Intermittency in a cascade model for three-dimensional turbulence

M. H. Jensen, G. Paladin,* and A. Vulpiani*
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
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We discuss a possible mechanism for intermittency of the energy dissipation in a model for three-dimensional fully developed turbulence. We compute the structure functions for the velocity field and show that their behavior can be described in the context of a multifractal approach. We also compute the instantaneous maximum Lyapunov exponent and the corresponding (stability) eigenvector. Violent bursts of energy dissipation are related to a sudden increase of the instantaneous Lyapunov exponent, and simultaneous localization of its eigenvector on the high wave numbers at the end of the inertial range. In particular, we relate the correction to the $k^{-5/3}$ Kolmogorov law for the energy spectrum to the fractal dimension extracted by temporal sequences both of the instantaneous Lyapunov exponent and of the eigenvector.

I. INTRODUCTION

The small-scale statistics of three-dimensional fully developed turbulence is one of the fundamental problems in fluid mechanics. The phenomenological theory of Kolmogorov[1,2] gives a qualitatively correct description of the main mechanism acting in incompressible fluids at high Reynolds number Re. According to Richardson's scenario,[3] in turbulent flows there is a cascade transfer of energy toward small scales where the dissipation is due to molecular friction. The cascade is hierarchical in the sense that a disturbance on a certain scale receives its energy from a larger-scale disturbance and transfer it to smaller-scale disturbances. At the end of the cascade, the smallest disturbances are characterized by very large velocity gradients because the direct conversion of kinetic energy into heat is strongly localized.

The presence of a range of length scales where inertial forces are dominant and where viscous effects as well as the external forcing can be neglected suggests the existence of (universal) scaling laws.

Assuming that the rate of nonlinear transfer of energy is constant both in space and in the steps of the energy cascade, one obtains the classical Kolmogorov results. In fact, dimensional analysis shows that the Navier-Stokes equations have singular velocity gradients in the limit of infinite Reynolds number, i.e., the velocity difference $\delta u(l) \equiv |\mathbf{u}(\mathbf{x}+\boldsymbol{l})-\mathbf{u}(\mathbf{x})| \sim l^h$ with a singularity $h=\frac{1}{3}$. It follows that, in the inertial range, the velocity structure functions scale as

$$\langle \delta u(l)^Q \rangle \propto l^{\zeta_Q}, \quad \text{with } \zeta_Q = Q/3 \tag{1.1}$$

where $\langle\ \rangle$ is a spatial average. Nevertheless there are many experimental and numerical evidences[2,4,5] that strong fluctuations of the energy transfer and dissipation are present, leading to the existence of a whole spectrum of possible singularities. In particular, the exponents ζ_Q are different from their classical value $Q/3$ and appear to be nonlinear in Q. Indeed, the intermittency of the energy dissipation is a very important feature of turbulence, even if its theoretical understanding is still at the first steps. For instance, some fractal phenomenological approaches have been proposed in the last years.[6,7] We believe that a first goal is the connection between the correction to the Kolmogorov scaling with the dynamical properties of the time evolution generated by the Navier-Stokes equations.

For this reason, it is useful to analyze particular models of the energy cascade process, instead of the complete Navier-Stokes equations, using an approach to the intermittency problem firstly proposed by Obukhov,[8] Gledzer,[9] Siggia,[10] and developed by Grappin *et al.*[11] We thus hope to reproduce the main characteristics of the small-scale statistics of turbulence by a chaotic dynamical system with a limited number of degrees of freedom.

We consider a model for the energy cascade extensively studied by Yamada and Ohkitani.[12,13] The Fourier space is divided in N shells. Each shell k_n $(n=1,2,\ldots,N)$ consists of the wave numbers k such that $K_0 2^n < k \le K_0 2^{n+1}$. The Fourier transform of the velocity difference over a length scale $\approx k_n^{-1}$ is given by a corresponding complex variable u_n. The energy is $E=\sum_n |u_n|^2/2$ and its power spectrum is $E(k_n)=|u_n|^2/(2k_n)$. The Navier-Stokes equations are thus approximated by a dynamical system with $2N$ differential equations. Let us remark that severe limitations of this type of scalar models stems from considering only the modulus of the wave number. Indeed, one loses all the effects due to the geometrical structures of turbulent eddies by neglecting phases. Moreover, for numerical simplicity, one only considers the interactions of a shell with its first and second nearest neighbors. This is a sensible approximation of the nonlinear terms of the Navier-Stokes equations, if one assumes that the energy cascade is local in the k space, with exponential decreasing interactions between shells.[14]

The linear terms of the equations are given by the viscous damping $-\nu k_n^2 u_n$, where ν is the kinematic viscosity. The nonlinear terms responsible for the forward energy transfer are just quadratic in the velocity fields with coefficients proportional to the wave number of the shell. In the unforced inviscid limit, the energy as well as the volumes in phase space are conserved. These conservation laws also hold for the Euler equations and are a fundamental requirement of any realistic model. In fact, it seems to us rather difficult to find an improved intermediate approximation between this type of scalar models and direct simulations of the Navier-Stokes equations.

