . \tag{A9}$$

In the latter case, the impedance boundary-value problem reduces to a nonlocal version of the problem formulated in Ref. 11, Eq. (3.67); moreover, the operator $\check{Z}_{k_0}^{-1}+B^{-1}A$ is symmetric by Eqs. (30) and (A8), so that in this case the establishment of general existence and uniqueness of solutions to Eq. (A9) can be addressed simultaneously.

We shall, similar to the proof of Theorem 3.37 of Ref. 11, use the result that $\mathscr{S}(k_0\check{Z}_{k_0})$ is negative definite to establish uniqueness criteria for the generalized impedance boundary-value problem of Eq. (15). Let $N$ be a Hermitian operator in $\mathcal{F}(\partial\Omega)$ and let $\mu$ and $\nu$ be real numbers so that $\mu/\nu$ has the physical dimension of *length*. It is straightforward to establish the operator identity

$$\mathscr{S}[(\nu^2k_0^*\check{Z}_{k_0}^{\dagger}B^{\dagger}-\mu^2k_0A^{\dagger})N(A\check{Z}_{k_0}+B)]+\mathscr{S}[(\mu^2A^{\dagger}NA+\nu^2B^{\dagger}NB)k_0\check{Z}_{k_0}]$$
$$\equiv\mu^2\mathscr{S}(k_0^*B^{\dagger}NA)+\nu^2\check{Z}_{k_0}^{\dagger}\mathscr{S}(k_0^*B^{\dagger}NA)\check{Z}_{k_0}\ . \tag{A10}$$

The latter result suggests the following: we further restrict $A$ and $B$ to have the properties that there exist Hermitian $N$ and real $\mu,\nu$ such that

$$\mu^2A^{\dagger}NA+\nu^2B^{\dagger}NB=I_\partial\ , \tag{A11}$$

and such that the operator

$$\mathscr{S}(k_0^*B^{\dagger}NA)\ \text{ is positive semidefinite}\ . \tag{A12}$$

Suppose now that $\Psi_{k_0}^{+}$ is an outgoing-wave solution in $\Omega^{\text{ex}}$ such that its limits on $\partial\Omega$ satisfy the homogeneous impedance boundary conditions Eq. (15), that is,

$$\langle \mathbf{r}_\partial | A | \Psi_{k_0}^+ \rangle + \langle \mathbf{r}_\partial | B | \partial_n \Psi_{k_0}^+ \rangle = \langle \mathbf{r}_\partial | (A\check{Z}_{k_0} + B) | \partial_n \Psi_{k_0}^+ \rangle = 0_{\partial\Omega}(\mathbf{r}_\partial) . \quad \text{(A13)}$$

If we take the $\langle \partial_n \Psi_{k_0}^+|$, $|\partial_n \Psi_{k_0}^+\rangle$ matrix element of both sides of Eq. (A10), the first term on the left-hand side vanishes because of Eq. (A13), and we infer that, using Eqs. (A3) and (A11),

$$\langle \partial_n \Psi_{k_0}^+ | \mathcal{S}(k_0 \check{Z}_{k_0}) | \partial_n \Psi_{k_0}^+ \rangle = \mu^2 \langle \partial_n \Psi_{k_0}^+ | \mathcal{S}(k_0^* B^\dagger N A) | \partial_n \Psi_{k_0}^+ \rangle + \nu^2 \langle \Psi_{k_0}^+ | \mathcal{S}(k_0^* B^\dagger N A) | \Psi_{k_0}^+ \rangle . \quad \text{(A14)}$$

If we imagine that there is a $\langle \mathbf{r}_\partial | \partial_n \Psi_{k_0}^+ \rangle$ that is not identically zero and that satisfies Eq. (A13), we have an immediate contradiction, since the left-hand side of Eq. (A14) must be negative, while Eq. (A12) implies that the right-hand side of Eq. (A14) is non-negative. Thus the only $\langle \mathbf{r}_\partial | \partial_n \Psi_{k_0}^+ \rangle$ mapped into the zero function on $\partial\Omega$ as per Eq. (A13) is itself the zero function on $\partial\Omega$. Hence there are no nontrivial outgoing-wave solutions to the specified homogeneous exterior impedance boundary-value problem, that is, if the Green's function $G_{Rk_0}^+$ exists, it is unique.

