Parametric resonance in Rayleigh-Bénard convection with corrugated geometry

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(Received 14 August 1990)

Rayleigh-Bénard convection is considered with the top surface of the bounding volume sinusoidally corrugated. Parametric resonance is shown to occur if the wavelength of the corrugation is half the wavelength of the convection rolls that one would have observed with a flat surface.

The problem of flow of a fluid over a sinusoidally patterned surface was considered by Kelvin.[1] He found the remarkable result that if the flow speed exceeded a certain value the free surface of the fluid was sinusoidal and in phase with the bottom surface, but for flow speeds below this critical value the sinusoidal pattern of the free surface was out of phase with the bottom surface. Kelvin's calculation can be presented[2] as a resonance phenomenon, and the phase shift understood on the basis of the phase shift associated with resonances. In this work, we put Kelvin's geometry in a new setting, namely Rayleigh-Bénard convection and predict the existence of a parametric resonance.

In the usual Rayleigh-Bénard convection, a fluid of Prandtl number σ ($\sigma=\nu/\Lambda$, where ν is the viscous diffusion and Λ is the thermal diffusion) and coefficient of thermal expansion α is taken between two infinite parallel plates a distance d apart (in the direction of gravity, g) and heated from below maintaining a temperature difference ΔT between the plates. Convection begins when the Rayleigh number $R=\alpha(\Delta T)d^3g/\Lambda\nu$ exceeds a certain critical value R_c which depends only on the boundary conditions on the top and bottom plates for infinitely large plates. A variant of this problem is to have a time-dependent heating and to study the effect of this time dependence on the threshold Rayleigh number and the flow pattern in the convective state.[3–6] We introduce a different variant by giving one of the plates (say the top one) a sinusoidally modulated shape (see Fig. 1). The effect of this spatial modulation on the threshold can be significantly larger than that due to temporal modulations studied so far.

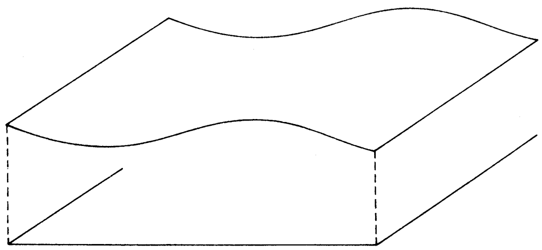

FIG. 1. The proposed geometry for a Rayleigh-Bénard convection experiment. The bottom surface is flat while the top surface is corrugated.

In the absence of convection, heat transport occurs by conduction alone, and the temperature profile is given by

$$T=T_s=T_1-\Delta T\frac{z}{d} , \tag{1}$$

where T_1 is the temperature of the lower plate which is at $z=0$ and T_2 is the temperature of the upper plate at $z=d$, with $\Delta T=T_1-T_2$. The mathematical formulation of the problem involves the study of the stability of the state $T=T_s(z)$ and velocity $\mathbf{v}=0$. To do so, one considers small perturbations, $\delta\mathbf{v}$ in the velocity and δT in the temperature, $\delta T=T(\mathbf{r})-T_s(z)$. The Navier-Stokes equation and the heat conduction equation are linearized in $\delta\mathbf{v}$ and θ under the Boussinesq approximation to yield the equations for linear stability analysis,

$$\nabla^2\left[\nabla^2-\frac{\partial}{\partial t}\right]w=R\,\nabla_1^2\theta , \tag{2}$$

$$\left[\nabla^2-\sigma\frac{\partial}{\partial t}\right]\theta=w . \tag{3}$$

In the above equations all variables are dimensionless. Spatial variables are scaled by d, time by d^2/ν, velocity by Λ/d, and temperature by ΔT. The z component of the velocity is w and $\theta=\delta T/\Delta T$. The derivative $\nabla_1^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. For an infinitely extended system, we must have translational invariance and hence periodic structure in the x-y plane. Hence θ and w are of the forms