In Sec. II, we recall the main features of the scalar model and compute its structure functions. We find that there are intermittency corrections to the predictions of the Kolmogorov theory which can be interpreted in the framework of the multifractal approach. Indeed, the scaling exponents ζ_Q are not linear in Q and seem to be quite close to the values obtained in experiments.[4]

In Sec. III we characterize the temporal intermittency of the system. We calculate the time sequence of the energy dissipation. Using the Taylor hypothesis, we then show that it concentrates on a fractal structure. Its dimension estimated by the Grassberger-Procaccia algorithm is found to be equal to the value estimated by the structure functions.

We also study the temporal behavior of the instantaneous maximum Lyapunov exponent and of the corresponding stability eigenvector. Our numerical results provide clear evidence that the burst of the energy dissipation are due to a sharp increase of the instantaneous maximum Lyapunov exponent and a simultaneous localization of its eigenvector in the dissipative shells consisting of the wave numbers at the end of the inertial range.

In Sec. IV one finds concluding remarks.

II. STRUCTURE FUNCTIONS OF THE SCALAR MODEL

The equations of the cascade model with N shells can be written as[12]

$$\left[\frac{d}{dt}+\nu k_n^2\right]u_n=i(a_n u_{n+1}^* u_{n+2}^* + b_n u_{n-1}^* u_{n+1}^* + c_n u_{n-1}^* u_{n-2}^*)+f\delta_{n,4} \qquad (2.1)$$

where ν is the viscosity, f is a forcing (here on the fourth mode). In the following, we shall use the notation

$$\frac{du_n^R}{dt}=F_{2n-1}(\mathbf{u}), \quad \frac{du_n^I}{dt}=F_{2n}(\mathbf{u}) ,$$

where u_n^R (u_n^I) denotes the real (imaginary) part of u_n.

The coefficients of the nonlinear terms of (2.1) follow from demanding energy and phase-space conservation in the inviscid case without forcing ($\nu=f=0$):

$$a_n=k_n, \quad b_n=\frac{-k_{n-1}}{2}, \quad c_n=\frac{-k_{n-2}}{2} ,$$
$$b_1=b_N=c_1=c_2=a_{N-1}=a_N=0 . \qquad (2.2)$$

Notice that the unstable fixed point of Eqs. (2.1) when $\nu=f=0$ is given by the Kolmogorov scaling $u_n \propto k_n^{-1/3}$.

The time evolution of the dissipative system (2.1) is chaotic and confined on a strange attractor in the $2N$-dimensional phase space, with an information dimension proportional to the number of degrees of freedom.[12]

The energy spectrum is observed to scale as $k^{-\alpha}$, in an inertial range of wave numbers with an exponent $\alpha=1+\zeta_2$ not depending on N, ν, or the particular type of forcing, but not exactly equal to the value $\frac{5}{3}$ expected by applying dimensional arguments which neglect the role of intermittency.

By a numerical integration of Eqs. (2.1) with $N=27$ and 19 shells, we have computed the structure functions (1) for positive integer moments up to $Q=12$. In the following, the averages $\langle\ \rangle$ obtained in the context of our model are time averages. Figure 1 shows the scaling of $\ln\langle|u_n|^Q\rangle$ with $\ln k_n$ (the slope is $-\zeta_Q$), which is clearly different from the Kolmogorov result $\langle|u_n|^Q\rangle \propto k_n^{-Q/3}$. Moreover, in Fig. 2 one sees that the exponents ζ_Q are not linear in Q. In particular the correction to the Kolmogorov law $\alpha=\frac{5}{3}$ for the energy spectrum is very close to the experimental value,[4] since we find $\zeta_2=\alpha-1=0.70\pm0.01$. We want to mention that Ohkitani and Yamada[13] have recently computed the variance of the energy flux. Its scaling exponent is found to be $\mu\approx0.3$, which is slightly different from our result for $\mu=\zeta_6-2=0.21\pm0.05$.

It is also interesting that our data for $Q\geq1$ can be fitted by a simple random β model:[7]

$$\zeta_Q=Q/3-\ln_2[1-x+x(\tfrac{1}{2})^{1-Q/3}], \quad x=0.12 \qquad (2.3)$$

where only two possible kinds of fragmentation are assumed in the cascade process: a disturbance generates either vorticity sheets (with probability x) or space filling disturbances, as in the Kolmogorov theory (with probability $1-x$). Let us recall that the experimental data of Anselmet *et al.*[4] can be fitted with a very similar value,[7] i.e., $x=0.125$.

The intermittency of the energy dissipation exhibited by the model is therefore consistent with the multifractal approach.[15] It is an open question to understand whether the exact values of ζ_Q (at least for not too large Q) do not depend on the details of the nonlinear interactions of the equations, and there exists a form of universality in the energy cascade, stronger than generally believed.

