Conditions for the existence of periodic solutions to integrable two-dimensional Hamiltonian systems

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An apparent connection between integrability and the existence of a periodic solution of the variational equations around the straight-line solutions of planar Hamiltonian systems has been suggested [F. T. Hioe, Phys. Rev. A **39**, 2628 (1989)]. Such solutions always exist if the gradient of the second integral and the Hamiltonian are independent on the straight lines. In any other case, the existence of such periodic solutions can be shown only if the second integral satisfies certain conditions.

In a recent paper,[1] Hioe suggests an apparent close relation between the existence of a periodic solution of the variational equations around the straight-line solutions of a planar Hamiltonian system (termed "stability of type 1") and integrability. More specifically it is found that in the system of two coupled quartic oscillators,

$$V(x,y)=\tfrac{1}{2}(Ax^2+By^2)+Dx^4+2Cx^2y^2+Ey^4\ , \qquad (1)$$

such solutions exist only in the six known integrable cases, while in the Hénon-Heiles system,

$$V(x,y)=\tfrac{1}{2}(Ax^2+By^2)+Cx^2y-\frac{D}{3}y^3\ , \qquad (2)$$

they exist in two of the three known integrable cases and also in six cases which are presumably nonintegrable.

In this Brief Report we demonstrate the exact relation between the existence of a second integral of motion and a periodic solution of the variational equations and provide a straightforward interpretation of the results obtained by Hioe in all except one of the integrable cases.

We consider a planar Hamiltonian system

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+V(x,y) \qquad (3)$$

and suppose that it admits

$$y=\varphi(t),\quad \dot{y}=\dot{\varphi}(t),\quad x=\dot{x}=0 \qquad (4)$$

as a straight-line solution (SLS). Let $\zeta_1,\zeta_2,\xi_1,\xi_2$ be the variations to $x,\dot{x},y,\dot{y}$, respectively, around the above SLS. The corresponding variational equations (VE) are

$$\dot{\xi}_1=\xi_2,\quad \dot{\xi}_2=-V_{yy0}\xi_1\ , \qquad (5a)$$

$$\dot{\zeta}_1=\zeta_2,\quad \dot{\zeta}_2=-V_{xx0}\zeta_1\ , \qquad (5b)$$

where a subscript x or y denotes partial differenetiation and 0 denotes that the corresponding quantity is computed on the SLS (4).

Equations (5a) are the tangent (TVE) and (5b) the normal variational equations (NVE). We define the vector operators

$$\nabla_T=\left[\frac{\partial}{\partial y}\ \ \frac{\partial}{\partial \dot{y}}\right]^T,\quad \nabla_N=\left[\frac{\partial}{\partial x}\ \ \frac{\partial}{\partial \dot{x}}\right]^T, \qquad (6a)$$

$$\nabla=\left[\frac{\partial}{\partial x}\ \ \frac{\partial}{\partial \dot{x}}\ \ \frac{\partial}{\partial y}\ \ \frac{\partial}{\partial \dot{y}}\right]^T, \qquad (6b)$$

and the scalar operators

$$D_T=\xi\cdot\nabla_T,\quad D_N=\zeta\cdot\nabla_N\ , \qquad (7)$$

where $\xi=(\xi_1\ \ \xi_2)^T$ and $\zeta=(\zeta_1\ \ \zeta_2)^T$. The following theorems are known.[2-4].

(i) The TVE admit a periodic solution

$$\xi=\omega(\nabla_T H)_0=(\dot{\varphi}\quad -V_{y0})^T \qquad (8)$$

with

$$\omega=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \qquad (9)$$

(ii) The monodromy matrix Δ_T of the TVE is symplectic and possesses a pair of unit eigenvalues.

(iii) $(D_T H)_0$ is an integral of the TVE.

Theorem (i) explains why in every case periodic solutions of the TVE have been found by Hioe. Let Δ_T in the basis ξ_i be

$$\Delta_T=\begin{bmatrix}a & b\\ c & 2-a\end{bmatrix} \qquad (10)$$

with $a(2-a)-bc=1$. If t_0 in (5) is selected such that $\dot{\varphi}(t_0)=0$, then by acting Δ_T on the integral (iii) we obtain

$$[\xi_1(t_0)(a-1)+\xi_2(t_0)b]V_{y0}=0 \qquad (11)$$

and since $\xi_i(t_0)$ are arbitrary initial conditions and $V_{y0}\neq 0$ we obtain $a=1$, $b=0$ and the corresponding periodic solution is $\xi_1=k\dot{\varphi}(t)$ with $k=$const, which is exactly the one obtained in all cases by Hioe.

Let $I(x,y,\dot{x},\dot{y})$ be a single-valued analytic integral of motion of system (3), such that ∇I is independent of ∇H on the SLS. Then concerning the NVE, the following theorems are also known.[2-4]

(iv) $\zeta=\omega(\nabla_N I)_0$ is a periodic solution of the NVE.

