PHYSICAL REVIEW A VOLUME 43, NUMBER 2 15 JANUARY 1991

Lévy walk in lattice-gas hydrodynamics

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(Received 6 July 1990)

In order to model turbulent channel flow, and establish a closure approximation in lattice-gas hydrodynamics, I implement a Lévy walk at the microscopic level. The algorithm is described and discussed, and results on velocity profile flattening and its shape are presented.

I. INTRODUCTION

The task of deriving the values of macroscopic quantities for large systems from an underlying microscopic model is daunting. Most often the microscopic model is overly complicated. However, for incompressible fluid phenomena a rather simple model, namely lattice-gas hydrodynamics, exists and has been tested in many simple flow situations.[1] In lattice-gas hydrodynamics point particles move from site to site on a hexagonal lattice, and undergo energy and momentum conserving collisions at sites. Upon coarse graining particle velocity and density, one recovers the Navier-Stokes equation.[2]

Here I wish to use the model to investigate the dynamics of turbulence. Lattice-gas hydrodynamics has been successfully used before in two-dimensional fully developed turbulence to study the evolution of an initially given vorticity field.[3] However, I am interested in a different issue and will be concerned with the relatively simple case of turbulent channel flow. This flow is characterized by a logarithmic velocity profile which depends on a number of constants determined from experiment. The logarithmic profile is derived from scaling arguments due to von Kármán and Prandtl.[4] The main ingredient is that of a mixing length, which increases with distance from the channel wall, over which momentum exchanges transverse to the flow take place. The same basic idea underlies our understanding of fully developed turbulence, namely, that momentum and energy exchanges occur over many different length scales. It is this phenomenon of mixing that leads to enhanced diffusion in turbulent flows.

In Ref. 5 a statistical mechanics approach to enhanced diffusion was proposed, based on Lévy distributions, which were called Lévy walks. Imagine a particle jumping from site to site in a flow. If the mean square displacement is finite the probability density will ultimately tend to a Gaussian. If it is not it will tend toward a Lévy distribution. A Lévy walk is a generalization, where to a Lévy distribution for distances is added a completion time for each jump. The probability density is thus a function of both space and time, and describes the probability of jumping a distance x in a time t. This time is taken to increase with distance and it was shown in Ref. 5 that, if the corresponding dependence is chosen correctly, enhanced diffusion characteristic of turbulence indeed results. For a sufficiently slowly falling probability distribution for distances, one recovers Richardson's law,[6] namely, that the mean square distance increases cubically in time. Additional enhancement of diffusion takes place if—as experiments suggest—it is assumed that turbulent phenomena take place on a fractal set.[5]

One is led to propose that the same behavior is obtained when a Lévy walk is implemented at the microscopic level. Lattice-gas hydrodynamics lends itself naturally to this implementation, since particles can be exchanged among lattice sites according to some space- and time-dependent distribution. This is the aim of this work: to study the implementation of Lévy walks in lattice-gas hydrodynamics in the case of pressure driven turbulent channel flow. The agreement of our results with experiment—at least as far as the velocity profile is concerned—shows that Lévy walks in lattice-gas hydrodynamics provide the equivalent of closure approximations[7] to the Navier-Stokes equations in the description of turbulent flows. Their implementation in a lattice gas may provide a testing ground for mechanisms of enhanced turbulent diffusion.

In Sec. II the algorithm and in Sec. III results on velocity profiles are described. Section IV contains a discussion of results and conclusion.

II. LÉVY WALK ALGORITHM

The ingredients of the lattice-gas algorithm are point particles that move from site to site of a hexagonal lattice. At a site they undergo two-particle, three-particle symmetric and asymmetric, and four-particle collisions whenever these conserve momentum and energy. There are no rest particles. Details of the algorithm being used can be found in Ref. 8.

A flow is set up in a two-dimensional channel in direction y. Transverse to direction y, at the channel walls, no slip boundary conditions are imposed. Microscopically they correspond to having particles which impinge on the wall bounce back into their incoming direction. At the inlet and outlet of the channel boundary conditions are periodic. However, in order to maintain the flow, certain particle configurations at the channel entrance which contain particles moving in a direction opposite to the flow are rearranged in a way which amounts to injecting momentum into the flow. This procedure leads to Poiseuille flow with the usual parabolic velocity profile depending on the transverse direction only.[8] Macroscop-

ic quantities such as density and velocity are obtained by averaging the corresponding quantities over regions of the lattice. Since the flow is essentially one dimensional these regions correspond for Poiseuille flow to strips along the whole channel in the direction of the flow, of a given width in the transverse direction. The width of the channel divided by the width of a strip gives the number of points at which the (macroscopic) velocity is determined. Actually a number of lines at the entrance of the channel are not included in the strips, in order to give room for the injected momentum to be distributed among all particles. For the largest system considered which is of width 640 in lattice units, and 640 units long, strips are ten units wide, and at an average density of two particles per site, the coarse graining involves around 12 000 particles in a strip.