Let us recall that, in dynamical systems,[16] one usually defines the generalized Renyi dimension D_q. In turbulence,[15] they are related to the probability measure of a ball Λ of radius l centered in a given fluid point x, $\varepsilon(l)=\int_\Lambda d^3x\, \varepsilon^*(x)$, given by the density of energy dissipation $\varepsilon^*(x)$. In the inertial range the moments scale as

$$\langle \varepsilon(l)^Q\rangle \propto l^{(Q-1)D_Q+3} . \qquad (2.4)$$

By dimensional counting, one sees $\varepsilon(l)=l^3\delta u^3(l)/l$ so that the relation between dimensions and structure function exponents is

$$D_Q=\frac{\zeta_{3Q}+2Q-3}{Q-1} . \qquad (2.5)$$

D_0 is the fractal dimension of the set where the energy dissipation concentrates. We numerically find that

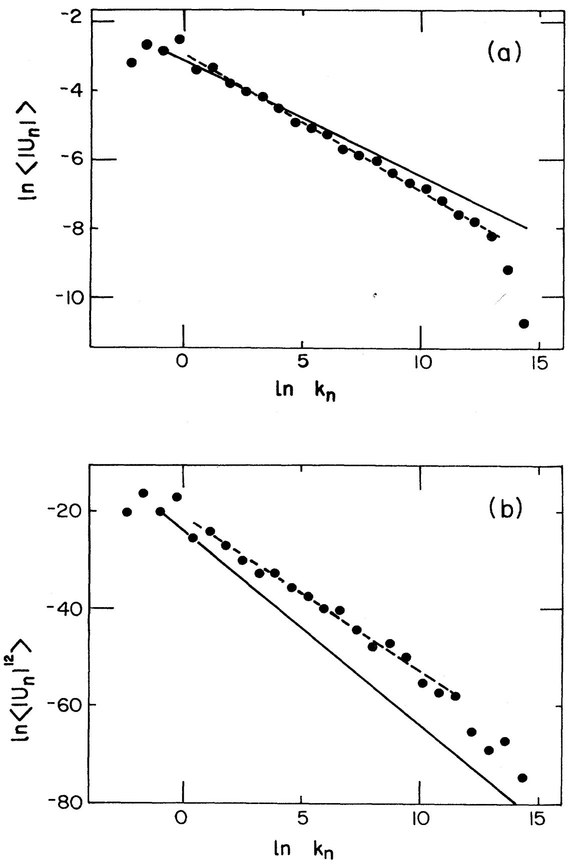

FIG. 1. Scaling of the structure functions. $\ln\langle |u_n|^Q\rangle$ vs $\ln K_n$ for (a) $Q=1$ and (b) $Q=12$ from a numerical integration of Eqs. (2.1) with $N=27$. We average over 10^4 steps where each step is 10^{-1} time unit. The solid lines with slopes respectively $-\frac{1}{3}$ and -4 indicate the Kolmogorov scaling laws. The dashed lines have slopes -0.39 and -3.18.

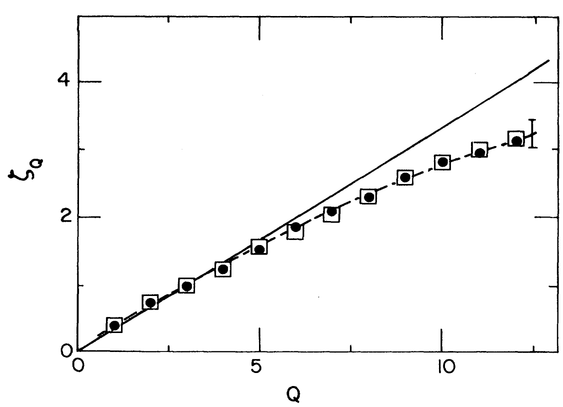

FIG. 2. The structure function exponents ζ_Q, plotted vs Q for positive integer Q up to $Q=12$. The squares are obtained by an integration of Eqs. (2.1) with $N=27$ shells, the circles with $N=19$. The errors are smaller than the size of the symbols for $Q<10$ and for $Q\geq 10$ are given by the bar close to the last point. The solid line is the Kolmogorov result $\zeta_Q=Q/3$; the dashed line is the random β model fit (2.3) with $x=0.12$.

$\lim_{Q\to 0}\zeta_Q=0$, i.e., $D_0=3$, in agreement with some experiments[17] and some recent numerical integrations of the Navier-Stokes equation with a 128^3 grid.[18] This result, however, is not in contradiction with the fractal nature of turbulence. Indeed the multifractal approach considers a hierarchy of singularities h and related fractal sets $S(h)$ of fluid points $\mathbf{x}$, such that $|\mathbf{u}(\mathbf{x}+\mathbf{l})-\mathbf{u}(\mathbf{x})|\sim l^h$. The fractal dimensions $D(h)$ of these sets are related to the exponents ζ_Q by the Legendre transformation[7,16]

$$\zeta_Q=\min_h[hQ-D(h)+3] \ . \tag{2.6}$$

If there are fluid regions (with fractal dimension 3) where the velocity gradients are not singular, that is

$$|\mathbf{u}(\mathbf{x}+\mathbf{l})-\mathbf{u}(\mathbf{x})|\sim l \ , \tag{2.7}$$

(2.6) implies that $\zeta_Q=Q$ for Q small enough. On the contrary, a nonzero fractal codimension (i.e., $\zeta_0\neq 0$) is obtained assuming $|\mathbf{u}(\mathbf{x}+\mathbf{l})-\mathbf{u}(\mathbf{x})|=0$ in the nonactive "laminar" regions instead of (2.7).