It remains to prove a solution to Eq. (A5) always exists, given suitable restrictions on $A$ and $B$, by showing that Eq. (A7) has only the trivial solution. If either $A$ is invertible or $B$ is invertible, the problem reduces to one involving symmetric operators, as in Eq. (A9), given that Eq. (A8) holds. An example of a physical problem for which neither of the corresponding operators $A$ or $B$ has an inverse occurs when the complete Green's function obeys sound-hard conditions on a part of $\partial\Omega$ and sound-soft conditions on the remainder of $\partial\Omega$; note, however, that this problem belongs to the class for which $\chi \in \mathcal{F}(\partial\Omega)$ such that

$$\text{both } A^\tau \chi = 0_{\partial\Omega} \text{ and } B^\tau \chi = 0_{\partial\Omega} \text{ implies } \chi = 0_{\partial\Omega} . \quad \text{(A15)}$$

Equivalently, we presume that there exists at least one real number $\rho$ with the physical dimension of *length* such that the Hermitian operator

$$\rho^2 A A^\dagger + B B^\dagger \text{ is positive definite} . \quad \text{(A16)}$$

We will now prove that a solution to Eq. (A5) always exists, given that $A$ and $B$ satisfy Eqs. (A8), (A11), (A12), and (A15) or (A16), and given the geometrical property that every open ball with bounding radius $a > 0$ and centered at $\mathbf{r}_\partial \in \partial\Omega$ contains a nonempty intersection with $\Omega$ as well as $\Omega^{\text{ex}}$. In fact, we suppose to the contrary that there is a solution $\chi \neq 0_{\partial\Omega}$ to Eq. (A7), and will arrive at a contradiction. Let $\Phi_{k_0}^+(\mathbf{r})$ be defined in terms of a certain superposition of single- and double-layer potentials, as follows:

$$\Phi_{k_0}^+(\mathbf{r}) = -[G_{k_0}^+ A^\tau \chi](\mathbf{r}) - \left[ \frac{\partial G_{k_0}^+}{\partial n_r} B^\tau \chi \right](\mathbf{r}) . \quad \text{(A17)}$$

The wave function $\Phi_{k_0}^+(\mathbf{r})$ satisfies the Helmholtz equation for all $\mathbf{r} \in \Omega \cup \Omega^{\text{ex}}$, and satisfies outgoing-wave conditions as $r \to \infty$. Let us compute the interior limiting values and derivatives of $\Phi_{k_0}^+(\mathbf{r})$ as $\mathbf{r} \to \mathbf{r}_\partial$. We find, with the aid of Table I and Eqs. (35) and (36), that

$$\Phi_{k_0}^+(\mathbf{r}_\partial -) = \tfrac{1}{2}[U_{k_0}(A^\tau + \check{Z}_{k_0}^{-1} B^\tau)\chi](\mathbf{r}_\partial) , \quad \text{(A18)}$$

$$\frac{\partial \Phi_{k_0}^+}{\partial n}(\mathbf{r}_\partial -) = \tfrac{1}{2}[(V_{k_0}^\tau - I_\partial)(A^\tau + \check{Z}_{k_0}^{-1} B^\tau)\chi](\mathbf{r}_\partial) . \quad \text{(A19)}$$

Since $\chi$ satisfies Eq. (A7), the right-hand side of both Eqs. (A18) and (A19) is $0_{\partial\Omega}(\mathbf{r}_\partial)$. With the aid of the jump conditions Eqs. (24)–(27), we find now that

$$\Phi_{k_0}^+(\mathbf{r}_\partial +) = [B^\tau \chi](\mathbf{r}_\partial) , \quad \text{(A20)}$$

$$\frac{\partial \Phi_{k_0}^+}{\partial n}(\mathbf{r}_\partial +) = -[A^\tau \chi](\mathbf{r}_\partial) . \quad \text{(A21)}$$

Thus the exterior limiting values and normal derivatives of $\Phi_{k_0}^+(\mathbf{r})$ satisfy the homogeneous impedance boundary conditions Eq. (A13):

$$[A\Phi_{k_0}^+](\mathbf{r}_\partial +) + \left[ B \frac{\partial \Phi_{k_0}^+}{\partial n} \right](\mathbf{r}_\partial +) = [(AB^\tau - BA^\tau)\chi](\mathbf{r}_\partial) = 0_{\partial\Omega}(\mathbf{r}_\partial) , \quad \text{(A22)}$$

according to Eqs. (A20), (A21), and (A8). But we proved earlier that any outgoing-wave solution with exterior limiting values and normal derivatives satisfying Eq. (A13) must be identically zero in $\Omega^{\text{ex}}$ if, as we have presumed, Eqs. (A11) and (A12) are satisfied. Accordingly, we have from Eqs. (A20) and (A21) that the function $\chi$ satisfies both