$$\theta=\mathrm{Re}\, f(z)e^{i(a_1x+a_2y)} \tag{4}$$

and

$$w=\mathrm{Re}\, g(z)e^{i(a_1x+a_2y)} \tag{5}$$

leading to

$$(D^2-a^2)\left[D^2-a^2-\frac{\partial}{\partial t}\right]g=-Ra^2f \tag{6}$$

and

$$\left[D^2-a^2-\sigma\frac{\partial}{\partial t}\right]f=g , \tag{7}$$

where $D = d/dz$. One needs to specify boundary conditions on f and g. If the plates are good thermal conductors, then $g=0$ on $z=0$ and $z=1$ and as for the velocity field the more mathematically tractable situation is that of stress-free boundary conditions which imply $w = D^2 w = 0$ on $z=0$ and $z=1$. The conduction state is unstable if the time dependence of f and g is exponentially growing, i.e., of the form $e^{\lambda t}$, where $\text{Re}\lambda > 0$. At the transition point $\text{Re}\lambda = 0$. This particular instability is rigorously known to be a stationary bifurcation and hence at the transition $\text{Im}\lambda = 0$ as well. Consequently, the critical R $(=R_c)$ is found from self-consistent solutions of

$$(D^2 - a^2)^2 g = -R_c a^2 f \ , \tag{8}$$

$$(D^2 - a^2) f = g \ , \tag{9}$$

under the boundary conditions mentioned above. Immediately one finds $R_c = (\pi^2 + a^2)^3 / a^2$, with the minimum value obtained for $a^2 = \pi^2/2$, giving

$$R_c = 27\pi^4/4 \ . \tag{10}$$

The linear stability analysis does not say anything about the plan form. However, it is possible experimentally to obtain cylindrical rolls by suitably biasing the external conditions. In the subsequent discussion we will consider a plan form of parallel rolls [see Fig. 2(a)].

We now provide an explanation for the resonance which we talked about in the opening paragraphs. The result of Eq. (10) can be understood as the condition where the energy lost due to viscous dissipation is exactly balanced by the energy released by buoyancy. Let us now look at the situation shown in Fig. 2(b) where the spatial modulation of the top plate has a wavelength which is half that of the rolls that would be formed in the absence of the modulation. It should be apparent from the figure that the energy E_b released due to buoyancy will be increased since $E_b = \langle w\theta \rangle$ (the angular bracket denotes average over horizontal coordinates) and at the points where w is biggest, the geometry produces a higher gradient (compared to the unmodulated case). If the modulation size is of $O(\epsilon)$, the excess energy thus created should be $O(\epsilon)$ and hence it should be possible to sustain the convection at a lower critical value of the Rayleigh number. The lowering of R_c should be $O(\epsilon)$. In the rest of the paper we establish this fact in perturbation theory.

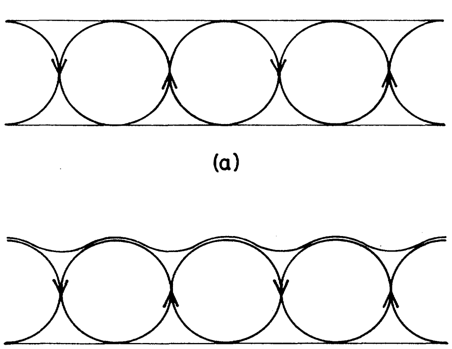

FIG. 2. (a) Section of the cylindrical flow pattern in the standard Rayleigh-Bénard geometry. (b) Section of the flow pattern in the corrugated geometry with the wavelength of the pattern twice the wavelength of the corrugation, the case for which parametric resonance is obtained.