Nevertheless, the relevant dimensionality is the fractal dimension of the probability measure (information dimension) D_1 rather than the dimension of its support. D_1 is given by the derivative of ζ_Q around $Q=3$. In fact, it can be shown[15] that the most probable behavior of the velocity gradients is given by the singularity $\bar{h}=(D_1-2)/3$. This means that the probability of finding a singularity $h\neq\bar{h}$ vanishes when $\text{Re}\to\infty$. We have estimated $3-D_1=0.08\pm 0.02$, which can be compared with the value $3-D_1=0.13$ obtained by different experiments[4,17] in real turbulence.

III. SPATIAL AND TEMPORAL INTERMITTENCY

In this section we want to link the multifractal corrections discussed in the previous section with the behavior of the instantaneous maximum Lyapunov exponent and of its eigenvector.

Indeed, it has been conjectured[12] that the scaling laws of the model could be related to the features of the characteristic Lyapunov exponents[19] $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{2N}$ and the corresponding (Lyapunov) eigenvectors $\mathbf{f}_1,\mathbf{f}_2,\ldots,\mathbf{f}_{2N}$. They are defined by considering a linear variational equation of the form

$$\frac{dz_i}{dt}=A_{i,j}z_j, \quad i,j=1,\ldots,2N \tag{3.1}$$

for the time evolution of an infinitesimal increment $\mathbf{z}=\delta\mathbf{U}$, where $A_{n,j}\equiv\partial F_n/\partial U_j$ is the Jacobian matrix of Eqs. (2.1), and $\mathbf{U}=(u_1^R,u_1^I,\ldots,u_N^R,u_N^I)$. The solution for the tangent vector $\mathbf{z}$ can thus be formally written as $\mathbf{z}(t_2)=M(t_1,t_2)\mathbf{z}(t_1)$, with $M=\exp(\int_{t_1}^{t_2}A(\tau)d\tau)$. The orthonormal Lyapunov basis is then given by the $2N$ eigenvectors $\mathbf{f}_i$ of the matrix $M^\dagger M$ in the limit $t\to\infty$, and depends on the starting point $\mathbf{U}_0$ in the phase space. It is also possible[20] to introduce a stability basis $\mathbf{e}_i$ given by the eigenvectors of the matrix M. Note that a generic tangent vector $\mathbf{z}(t)$ is projected by the evolution along $\mathbf{e}_1$ [i.e., $\mathbf{z}(t)=c\exp(\lambda_1 t)\mathbf{e}_1$] a part of corrections $O(\exp(-|\lambda_1-\lambda_2|t)$.

Yamada and Ohkitani[12] have computed the time averages of the Lyapunov basis. In fact, $\langle |f_i(k_n)|^2 \rangle$ gives a measure of the localization of the instability related to the ith Lyapunov exponent on the k_n shell in the wave-number space, defining $|f_i(k_n)|^2 = |f_i(2n-1)|^2 + |f_i(2n)|^2$.

In particular, there is a strong correspondence between Lyapunov eigenvectors of the last negative Lyapunov exponents and dissipative modes following the end of the inertial scaling range. This result is somewhat expected since the viscous damping is responsible for the strongest contraction rates, so that $\lambda_{2i} \approx \lambda_{2i-1} \propto -\nu k_i^2$ for $i \approx N$.

More interesting, a large part of the Lyapunov exponents is found to be very close to zero. Their eigenvectors are concentrated, although in a less sharp way, in the inertial wave numbers. This *weak correspondence* indicates that power scaling laws are probably connected to a very large number of small (or zero) Lyapunov exponents in turbulence. Ruelle[21] has obtained a similar theoretical result for the (discrete) spectrum of the operator which linearizes the Navier-Stokes equations, in our model the Jacobian matrix A of Eqs. (3.1). He has shown that the number of operator eigenvalues γ_i with average close to zero diverges in the "unifractal" β model but only if the intermittency is strong enough, i.e., if $D_0 \le 2.6$. This result is not in disagreement with the fact that in our model $D_1 \approx 2.9$, while there is a finite fraction of Lyapunov exponents close to zero. Indeed, the only relation between Lyapunov exponents and average eigenvalues is the bound:

$$\sum_{i=1}^{m} \lambda_i \le \sum_{i=1}^{m} \langle \gamma_i \rangle \ . \tag{3.2}$$

The *weak correspondence* suggests that power laws appearing in the small-scale statistics of turbulent flows could be connected to a situation of "weak" chaos.[22] It is interesting to remark that a large number of zero Lyapunov exponents is also present in chaotic Hamiltonian systems near integrability,[22] where there are slow relaxation phenomena with characteristic times unrelated to the maximum Lyapunov exponent.