The Lévy walk algorithm for a system of width $2L$ goes as follows. A number l between one and L in lattice units is drawn according to the normalized algebraic probability distribution

$$p(l)=Kl^{-z} \tag{1}$$

where $K=(z-1)/(1-L^{1-z})$, and where the exponent z characterizes the fall-off at large distances.

The exchange of particle momenta will take place between two sites at equal distance l from a randomly chosen site $\mathcal{S}$ along the common line of lattice sites perpendicular to the flow to which $\mathcal{S}$ belongs. Since the exchange involves all particles at each site, total momentum is conserved. For a chosen l the number of exchanges depends, however, on the value of l. That number is the smaller the larger l. The relationship chosen is simply linear, a maximum l of 300 (when $L=320$) leading to one exchange, whereas a minimum l of 1 gives rise to 300 exchanges. Each of these exchanges occurs at a different randomly chosen site $\mathcal{S}$ through the bulk of the flow. Since all of these exchanges take place between two updates (translation plus collision) of the lattice gas, the characteristic time associated with an exchange of given l is smaller for exchanges that occur over small distances than for those that occur over long distances. This corresponds precisely to the time-dependent aspect of a Lévy walk, where the time it takes to travel a given distance increases with that same distance. I will comment in the last section on the choice of linear relationship between distance and number of exchanges.

One moment's thought will convince one that the effect of momentum exchanges is to flatten the velocity profile as compared to the parabolic one. This is exactly what happens in turbulent channel flow. Of course the flattening can occur only if momentum can be exchanged between far apart sites. Thus if the exponent z becomes too large in Eq. (1) nothing will happen. The exponent I use in the simulations is $z=1.35$, which for an infinite system would correspond to a Lévy distribution without a first moment. From Eq. (1) the average l can be computed to be (to leading order in L)

$$\bar{l}\sim[(z-1)/(2-z)]L^{2-z}\,. \tag{2}$$

The average l increases less than linearly with the size of the system. For $z=1.35$ its value is equal to 23 in lattice units for a system of half-width equal to 320.

One last issue has to be clarified. Once a distance l has been drawn, there will be sites $\mathcal{S}$ which are closer to one of the channel walls than l. In such a case the exchange cannot take place and a new site has to be chosen. Thus the choice of sites $\mathcal{S}$ corresponds to a probability distribution which effectively is not uniform. This can be checked by counting how often site $\mathcal{S}$, which is halfway between the sites whose momenta are being exchanged, occurs in any of the 32 bins, each ten lattice units wide, that fill up half the system from say the left wall to the channel center line. (The flow is symmetric about the center line.) Take bin 1 to be at the wall and bin 32 to be at the center. Numerical results show that site $\mathcal{S}$ falls approximately with equal probability into bins 24–32, 2–3 % less often into bins 13–23, and 14% less often into bin 4, which is separated from the wall by about twice the average l. These results can be recovered in an approximate one-dimensional calculation, in which the fact that point $\mathcal{S}$ is chosen anywhere in the two-dimensional channel is ignored, together with the fact of macroscopic averaging, and where the relationship between the choice of l and the number of exchanges is neglected. Namely, call $p(x;l)$ the conditional probability density for x given l, where x denotes the distance of $\mathcal{S}$ from the left wall. It varies between 1 and L in lattice units, which is also the range of variation of l. Then

$$p(x;l)=1/L$$

if $l<x$, and

$$p(x;l)=0$$

otherwise.

Consequently the normalized probability density $P(x,l)$ for x and l is given by

$$P(x,l)=[1/(1-\bar{l}/L)]p(x;l)p(l)\,.$$

It is then straightforward to calculate the probability of x for any l, and therefore the ratio $R(x)$ of the probability of x over that of L, which is given by

$$R(x)=(1-x^{1-z})/(1-L^{1-z})\,.$$

This probability is slowly decreasing as one moves from the center of the channel to the wall. It should be compared with the simulation results quoted above. The calculation gives the following: for $x=120$, i.e., bin 12, one finds $R(x)=0.96$, whereas for bin 4 one has $R(x)=0.84$. Both of these results can be considered in very good agreement with the numerical results in view of the approximations made, which are enumerated above.

