Partially coherent propagation-invariant beams: Passage through paraxial optical systems

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The cross-spectral-density function characterizing a general propagation-invariant wave field satisfies the associated (planar) paraxial equation. Building on this result, we solve exactly the problem of the passage of such fields through arbitrary lossless optical systems characterized by their *ABCD* matrices. It is shown that the wave field remains shape invariant in all such systems, but that the property of strict propagation invariance is conserved only if the system is afocal.

Diffraction-free fields[1,2] are exact solutions of the Helmholtz equation that propagate in free space without any modification of the transverse intensity distribution. Well-known examples are the extremely narrow J_0 beams and other nondiffracting Bessel beams. The passage of such coherent fields through first-order optical systems characterized by their *ABCD* matrices[3] has been investigated in Ref. 4 using group theory; paraxial optical transformations belong to a metaplectic group that is isomorphic to the symplectic group of geometrical-optics ray-transfer matrices.[5,6] A conclusion of Ref. 4 was that diffraction-free beams remain nondiffracting in all such systems. A fundamental difficulty arises, however, from the fact that the nondiffracting fields are not solutions to the parabolic equation as is required by wave propagation in *ABCD* systems. In Ref. 4 this difficulty is handled by taking in free flight the paraxial approximation of the diffraction-free beam, i.e., by approximating the dispersion surface, a sphere of radius k, by a paraboloid of the same curvature. This procedure creates an extraneous, beam-width-dependent phase factor for the field, though naturally it will not affect the beam intensity.

In recent papers[7] we have investigated the extension of the concept of propagation invariance into the domain of partially coherent optics. We made use of the newly developed theory of partial coherence in the space-frequency domain[8] and derived, in particular, an explicit expression for optical wave fields that are characterized by a cross-spectral-density function[9] that remains invariant in any plane transverse to the propagation direction. The main purpose of this Brief Report is to investigate the passage of these propagation-invariant partially coherent fields through *ABCD* optical systems. The transfer operators of such systems are canonical transforms[10] that form a metaplectic group and admit, in general, an integral representation that is identical to the generalized Huygens-Fresnel integral.[10,11] Hence, in essence, this work extends the results of Ref. 4 into stochastic wave fields in terms of familiar laser-physics techniques.[12] It turns out also that the fundamental difficulty mentioned above vanishes (even in the fully coherent case) when we consider the propagation of the cross-spectral density.

Within the context of second-order coherence theory, the free-space cross-spectral-density function of a statistically stationary (scalar) wave field obeys the coupled Helmholtz equations[8,9]

$$(\nabla_j^2+k^2)W(\mathbf{r}_1,\mathbf{r}_2)=0, \quad j=1,2 \tag{1}$$

where $\mathbf{r}=(\boldsymbol{\rho},z)=(x,y,z)$, $k=\omega/c$ is the wave number, and the explicit dependence on the frequency ω will be omitted throughout. It was shown in Ref. 7 that the solutions to Eqs. (1), which moreover satisfy the condition of propagation invariance

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;z)=W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;0) \tag{2}$$

for all values of $z\geq 0$, must necessarily be of the form

$$W(\mathbf{r}_1,\mathbf{r}_2)=\int_0^{k/2\pi}\int_0^{2\pi}\!\!\int f^2 S(f,\theta_1,\theta_2)\exp\{i(z_2-z_1)[k^2-(2\pi f)^2]^{1/2}\} \times\exp[i2\pi f(x_2\cos\theta_2-x_1\cos\theta_1+y_2\sin\theta_2-y_1\sin\theta_1)]\,df\,d\theta_1\,d\theta_2\,, \tag{3}$$

where $S(f,\theta_1,\theta_2)$ is an arbitrary function. We note that Eq. (2) poses an invariance condition on both the transverse intensity and the transverse spatial (spectral) coherence distributions.[9,13]

The general expression (3) contains as special cases the usual coherent diffraction-free fields[1] with cross-spectral density of the factored form[14] $W(\mathbf{r}_1,\mathbf{r}_2)=U^*(\mathbf{r}_1)U(\mathbf{r}_2)$. Setting

$$S(f,\theta_1,\theta_2)=(2\pi/\alpha)^2F^*(\theta_1)F(\theta_2)\delta(f-\alpha/2\pi)$$

we readily find from Eq. (3) that

$$U(\mathbf{r})=\exp(i\beta z)\int_0^{2\pi}F(\theta)\exp[i\alpha(x\cos\theta+y\sin\theta)]d\theta\ , \tag{4}$$

where $F(\theta)$ is arbitrary and the condition $\alpha^2+\beta^2=k^2$ must hold for $U(\mathbf{r})$ to be a solution of the Helmholtz equation. Expression (3) also contains nontrivial partially coherent solutions such as the J_0 Bessel-correlated field[15]

$$W(\mathbf{r}_1,\mathbf{r}_2)=\exp[i\beta(z_2-z_1)]J_0(\alpha|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|) \tag{5}$$