Now, in turbulence there is also a large positive maximum Lyapunov exponent, which must be proportional to the inverse of the smallest characteristic time of the system, the Kolmogorov turnover time, using dimensional arguments.[23] In light of the previous discussion, it is natural to expect that it gives origin to the intermittent corrections to the power laws connected with the almost zero Lyapunov exponents. As there is a correspondence between smallest Lyapunov exponents and dissipative small length scales, zero Lyapunov exponents and inertial range, one could imagine that there is again a rough correspondence between λ_1 and the large length scales. However, this is not true in the cascade model where the average of the first eigenvector $\mathbf{f}_1$, as well as of $\mathbf{e}_1$, is not concentrated on the small wave numbers, but spreads in the whole inertial range.[12] Our intermittent corrections to the Kolmogorov scaling laws cannot be simply related to the instability of the large scales (energy containing eddies) but must be related to a more complex mechanism involving the energy transfer in the whole inertial range.

To make this idea more precise, we have computed the response after a time τ to an infinitesimal perturbation, defining an instantaneous maximum Lyapunov exponent as

$$\chi_\tau(t) \equiv \frac{1}{\tau} \ln \left| \frac{z(t+\tau)}{\mathbf{z}(t)} \right| \ . \tag{3.3}$$

Note that $\lambda_1 = \lim_{T\to\infty} (1/T) \int_0^T dt\, \chi(t)$. We have found $\lambda_1 = 0.169 \pm 0.003$, for $N=19$ and $\nu = 10^{-6}$, in good agreement with the independent results of Ref. 12. The value of χ is an indication of the global chaoticity of the system, at a given instant. In order to determine how the chaoticity is distributed among the wave-number shells, we estimate the components of the stability eigenvector $\mathbf{e}_1$, defining $|e_1(k_n)|^2 = |e_1(2n-1)|^2 + |e_1(2n)|^2$. The value of $p(n) \equiv |e_1(k_n)|^2 / \sum_j |e_1(k_j)|^2$ can thus be interpreted as the fraction of the largest instability localized over the shell k_n. Note that $\mathbf{e}_1$ is different from $\mathbf{f}_1$, but has a direct geometrical interpretation, and seems to us more relevant from a physical point of view.

For simplicity we have focused our attention only on one dissipation wave number (practically a k_D close to the end of the inertial range, for instance k_{15} in a numerical integration with $N=19$). We have thus computed three temporal sequences for 10^4 time units, considering $N=19$ shells: the instantaneous Lyapunov exponent χ, the energy dissipation E_D estimated by $|u_{15}|^2$, and the chaoticity fraction on the shell k_D estimated by $p_D \equiv p(k_{15})$, as shown in Fig. 3. They present a very strong temporal intermittency. The average values are small compared to the large deviations which appear frequently. It is quite impressive that $p(k_{15})$ can arrive up to 0.7, while its time average is $\approx 2.1 \times 10^{-2}$.

The peaks of the three sequences are moreover very correlated. This indicates that instantaneously the chaoticity concentrates on dissipative wave numbers, in correspondence with high values of the energy dissipation and of the instantaneous Lyapunov exponent. Defining $\delta y \equiv y - \langle y \rangle$, we have computed the correlation coefficients for E_D and p_D,

$$\frac{\langle \delta E_D \delta p_D \rangle}{(\langle \delta E_D^2 \rangle \langle \delta p_D^2 \rangle)^{1/2}} \ , \tag{3.4}$$

and similarly for E_D,χ and p_D,χ which are, respectively, 0.38, 0.25, and 0.23. The correlations decay very fast in time. For instance, Fig. 4(a) shows

$$C_{E_D,p_D}(\tau) = \frac{\langle \delta E_D(t+\tau) \delta p_D(t) \rangle}{\langle \delta E_D \delta p_D \rangle} \ . \tag{3.5}$$

The correlation between energy dissipation and instantaneous Lyapunov exponent, $C_{C_D,\chi}$, as well as the correlation between tangent vector and instantaneous Lyapunov exponent, $C_{pD,\chi}$, are also fast decaying with the time delay τ. Nevertheless, there is a strong anticorrelation after a delay of order one time unit, as shown in Fig. 4(b). This means that a chaoticity burst is followed by a strong contraction rate (i.e., a negative χ).