III. VELOCITY PROFILE

A. Profile flattening

The main result of transverse momentum exchanges is a flattening of the parabolic velocity profile. This flattening is the signal of turbulent channel flow. In Fig. 1 are shown the numerical results, for a system of 640×640, of

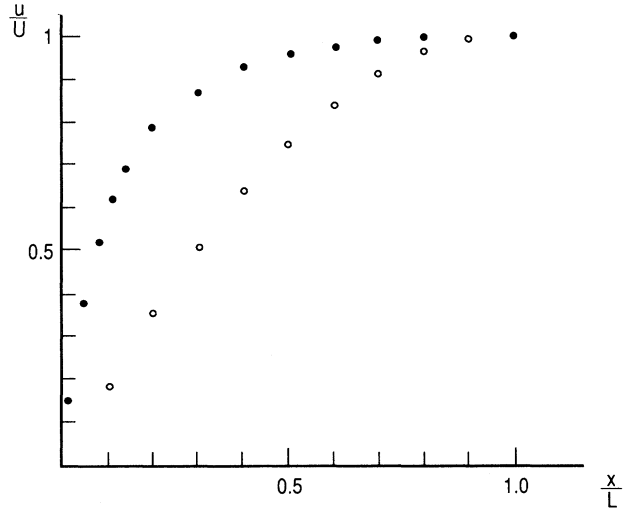

FIG. 1. Fluid velocity u normalized by the center line velocity U, as a function of the distance x to the wall, normalized by the channel half width L (solid circles). Here the exponent z in Eq. (1) is equal to 1.35, and the system studied is 640×640 in lattice units. As a comparison the usual parabolic profile for pressure driven flow is given (open circles).

fluid velocity normalized by the maximum velocity at the channel center. Because of symmetry only half of the profile is shown. The flattening is significant when compared with a parabolic profile. In shape it corresponds to the profile observed in smooth pipe flow[4] at Reynolds numbers of the order 10^5, although for the flow studied numerically the actual Reynolds number is only a few hundred. In this sense the microscopic implementation of a Lévy walk is equivalent to a closure approximation for the Navier-Stokes equation. Because it entails creating turbulent flow through the mechanism of momentum exchanges on many different scales, it enables one to reproduce experimental turbulent flow as it is observed at much higher Reynolds number. The data in Fig. 1 are obtained for an exponent z equal to 1.35 in the probability density distribution for length scales, Eq. (1). If the exponent increases there is less momentum exchange at large scales. As a result the profile is less flat. This is shown in Fig. 2 for a much smaller system (180×180) where exponent $z=1.35$ is compared with exponent $z=1.55$. Whereas the $z=1.35$ profile is compatible with the corresponding one for the larger system of Fig. 1, the $z=1.55$ one is much less flat. Clearly if z increases even further, the profile will remain parabolic because momentum exchanges over small distances will not affect it.

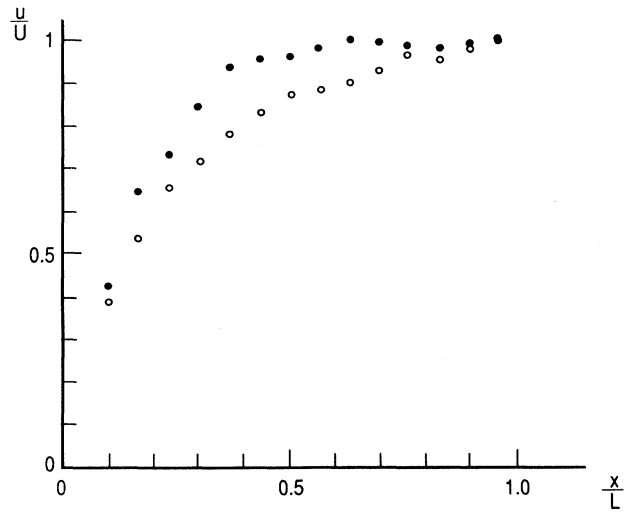

FIG. 2. A comparison for different exponents z of fluid velocity vs wall distance, normalized as in Fig. 1. Solid circles correspond to $z=1.35$, and open circles to $z=1.55$. The system here is 180×180 in lattice units. The data in this figure do not have the statistical accuracy of those presented in Fig. 1.