The bounding surfaces in this new Rayleigh-Bénard problem are $z=0$ and $z=1+\epsilon\cos kx$, where $\epsilon \ll 1$. The conduction state is now to be determined under the condition that the temperature is maintained at $T=T_1$ on $z=0$ and at $T=T_2$ on $z=1+\epsilon\cos kx$. To $O(\epsilon)$, the profile can be written down by inspection as

$$T_s(x,z) = T_1 - (\Delta T)z + \epsilon \Delta T \cos kx \sinh(kz)/\sinh k \ . \tag{11}$$

To establish Eq. (11) as the first term of a perturbation series, we carry out a transformation which will be used throughout. The temperature profile in the conduction state is the solution to

$$\frac{\partial^2 T_s}{\partial x^L} + \frac{\partial^2 T_s}{\partial z^2} = 0 \tag{12}$$

under the boundary conditions

$$T_s = T_1 \quad \text{on } z=0 \tag{13}$$

and

$$T_s = T_2 \quad \text{on } z = h(x) = 1 + \epsilon \cos kx \ . \tag{14}$$

The transformation is to use the variable $\bar{z}$ defined by

$$\bar{z} = z/h(x) \tag{15}$$

such that in terms of $\bar{z}$, the boundaries are at $\bar{z}=0$ and $\bar{z}=1$. For the derivatives the following transformations hold

$$\left[\frac{\partial f}{\partial z}\right]_x = \frac{1}{h(x)} \left[\frac{\partial f}{\partial \bar{z}}\right]_x \tag{16}$$

and

$$\left[\frac{\partial f}{\partial x}\right]_z = \left[\frac{\partial f}{\partial x}\right]_{\bar{z}} - \frac{\bar{z}h'(x)}{h(x)} \left[\frac{\partial f}{\partial \bar{z}}\right]_x \ , \tag{17}$$

where the prime denotes differentiation with respect to x. Using Eqs. (16) and (17) repeatedly, Eq. (12) becomes

$$h^2 \left[\frac{\partial^2 T_s}{\partial x^2}\right]_{\bar{z}} + \frac{\partial^2 T}{\partial \bar{z}^2} = 2h'h\bar{z}\frac{\partial^2 T_s}{\partial x \partial \bar{z}} + hh''\bar{z}\frac{\partial T_s}{\partial \bar{z}} - 2h'^2\bar{z}\frac{\partial T_s}{\partial \bar{z}} - h'^2\bar{z}^2\frac{\partial^2 T_s}{\partial \bar{z}^2} \ . \tag{18}$$

The right-hand side is of $O(\epsilon)$ and higher, being proportional to derivatives of $h(x)$. On the left-hand side, $h(x)$ contains terms of $O(1)$ and higher-order terms in ϵ. The zeroth order solution is clearly

$$T_s^{(0)} = T_1 - (\Delta T)\bar{z} \tag{19}$$

and the complete profile is obtained as the series

$$T_s = T_s^{(0)} + \epsilon T_s^{(1)} + \cdots \; . \tag{20}$$

Clearly $T_s^{(1)}$ and the other terms have to satisfy $T_s^{(1)}=0$ on $\bar{z}=0$ and $\bar{z}=1$. Equating $O(\epsilon)$ terms in Eq. (18) one finds

$$\frac{\partial^2 T_s^{(1)}}{\partial \bar{z}^2} + \frac{\partial^2 T_s^{(1)}}{\partial x^2} = hh''\bar{z}\frac{\partial T_s^{(0)}}{\partial \bar{z}} = (\Delta T)\bar{z}k^2 \cos kx \tag{21}$$

leading to

$$T_s^{(1)} = \Delta T \left[\frac{\sinh k\bar{z}}{\sinh k} - \bar{z} \right] \cos kx \; . \tag{22}$$

We can easily see that Eqs. (20) and (22) are equivalent to Eq. (11) to $O(\epsilon)$.