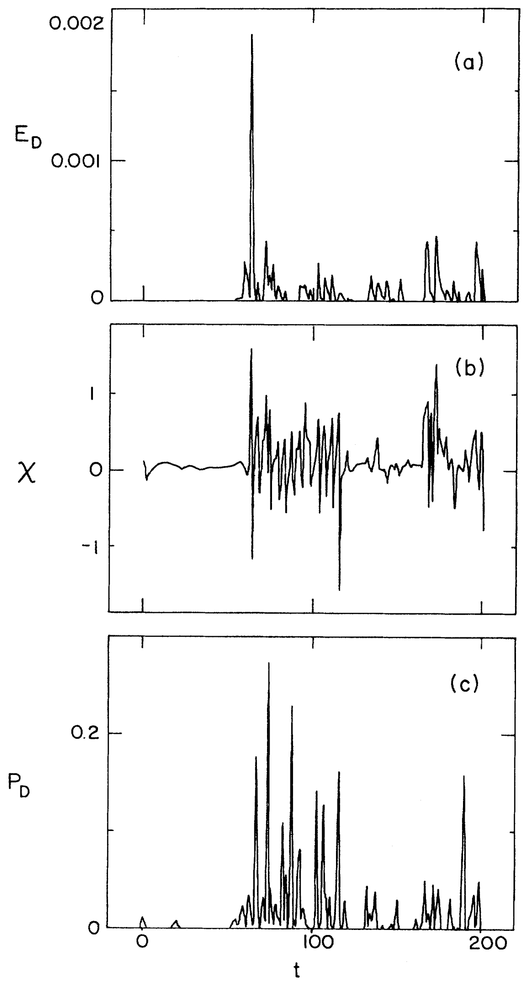

FIG. 3. Temporal sequences (200 time units) of (a) E_D, (b) the instantaneous Lyapunov exponents χ, and (c) P_D for $N=19$. The sequences show that the energy bursts are in correspondence with large deviations of χ and localization of its eigenvector on the dissipative modes. Note the laminar phase (very small E_D values) during the first 50 time units corresponding to almost vanishing p_D and χ.

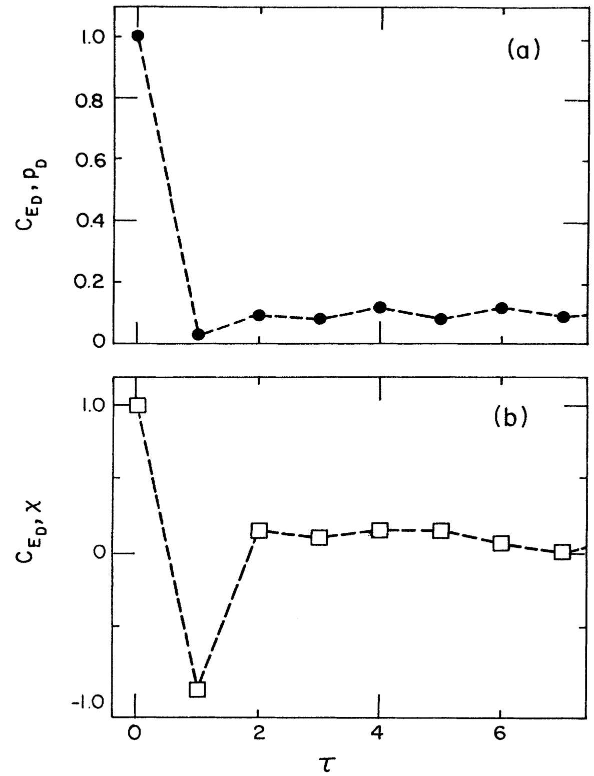

FIG. 4. From a temporal sequences of 10^4 time units for $N=19$ shells: (a) The time decay of the correlation $C_{E_D,p_D}(\tau)$ between E_D and p_D. (b) The time decay of the correlation $C_{E_D,\chi}(\tau)$ between E_D and χ. Note the strong anticorrelation after $\tau=1$ time unit.

This feature is clearly exhibited from the time sequence of χ shown in Fig. 3. In fact, a high value of E_D (energy burst) is followed by a very small value because of the forward energy transfer and dissipation. If $|u_n|$ is small for $k_n \sim k_D$, the dominant terms of the Jacobian matrix $A_{i,j}$, ruling the tangent vector evolution (3.1), are given by the viscous linear part of Eqs. (2.1). The behavior is thus of laminar type for a short period following the passage of the energy burst.

In Fig. 5 we show $p(n)$ as a function of n at two different instants which belong, respectively, to a laminar phase (very low values of E_D and $\chi \approx 0$) and to a chaotic phase (energy burst plus a large χ value). In the first case, the components of $\mathbf{e}_1$ spread around the forced mode and over the whole inertial range in contrast with the second case where $p(n)$ has values significantly

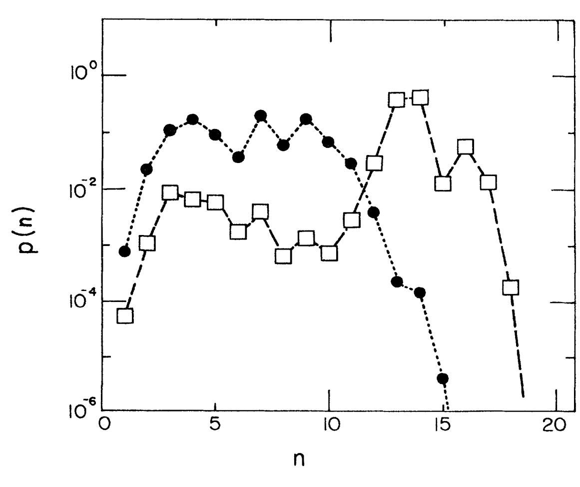

FIG. 5. Distribution of the global chaoticity among the shells during a laminar and a chaotic phase. $\ln p(n)$ is plotted vs n for $N=19$. the crossed circles (linked by a dotted line) correspond to a "laminar" instant ($E_D=4.2\times10^{-6}$, $\chi=0.12$) and the squares (linked by a dashed line) correspond to a burst ($E_D=0.39$, $\chi=1.35$). Note that the average values are $\langle E_D\rangle=(2.09\pm0.02)\times10^{-2}$ and $\lambda_1=\langle\chi\rangle=0.169\pm0.003$.

different from zero only for few shells k_n around k_D.