B. Logarithmic velocity distribution

Turbulent channel or pipe flow is characterized by a logarithmic velocity profile, given by

$$u/v^*=A\ \ln xv^*/\nu+B \tag{3}$$

where x varies from close to the wall to the center line.

This form of the profile results from similarity assumptions on turbulent shear stress, or from the introduction of a mixing length which increases linearly with distance from the wall close to it.[4] In Eq. (4), ν denotes the kinematic viscosity, and A and B are two empirically determined constants, where $A=2.5$ and $B=5.5$, which are the same for channel and pipe flow. v^* is a characteristic velocity which determines shear at the wall, shear itself being balanced by the overall pressure drop driving the flow. The logarithmic velocity profile is valid only outside the viscous sublayer, the width of which is of order ν/v^*.

As is customary I take the value of v^* from empirical fits to the u/U data for smooth pipes. These fits use the form[4]

$$u/U=(x/L)^{1/n} \tag{4}$$

where U is the maximum velocity at the channel center.

The exponent n varies between 6 and 10 for Reynolds numbers between 10^3 and 10^6. There is no single, exponent, however, which fits data of Fig. 1, although a varying exponent between 6 and 10 is adequate. This is no surprise. There is too much uncertainty in the data of Fig. 1, coming from the statistical uncertainty in u/U which is of order 1–2 %, but more so from the uncertainty in x/L since each point can be located anywhere in a bin of width 10. Small variations in velocity or position are magnified because of the high value of n. However, the value of v^* turns out to be rather insensitive to the exact value of n at the 10–20 % level of uncertainty. I therefore use the following expression[4] to calculate v^*:

$$v^*=(U/8.74)^{n/n+1}(\nu/L)^{1/n+1}\ . \tag{5}$$

With $\nu=0.16$, $L=320$, $U=0.27$, and $n=7$ one finds $v^*=0.0185$, whereas for $n=8$ one has $v^*=0.0195$, and for $n=10$, $v^*=0.021$. In the following I will thus take $v^*=0.02$.

My results for the velocity divided by v^* are shown in Fig. 3 as a function of $\ln x$. There is no reason they should follow Eq. (3). However, my data have the right features for turbulent channel flow. First, and not surprisingly in view of the results described in Fig. 1, u/v^* has the right magnitude. In particular for $\ln x$ greater than 4 the results lie between 10 and 15. Second, and most importantly, the shape of the curve is right compared with either experiment or three-dimensional numerical results based on the Navier-Stokes equation.[9] Namely, within the viscous sublayer, flow velocity is determined by viscous friction and is therefore linear in x. This region is followed by a transition region, where both laminar and turbulent friction coexist, which precedes the logarithmic region proper. These three regions are seen in Fig. 3.

In our case the thickness of the viscous sublayer is eight lattice units (the width of a bin) corresponding to $\ln x=2$. u/v^* is exponentially rising as a function of $\ln x$ in this domain. This behavior is visible in Fig. 3 in the appropriate domain and is followed by a transition region until there is a change of slope for $\ln x$ greater than 5.5. This is exactly what is observed in the experimental data.[4] Unfortunately, with a system of half-width 320 in lattice units, one reaches barely into the fully turbulent region where the logarithmic profile dominates. According to experiment this region starts when $\ln x$ is greater than 5.75, which is the limit of our system. The logarithm being a very slowly increasing function of x, a quite larger system than 320 would be required to fully establish the velocity profile in the turbulent region. It is not possible from the high $\ln x$ data in Fig. 3 to extract with any precision the slope of u/v^* versus $\ln x$, which empirically is equal to 2.5. For reference I have drawn a corresponding line of slope equal to 1.