To study the stability of this new conduction state, we obtain the analogs of Eqs. (2) and (3) using the variables $\bar{z}$ and x. The parallel rolls plan form eliminates the y dependence. The time dependence is dropped as the instability is going to be stationary. Straightforward algebra leads to

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \bar{z}^2} \right]^2 w - R_c \frac{\partial^2 \theta}{\partial x^2} = \left[1 - \frac{1}{h^4} \right] \frac{\partial^4 w}{\partial \bar{z}^4} + 2\left[1 - \frac{1}{h^2} \right] \frac{\partial^4 w}{\partial \bar{z}^2 \partial x^2} + 4\bar{z}h' \frac{\partial^4 w}{\partial x \partial \bar{z}^3} + 4\bar{z}h' \frac{\partial^4 w}{\partial x^3 \partial \bar{z}} + 6h' \frac{\partial^3 w}{\partial x \partial \bar{z}^2}$$
$$+ 6\bar{z}h'' \frac{\partial^3 w}{\partial x^2 \partial \bar{z}} + 2\bar{z}h'' \frac{\partial^3 w}{\partial \bar{z}^3} + 4h' \frac{\partial^2 w}{\partial \bar{z}^2} + 4\bar{z}h''' \frac{\partial^2 w}{\partial \bar{z} \partial x} + \bar{z}h''' \frac{\partial w}{\partial \bar{z}} \; , \tag{23}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \bar{z}^2} \right] \theta - w = \bar{z}h' \frac{\partial \theta}{\partial z} + \epsilon k w \cos(kx) \cosh k\bar{z} / \sinh(k) - \epsilon k u \sin(kx) \left[\frac{\sinh k\bar{z}}{\sinh k} - 1 \right] . \tag{24}$$

The right-hand side is of $O(\epsilon)$. Perturbation theory proceeds along standard lines[7] at this point. We expand

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots \; , \tag{25a}$$

$$\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \cdots \; , \tag{25b}$$

$$R_c = R_0 + \epsilon R_1 + \epsilon^2 R_2 + \cdots \; . \tag{25c}$$

At $O(1)$, we obtain the solution exhibited in Eq. (10), with $w_0 = A \cos a_c x \sin \pi \bar{z}$ and $\theta_0 = A \cos a_c x \sin \pi \bar{z} / (\pi^2 + a_c^2)$, where A is some constant. At $O(\epsilon)$, we are faced with an equation of the form

$$\mathcal{L} V = N \; , \tag{26}$$

where V is a two-component vector with components w_1 and θ_1. The column vector N is constructed from Eqs. (23) and (24) by collecting all the $O(\epsilon)$ terms and $\mathcal{L}$ is the operator

$$\begin{bmatrix} \nabla^4 & -R_0 \nabla_1^2 \\ \nabla^2 & -1 \end{bmatrix} ,$$

where $\nabla^2 = \partial^2/\partial \bar{z}^2 + \partial^2/\partial x^2$ and $\nabla_1^2 = \partial^2/\partial x^2$. The operator $\mathcal{L}$ has a zero eigenvector which is the $O(1)$ solution. Consequently, for the solvability of Eq. (26), the right-hand side has got to be orthogonal to the left eigenvector of $\mathcal{L}$ and that leads to

$$R_1 = \begin{cases} -\frac{10}{9} R_0 & \text{if } k = 2a_c \; , \\ 0 & \text{otherwise} \; . \end{cases} \tag{27}$$

Hence, to $O(\epsilon)$, we find the critical Rayleigh number for a modulation with $k = 2a_c$

$$R_c = \frac{27\pi^4}{4} \left[1 - \tfrac{10}{9}\epsilon \right] . \tag{28}$$

This reduction in the threshold at $O(\epsilon)$ is the parametric resonance we described above. In a more realistic calculation the stress-free boundaries have to be replaced by rigid boundaries, which changes the numbers for R_0 and R_1 without changing the qualitative situation in any way. In summary, we have shown that the conventional Rayleigh-Bénard experiment done with a spatially modulated boundary can produce an observable parametric resonance when the modulation wavelength is half the wavelength of the convection rolls.

Conversations with Professor R. A. Ferrell, T. Shinbrot, and Dr. P. G. J. Lucas are gratefully acknowledged.

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