As discussed above the intermittency gives rise to corrections to Kolmogorov theory leading to a concentration of the energy dissipation on a fractal set with dimension $D_1 \approx 2.9$, which is not space filling. One may therefore naturally ask: Where is this fractal in the model? To get an idea, we analyze a time sequence of the velocity difference in one of the modes. As we are interested in dissipation, we consider a mode k_D at the end of the inertial range, corresponding to the Kolmogorov length scale. This sequence was discussed above and $E_D = |u_{15}(t)|^2$ is shown in Fig. 3(a). Clearly, the energy content of the sequence is concentrated in isolated bursts and since we are at the dissipation scale, the energy dissipation will also be concentrated in the bursts. Actually, it seems that the "peak structure" shows self-similar behavior. To test this quantitatively, we choose a "gate" u_g^2 and take into account only those bursts whose value of $E_D(t)^2$ is greater than u_g^2 (see Fig. 6). At $N=19$ we have used a time sequence over 10^4 time units (the maximum Lyapunov exponent is $\lambda_1 = 0.169$) with a recorded value at each time unit. With $u_g^2 = 10^{-4}$, we get a Cantor-like set consisting of $M = 1181$ points. To estimate its dimension d_f, we use the Grassberger-Procaccia algorithm[24] for the correlation integral by evaluating the number of pairs of points separated by a distance less than τ, i.e.,

$$I(\tau) = \sum_{i,j=1}^{M} \Theta(\tau - |t_i - t_j|) \tag{3.6}$$

where Θ is the Heaviside step function and t_i are the times for which $|u(k_D)|^2$ is larger than u_g^2. The squares in Fig. 7 show a plot of $I(\tau)$. A good scaling is found over around three decades indicating a power law $I(\tau) \sim \tau^x$ with an exponent $x = 0.92 \pm 0.02$. In principle one has only the bound $x \le d_f$, but our set is very close to a homogeneous fractal, with respect to the weight given by the point density, so that $x \approx d_f$. It is interesting to remark that a similar behavior is exhibited by the random β model which is multifractal with respect to the energy dissipation but not with respect to the probability measure given by the number of active eddies.[15]

The exponent x appears to be independent of the gate, at least when it is chosen within some reasonable limits. We have performed similar calculations for the fractal structure of the bursts in time sequences of $p_D(t)$ and of the instantaneous maximum Lyapunov exponent $\chi(t)$. Of course, the gate is changed now according to the typical values of the signals. The corresponding curves for the correlation integrals are also shown in Fig. 7. Again, reasonable scaling is found, giving an exponent $x = 0.94 \pm 0.03$ for p_D and $x = 0.94 \pm 0.03$ for χ. So the fractal dimensions of the burst structure are, within errors, equal in the three cases, giving further evidence that bursts in each of the three quantities are strongly correlated. To relate the fractal structure of the time sequences to a fractal structure in space, we invoke the so-called Taylor hypothesis[25] saying that the statistics extracted by time measurements is similar to that extracted by space sampling at fixed time. We thus would say that the intermittent structures shown in Fig. 3 are equivalent

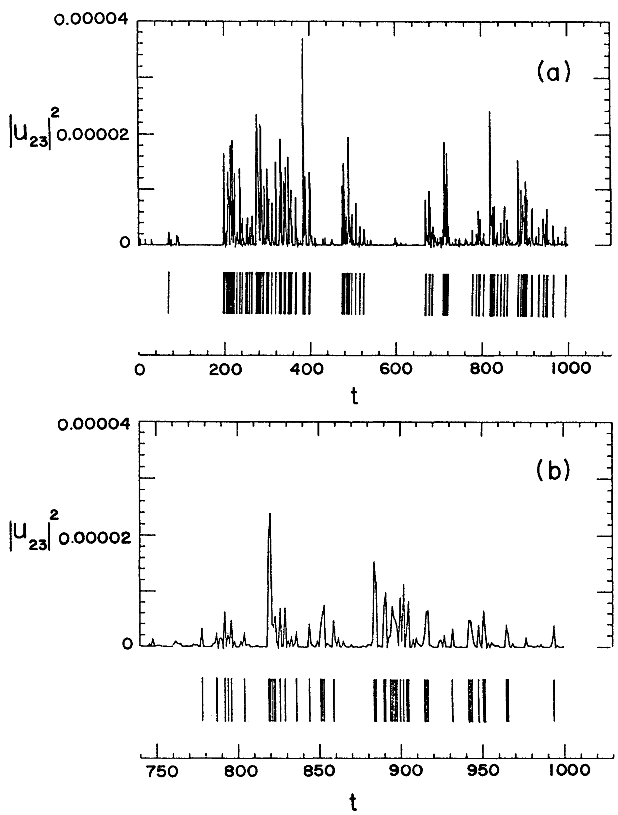

FIG. 6. Construction of the fractal set of the energy dissipation, estimated by $|u_{23}|^2$ for $N=27$. When $|u_{23}|^2 \ge 2 \times 10^{-6}$, a vertical bar is drawn under the plot of the time sequence. (a) is a sequence from $t=0$ to $t=1000$, (b) is an enlargement of (a) from $t=740$ to $t=1000$. Note the self-similar appearance of the set.