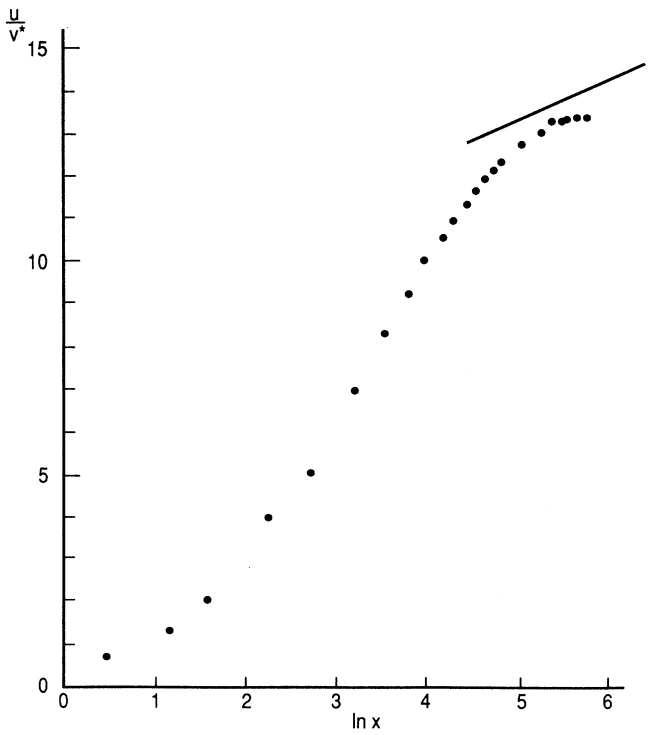

FIG. 3. Fluid velocity u normalized by the characteristic velocity v^* as a function of $\ln x$. The straight line, which corresponds to $u/v^*=\ln x+\text{const}$, is drawn for comparison.

It is remarkable that from the microscopic, Lévy-walk-based, approach which, though embodying some of the same physical ideas, is totally different from the macroscopic one, rather good agreement with experimental data is found. How much my profile follows exactly the logarithmic form of Eq. (3) cannot be decided because simulation results do not reach far enough into the region where turbulent shear stress dominates.

IV. DISCUSSION AND CONCLUSION

This work is an attempt to devise a closure mechanism in lattice-gas hydrodynamics. The mechanism proposed is that of microscopic Lévy walks. Results are encouraging for turbulent channel flow: the fluid velocity profile flattens and, most importantly, its shape and magnitude from the wall to the channel center line are in reasonable agreement with experimental data. The Lévy walk algorithm contains one parameter only, once the relation connecting the number of momentum exchanges to distances is chosen. This parameter is the exponent in the probability density distribution for distances, Eq. (1). The flattening of the velocity profile depends sensitively on this exponent. If it is too large there is no flattening at all, because momenta exchanges do not occur over large enough distances.

I have done a study on how the results reported here depend on the form of the above-mentioned relationship. I have examined the extreme but simple case where the number of exchanges is fixed whatever the distance over which momentum exchange takes place. Since, for the most frequently drawn small distances, the velocity profile is not affected by any exchange, this procedure captures the essence of the Lévy walk, if—for a given system size—the effective number of exchanges is chosen appropriately. Results are essentially the same as for the more elaborate case reported on here. Therefore, whereas for fully developed turbulence the result for maximally enhanced diffusion depends on the value of the exponent which relates the time it takes to jump a distance x to a power of x[5], it appears that for the microscopic implementation in a finite system considered here, results are insensitive to the precise form of the relationship. The advantage is that the simulation then depends sensitively on one parameter only, namely, as already stressed, on the exponent in the algebraic distance distribution.

I have also considered a different algorithm for momentum exchanges. Whereas I have reported on a method where a distance l is drawn before a point is

chosen where exchanges over l take place, in the second approach a point is chosen first, and then l drawn from an algebraic distribution which ranges from 1 to the distance of that point to the nearest wall. Again results are essentially the same.

A Reynolds tensor component connecting longitudinal and transverse fluctuations can be defined as the average of longitudinal velocity fluctuations taking place per unit time across the distance over which momentum exchange occurs. I have obtained it numerically. This Reynolds tensor component is—as expected—antisymmetric with respect to the channel center line, but the statistical noise in the data is too large to extract reliably its magnitude which is of the order of v^{*2}.[10]

As a conclusion one can say that the method proposed to implement a closure approximation in lattice-gas hydrodynamics appears successful. The method relies on microscopic Lévy walks for creating the enhanced momentum diffusion required to describe turbulent flows. Further studies will be necessary to explore issues not clarified or touched upon here. In particular, while it is clear that the velocity profile is not sensitive to the precise relationship between time and distance for segments of the Lévy walk, this might mot be so either for other quantities, such as the increase of separation between fluid elements with time, or for other turbulent situations where the proposed microscopic closure could prove itself useful.

ACKNOWLEDGMENTS

I am very grateful to Yves Pomeau for his interest in this work and for helpful discussions. I also thank Stephane Zaleski for discussions. This work was supported by the U. S. Department of Energy (Contract No. DE-F-G02-88ER13916A000), and benefitted from computer time provided by The Ohio Supercomputer Center.

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