RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 43, NUMBER 12 15 JUNE 1991

Finite-size effects in a sandpile

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(Received 21 March 1991)

We examine how the size of a sandpile can affect the nature of the avalanches occurring on its surface. We propose a simple explanation for why, if a pile is made sufficiently small, the distribution of avalanches changes from what is observed in large systems and give a criterion for how small the pile must be in order for this to happen. This explanation does not support the conclusion that avalanches show self-organized critical behavior. We also present data which indicate that the distributions of avalanches in very narrow sandpiles are qualitatively the same as those found in large piles.

It has been suggested that an avalanche in a pile of sand is the prototypical example of a phenomenon called self-organized criticality.[1] Experiments[2] carried out on actual sandpiles, however, have suggested that such avalanches do not behave in a critical manner at all but have hysteresis behavior reminiscent of a first-order phase transition. The sandpile remains stationary as θ, the angle that the top surface makes with respect to the horizontal increases above the angle of repose θ_r. An avalanche, which returns the pile to θ_r where it is again stable, is initiated only when θ reaches the maximum angle of stability θ_m. Between θ_r and θ_m the pile is metastable.

A recent experiment[3] has indicated that when the pile of sand is sufficiently small, the behavior of the avalanches changes qualitatively. For these small piles the distribution of avalanche sizes, $\rho(m)$, appears to obey finite-size scaling. The distributions for piles of different sizes could be superimposed on top of one another to give a single, smooth curve. When the size of the pile was increased this scaling behavior disappeared and the hysteresis described above was again observed. These data have been taken as vindication of the original conjecture that sandpiles do indeed display self-organized critical behavior.[3-5] It therefore becomes important to understand what is the role of finite-size effects in such granular systems.

We will first address the question of what role the length of a sandpile plays in determining the behavior of an avalanche in order to see if we can understand why, for very small sandpiles, the hysteretic behavior disappears. We note first that the difference between the maximum angle of stability and the angle of repose is typically a few degrees:

$$\delta \equiv \theta_m - \theta_r \approx 2^\circ . \quad (1)$$

We argue that finite-size effects must enter to destroy the observance of a first-order transition when the addition of a *single* particle, of diameter d, will bring the pile from a condition of stability, where $\theta < \theta_r$, to a condition of instability, where $\theta > \theta_m$. As can be seen trivially in Fig. 1 this will occur when the length of the pile, L, is such that

$$L \lesssim d/\delta \quad (2)$$

which for $\delta \approx 2^\circ$ gives $L \lesssim 30d$. Below this value for L, the pile is not large enough to be able to distinguish the difference between the two angles θ_m and θ_r and consequently hysteresis can no longer occur. In the experiments of Held *et al.*,[3] the transition between the large sys-

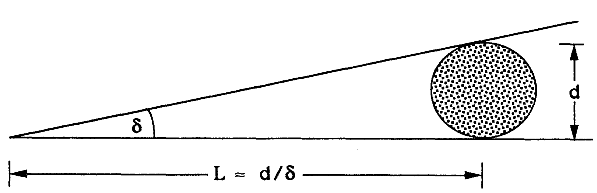

FIG. 1. Illustration of the criterion for when finite-size effects should become important in a sandpile. If the radius of the pile is smaller than $L \approx d/\delta$ then the pile can no longer distinguish between θ_r and θ_m: a single grain added at the top of the pile will bring the slope from a stable configuration below θ_r to an unstable one above θ_m. For $\delta \approx 2^\circ$, $L \approx 30d$.

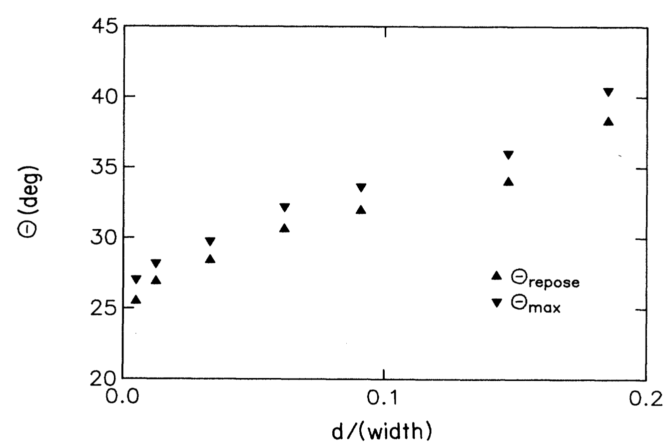

FIG. 2. The two angles θ_r and θ_m, as a function of d/w, where w is the width of the sandpile and d is the average diameter of a grain of sand. The pile, which is held between two smooth parallel walls, has a length L which is much larger than d/δ.

tems where hysteresis occurs and small systems where finite-size scaling was applied, did occur between $L=30d$ and $60d$, very close to the value we have just calculated. This suggests that what they saw was not the onset of true critical behavior in the sense of a second-order phase transition. Similar fluctuations might be observed in a ferromagnet in a magnetic field below the Curie temperature if the sample were sufficiently small (i.e., a few spins). In that case, as well, the first-order transition would be masked by the finite size of the system.

So far we have examined what happens to the behavior of the sandpile if the length of the pile is short. Finite-size effects could also become important if the pile is very narrow. In such an arrangement the sand is held between two parallel walls a distance w apart. The walls could induce finite-size effects either by confining the geometry of the pile to be effectively one-dimensional or by introducing friction at these lateral boundaries. In our experiment we have used smooth walls to minimize the effects of friction. In order to see whether the variation of w changes the nature of the avalanches we have measured the two angles θ_r and θ_m as w is varied. The results are shown in Fig. 2 for spherical grains of average diameter $d \approx 0.5$ mm. As w is varied, we see that, although the two angles themselves also vary and increase as w is decreased, the value of δ remains nonzero. This is in agreement with the data of Evesque and Rajchenbach.[6] This implies that the nature of the avalanches does not change qualitatively as the width of the sandpile is decreased.

We conclude that finite-size effects can become important in determining the nature of the avalanches in a sandpile only if the length of the pile becomes so small that the addition of a single grain of sand takes the pile from an angle below the angle of repose to an angle greater than the maximum angle of stability. The fluctuations that have been observed in experiments on small systems appear to be due simply to the system being too small for it to show the two distinct angles θ_r and θ_m that determine the behavior in the infinite system. In particular, this data on small size systems does not appear to support the application of self-organized critical behavior to avalanches in sandpiles.

This work was supported by NSF Materials Research Laboratory Grant No. DMR-8819860.

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