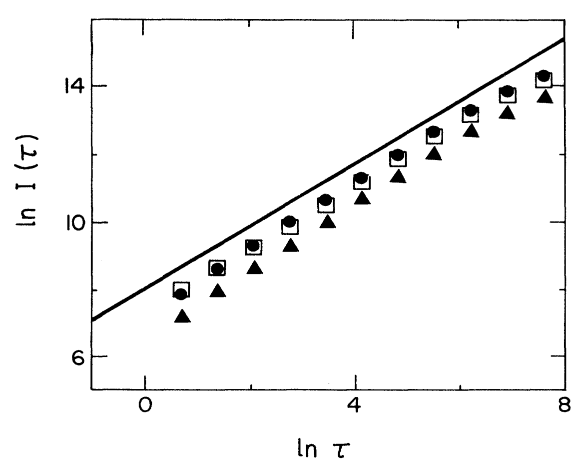

FIG. 7. Correlation integral (3.6) for $N=19$ shells. Scaling of $\ln I(\tau)$ vs $\ln\tau$ for E_D (squares), χ (circles), and p_D (triangles) for $N=19$. The gates are $u_g^2 = 10^{-4}$ (leading to $M=1181$ points), $p_g = 0.03$ ($M=912$ points), and $\chi_g = 0.3$ ($M=1206$ points). The solid line has slope 0.92.

to an intermittent space structure. To use this hypothesis, one needs a typical velocity v_f in order to relate a spatial segment L to a time segment τ by $L = v_f \tau$. For a turbulent fluid in a channel, v_f is just the average flow velocity. On the other hand, for homogeneous turbulence, one can assume that the large-scale structures (energy containing eddies) have the same effect of a mean flow on the small-scale behavior. In this case $v_f \sim \langle \sum_n |u_n|^2 \rangle^{1/2}$.

If we accept the Taylor hypothesis and make the assumption that the two other transverse spatial directions in the system do not have fractal structure, we obtain a dimension for the dissipation set of $D_f = 2 + x = 2.92$. This value is in perfect agreement with the value estimated by the structure functions in Sec. II, i.e., $D_1 = 2 + 3d\zeta/dQ|_{Q=3} = 2.92 \pm 0.02$, and is an *a posteriori* justification of the validity of the Taylor hypothesis in our model. Indeed, we expect $d_f = D_1 - 2$, because our fractal set contains by construction only the points where the energy dissipation is concentrated (i.e., $E_D \geq u_g^2$). In order to study the multifractal behavior exhibited by the structure functions, one should consider the scaling of the weight given by the value of E_D on each point of the fractal obtained by the time series. This would be a test of the Taylor hypothesis in a stronger form, i.e., a test of its validity not only for the typical events but also for the statistics of the large fluctuations of the energy dissipation. Because of the gate and of the absence of a weight on the points, we have taken into account only the most probable behavior connected with the information dimension D_1 of a probability measure.[15]

The numerical integrations of (2.1) and (3.1) described in Secs. II and III have been performed using two different methods: fourth-order Runge-Kutta and Burlirsch-Stoer, with 16 digits precision. We have considered $N = 19$ shells with $\nu = 10^{-6}$, $f = (1+i) \times 5 \times 10^{-3}$, $K_0 = 2^{-4}$, and $N = 27$ shells with $\nu = 10^{-9}$, $f = (1+i) \times 5 \times 10^{-3}$, $K_0 = 0.05$. These parameter values are the same ones used in Ref. 12.

IV. CONCLUSIONS

We have proposed a mechanism describing the intermittency of the energy dissipation in a cascade model for three-dimensional fully developed turbulence. Our numerical calculations show that the structure functions for the velocity field have an anomalous scaling which can be interpreted in the context of the multifractal formalism.

We have provided strong evidence that this result is connected to the temporal intermittency of the chaotic evolution exhibited by the dissipative dynamical system (2.1). In fact, the instantaneous maximum Lyapunov exponent has very large fluctuations. Its peak appear in correspondence to a sudden localization of its eigenvector $\mathbf{e}_1$ on the dissipative modes at the end of the inertial range, and to violent bursts of energy dissipation, as well. The three temporal sequences (energy dissipation E_D, component of $\mathbf{e}_1$ on a dissipative wave number p_D, instantaneous Lyapunov exponent χ) allow us to introduce a fractal set in space, via the Taylor hypothesis. Its dimensionality is in agreement with the dimensionality of the energy dissipation extracted by the structure functions.

We have discussed the severe limitations of the scalar cascade model as compared to real Navier-Stokes equations. Nevertheless extended numerical integrations[26] of the two-dimensional Navier-Stokes equations indicate that some features of the model are generic phenomena of fully developed turbulence. For this case, when, at a given instant, the vorticity gradients assume very large values in small zones of the fluid, the eigenvector corresponding to the maximum Lyapunov exponent has been found to be concentrated in those same zones.

Our results can thus be useful for a deeper comprehension of the fractal nature of turbulence using dynamical systems with a limited number of degrees of freedom.

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*Permanent address: Dipartimento di Fisica, Università de L'Aquila, via Vetoio, I-67010 Coppito L'Aquila, Italy.

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