$$A^\tau \chi = 0_{\partial\Omega} \text{ and } B^\tau \chi = 0_{\partial\Omega} , \quad \text{(A23)}$$

so Eq. (A15) implies that $\chi = 0_{\partial\Omega}$, contrary to our hypothesis. Hence, given our assumptions on $A$ and $B$, Eqs. (A6) and (A7) both have only the trivial solution, that is, the operator $A + B\check{Z}_{k_0}^{-1}$ maps $\mathcal{F}(\partial\Omega)$ onto itself in a one-to-one manner, and Eq. (A5) always has exactly one solution $\phi \in \mathcal{F}(\partial\Omega)$ for any $\psi \in \mathcal{F}(\partial\Omega)$. By definition, therefore, there is a unique operator $(A + B\check{Z}_{k_0}^{-1})^{-1}$ that acts as both a left and right inverse to $A + B\check{Z}_{k_0}^{-1}$.

The proof in the last paragraph used the presumed geometry of the obstacle in an essential way: if, for example, the obstacle had been at least in part a thin plate, or a thin screen with apertures, Eqs. (A18) and (A19) would have been meaningless for a part of $\partial\Omega$. We shall not attempt here to establish existence theorems for such geometries.

We remark that the special impedance boundary conditions analyzed in Ref. 11, Chap. 3.7, are of the type that all the above restrictions on $A$ and $B$ can be satisfied. In fact, we take $A = \lambda(\mathbf{r}_\partial) I_\partial$ and $B = I_\partial$, with $\text{Im}[k_0^* \lambda(\mathbf{r}_\partial)] \geq 0$ for all $\mathbf{r}_\partial \in \partial\Omega$. Then the choices $\mu$ = any real length, $\nu = 1, \rho = 0$, and

$$N = [\mu^2 |\lambda(\mathbf{r}_\partial)|^2 + 1]^{-1} I_\partial \quad \text{(A24)}$$

yield Eqs. (A8), (A11), (A12), and (A16).

If $k_0 \neq 0$ is real, the conditions Eqs. (A8), (A11), and (A12), together with Eq. (15) and an identity like Eq. (A10), can be manipulated to show that the integral over $\partial\Omega$ of the normal component of the time-averaged total acoustic energy flux derived from $G^+_{Rk_0}$ is never positive; that is, the obstacle can only absorb or reflect, and cannot amplify, any acoustic signal of frequency $k_0 c$ that impinges upon it. In this connection, no absorption of acoustic energy takes place for the case of $N$-type or $D$-type boundary conditions on $\partial\Omega$. Correspondingly, it can be verified by explicit computations that each of the operators $T^+_{Nk_0}$ and $T^+_{Dk_0}$, which can be inferred from Eqs. (59) and (60), satisfies the formal operator identity known as the (generalized) optical theorem—see Ref. 1, p. 90, Eq. (5.29). We shall not attempt here to relate these results to the unitarity properties of corresponding $S$ operators for acoustic signal scattering in the time domain, since, as noted in Ref. 7, acoustic-wave propagation takes place according to the wave equation, while the time-dependent theory of Ref. 1 was developed for the Schrödinger equation.

We note that if $k_0 = 0$, Theorem 3.12 of Ref. 11 breaks down, and other methods, such as those of Kellogg [Ref. 66, Chap. VIII, Sec. 1], can be used to establish some sufficient conditions for uniqueness of solutions to Eq. (A5). We state the following conditions without proving them: let $k_0 = 0$, with $A$ and $B$ such that Eq. (A11) can be satisfied with some Hermitian $N$ and real $\mu, \nu$. Then Eq. (A8) is needed for reciprocity, and the condition that the Hermitian operator

$$B^\dagger N A + A^\dagger N B \text{ is negative semidefinite} \qquad \text{(A25)}$$

is sufficient for uniqueness. A special case of this result is that obtained by adapting Theorem V on p. 214 of Ref. 66 to be a valid uniqueness condition for the corresponding exterior problem. Once uniqueness is guaranteed, the restrictions Eq. (A15) or (A16) on $A$ and $B$ are sufficient for the existence of a complete Green's function, provided that Eqs. (A18) and (A19) are geometrically meaningful for all of $\partial\Omega$ except on those possible isolated points or curves where $\partial\Omega$ is not smooth.

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