[choice $S(f,\theta_1,\theta_2)=(2\pi/\alpha)^2\delta(f-\alpha/2\pi)\delta(\theta_1-\theta_2)$ in Eq. (3)], which has a constant intensity but a sharply peaked transverse-spatial-correlation profile. Propagation-invariant wave fields that possess rapidly varying profiles of both the optical intensity and the spatial coherence also exist: one example is [cf. Ref. 7(b)]

$$W(\mathbf{r}_1,\mathbf{r}_2)=\exp[i\beta(z_2-z_1)][J_0(\alpha|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)-J_0(\alpha|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|)]\ , \tag{6}$$

which is obtained from Eq. (3) with $S(f,\theta_1,\theta_2)=(2\pi/\alpha)^2\delta(f-\alpha/2\pi)[\delta(\theta_1-\theta_2)-\delta(\theta_1-\theta_2+\pi)]$.

The propagation of an optical field from the input plane $z=0$ of an arbitrary $ABCD$ optical system to the output plane $z=z_0$ is governed by a canonical transfer operator[10] (a linear similarity transformation that preserves the commutation relation of canonically conjugate operators). Canonical-operator theory has been extensively studied in connection with first-order optical systems that may contain gain and loss, misalignments, tilts, and anisotropic media (see, for example, Refs. 6, 12, and 16). For axial-symmetric optical systems the action of the transfer operator may be expressed in a form that is mathematically identical to the extended Huygens-Fresnel integral:[10–12,17]

$$U(\boldsymbol{\rho},z_0)=-\frac{ik}{2\pi B}e^{ikL}\int\int_{-\infty}^{\infty}U(\boldsymbol{\rho}',0)\exp\left[\frac{ik}{2B}(D\rho^2-2\boldsymbol{\rho}\cdot\boldsymbol{\rho}'+A\rho'^2)\right]d^2\rho'\ , \tag{7}$$

where

$$L=\int_0^{z_0}n(0,z)dz \tag{8}$$

is the on-axis optical length of the system [$n(\boldsymbol{\rho},z)$ is the refractive index]. Although Eq. (7) is valid also for systems with gain or loss,[10] we shall simply consider lossless systems that are embedded in air; consequently all of the elements A, B, C, and D are real parameters and $AD-BC=1$. Interpreting $U(\boldsymbol{\rho},z)$ as an appropriate space-frequency domain field realization,[8] it then follows from the definition of the cross-spectral-density function (after some rearrangement) that

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;z_0)=(\lambda B)^{-2}\exp\left[-\frac{ikD}{2B}(\rho_1^2-\rho_2^2)\right]\times\int\int\int\int_{-\infty}^{\infty}W(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2';0)\exp\left[-\frac{ikA}{2B}[(\rho_1')^2-(\rho_2')^2]\right]\exp\left[\frac{ik}{B}(\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_1'-\boldsymbol{\rho}_2\cdot\boldsymbol{\rho}_2')\right]d^2\rho_1'd^2\rho_2'\ . \tag{9}$$

With the help of this and the previous expressions we may now readily elucidate the problems surrounding rigorous solutions (such as the Bessel beams) and paraxial approximations in optical $ABCD$ systems.

The main problem stems from the fact that while the general coherent diffraction-free field (4) is an exact solution of the Helmholtz equation, it does not satisfy the corresponding parabolic equation $(\nabla_\perp^2+2ik\partial/\partial z)V(\mathbf{r})=0$, where $V(\mathbf{r})=U(\mathbf{r})\exp(-ikz)$ and $\nabla_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. This latter equation, on the other hand, is mathematically equivalent to the Fresnel diffraction integral[12] and governs the free-flight propagation in $ABCD$ systems. Stated more specifically, Eq. (4) is not expressible in the form of Eq. (7) when $A=D=1$, $B=z_0$, and $C=0$. The problem can be avoided (cf. Ref. 4) by considering only the paraxial approximation of the diffraction-free field; this amounts to expanding $\beta\approx k-\alpha^2/2k$ in Eq. (4). Hence the price to be paid is a phase factor that depends on z and on α (which determines the beam size), but physically the phase has no effect on the optical intensity. More generally, we observe that these phase factors cancel when calculating, for example through statistical averaging of the diffraction-free beams,[7,8] the cross-spectral-density functions needed in expression (9).

Owing to the specific form of Eq. (9) and the condition (2) of propagation invariance, it is in fact appropriate to consider here directly the planar cross-spectral-density function $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;z)$, i.e., to set $z_2=z_1=z$. For general propagation-invariant wave fields (which include any correlated ensembles of the usual diffraction-free beams[7]) this function is, according to Eq. (3), simply

$$W(\boldsymbol{\rho}_1;\boldsymbol{\rho}_2;z)=\int_0^{k/2\pi}\int\int_0^{2\pi}f^2S(f,\theta_1,\theta_2)\exp[i2\pi f(x_2\cos\theta_2-x_1\cos\theta_1+y_2\sin\theta_2-y_1\sin\theta_1)]df\,d\theta_1d\theta_2\ . \tag{10}$$

The free-space propagation of the planar cross-spectral-density functions associated with paraxial partially coherent wave fields is governed by the differential equation[18]

$$\left[\nabla_{1\perp}^2-\nabla_{2\perp}^2-2ik\frac{\partial}{\partial z}\right]W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;z)=0\ . \tag{11}$$

On substituting Eq. (10) into Eq. (11) we see that the cross-spectral-density function of a general propagation-invariant field, which is an exact solution to a pair of coupled Helmholtz equations, is in any transverse plane remarkably also a solution to the associated (single) paraxial equation. Therefore, no formal difficulties arise when dealing with the problem of partially coherent wave propagation through arbitrary *ABCD* optical systems. We emphasize that the description of coherent wave fields with the aid of separable planar cross-spectral-density functions[14] $W(\mathbf{r}_1,\mathbf{r}_2)=U^*(\boldsymbol{\rho}_1,z)U(\boldsymbol{\rho}_2,z)$ solves the dilemma even in the fully coherent case.

We are now in a position to evaluate the expression that results when a propagation-invariant input field traverses a lossless *ABCD* optical system. On substituting Eq. (10) (which, in fact, is independent of z) into the integrand of Eq. (9) and performing the necessary integrations, we arrive at

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;z_0)=A^{-2}\exp\left[-\frac{ik}{2}\frac{C}{A}(\rho_1^2-\rho_2^2)\right]$$
$$\times\int_0^{k/2\pi}\int\int_0^{2\pi} f^2S(f,\theta_1,\theta_2)\exp[i2\pi f(x_2\cos\theta_2-x_1\cos\theta_1+y_2\sin\theta_2-y_1\sin\theta_1)/A]\,df\,d\theta_1 d\theta_2\ . \tag{12}$$

Clearly, the integral still represents a propagation-invariant wave field similar to the incident field up to a scaling factor of $1/A$. In addition, however, an exponential term is introduced that corresponds to a spherical wave front with the radius of curvature $R=A/C$. Strictly speaking, therefore, a propagation-invariant field is generated in the output plane of the system only if $C=0$, which in turn implies that the system is afocal,[3] with magnification A. Our definition (2) of propagation invariance reduces to the original intensity condition $I(\boldsymbol{\rho},z)=I(\boldsymbol{\rho},0)$ given in Ref. 1 for arbitrary fully coherent diffraction-free fields, and thus the conclusion obtained above also holds for these wave fields, including the J_0 Bessel beam discussed in Ref. 4.

Assuming that the *ABCD* system is a combination of some optical system characterized by a matrix $A'B'C'D'$ and a free-space propagation over a distance d, the combined matrix elements are $A=A'+dC'$, $B=B'+dD'$, $C=C'$, and $D=D'$. It is now seen from Eq. (12) that the partially coherent propagation-invariant field behind the output plane cannot satisfy the condition of propagation invariance unless $C'=0$; although the spatial correlation properties and the intensity distribution across any transverse plane after the fixed optical system are equal—up to a scale factor—to those of the input beam, this scale factor depends on the propagation distance d. Such behavior is characteristic, for example, of the classic Hermite-Gaussian laser modes[12] and (in variable-coherence optics) the so-called Gaussian Schell-model beams (see, e.g., Ref. 19). These beams are normally not considered diffraction free, but merely shape invariant.

In conclusion, using the well-known methods of physical optics, we have extended the results of Ref. 4 to cover the passage of all (coherent or partially coherent) propagation-invariant wave fields through arbitrary (lossless) *ABCD* optical systems. In particular, we have shown that in plane-to-plane propagation such wave fields do not distinguish between exact and paraxial evolution, and that these fields can remain propagation invariant only if the optical system is a focal.

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[17]Expression (7) is valid only if $B \neq 0$, i.e., if the optical system is not an imaging system. Cases when $B=0$ can be treated by a suitable limiting process, or explicitly by making use of the expression $U(\boldsymbol{\rho},z_0)=|A|^{-1}\exp[i(k/2)(C/A)\rho^2]U(\boldsymbol{\rho}/A,0)$, where A is the image magnification. We note that the final result will be identical to Eq. (12).

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