(v) The monodromy matrix Δ_N of the NVE is symplectic and possesses a pair of unit eigenvalues.

(vi) $(D_N I)_0$ is an integral of the NVE.

This latter integral is nontrivial since $(D_N I)_0=0$ implies linear dependence between ∇I and ∇H on the SLS.[4] If moreover $I_{x0}=0$ or $I_{\dot{x}0}=0$ at t_0, then by the same reasoning it can be shown that Δ_N is triangular. This result covers cases I, II, and IV for the quartic oscillator (the parameter relationships for $A:B:C:D:E$ are 1:1:1:1:1 for case I, 1:1:3:1:1 for case II, and 4:1:6:16:1 for case IV) and case 6 for the Hénon-Heiles system ($A/B=1$, $C/D=-1$) (numbering of cases refers to Tables I and II of Hioe).

In the remaining integrable cases, ∇I is dependent on ∇H or zero on the SLS and these cases need a separate treatment. The integrable case of the Hénon-Heiles system which does not admit a periodic solution of the NVE and case III ($A:B:C:D:E=1:4:6:1:16$) of the quartic oscillator correspond to $\nabla I=\mathbf{0}$ on the SLS, while in case VI ($A:B:C:D:E=0:0:3:8:1$) ∇I depends linearly on ∇H on the SLS, namely $\nabla I=8h\nabla H$ where h is the energy of the SLS. These cases however can be treated together since in the last case we may form another integral of motion by the relation[4,5]

$$I'=I-8h(H-h) \tag{12}$$

such that $\nabla I'=\mathbf{0}$ and at least some of the second-order derivatives of I or I' are different from zero on the SLS. In the following we omit the prime but we work with the new integral (12). We will use the following theorem.[4,5]

(vii) $(D_N^m I)_0$ is an integral of motion of the NVE, where m is an integer such that all the derivatives of I of order $n<m$ are zero on the SLS. If $(D_N^m I)_0=0$ then another integral of the NVE, homogeneous in the ξ_i of degree $k<m$ can always be found. In our case however this integral is nontrivial.

For the cases mentioned above $m=2$ while in the remaining cases V and 1, $m=4$. We will refer to these cases later. Let

$$\Delta_N=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{13}$$

with $ad-bc=1$ the monodromy matrix of the NVE.

Let I_{ij} stay for the $(i+j)$th partial derivative of I, i times with respect to x and j times with respect to $\dot{x}$, computed on the SLS at t_0. By acting Δ_N on the integral $(D_N^2 I)_0$ and taking into account that $I_{11}=0$ in all cases into consideration, we obtain the relations

$$(a^2-1)I_{20}+c^2 I_{02}=0\ , \tag{14a}$$

$$abI_{20}+cdI_{02}=0\ , \tag{14b}$$

$$b^2 I_{20}+(d^2-1)I_{02}=0\ . \tag{14c}$$

In case VI of the quartic oscillator $I_{02}=0$ also which implies $a^2=d^2=1$, $b=0$ so that Δ_N^2 has two unit eigenvalues and a periodic solution of the NVE exists. In the corresponding case of the Hénon-Heiles system, $I_{20}I_{02}\neq 0$ and Eqs. (14) yield $a=d$ and

$$a^2=1-c^2\frac{I_{02}}{I_{20}}\ . \tag{15}$$

From (15) we may find that if

$$h>A(4A-B^2)/8C^2 \tag{16}$$

then $a^2>1$ and the solutions of the NVE are exponentially unstable which is the case in the example given by Hioe in his Fig. 1. In case III of the quartic oscillation $I_{02}I_{20}\neq 0$ also and one cannot show the existence of a periodic solution of the NVE by the known integral of motion. These two potentials belong to a sequence of integrable potentials which may superposed without destroying integrability, found by Ramani, Dorizzi, and Grammatikos.[6]

In the last integrable cases, case V ($A:B:C:D:E=0:0:3:1:8$) and case 1 ($A/B=1/16$ and $C/D=-1/16$), all the derivatives of I up to the third order are zero on the SLS and the corresponding integral is $(D_N^4 I)_0$. By acting Δ_N on this integral and taking into account that $I_{13}=I_{31}=0$ we obtain the relations

$$(a^4-1)I_{40}+6a^2c^2I_{22}+c^4I_{04}=0\ , \tag{17a}$$

$$a^3bI_{40}+3(ad+bc)acI_{22}+c^3dI_{04}=0\ , \tag{17b}$$

$$a^2b^2I_{40}+6abcdI_{22}+c^2d^2I_{04}=0\ , \tag{17c}$$

$$ab^3I_{40}+3(ad+bc)bdI_{22}+cd^3I_{04}=0\ , \tag{17d}$$

$$b^4I_{40}+6b^2d^2I_{22}+(d^4-1)I_{04}=0\ . \tag{17e}$$

The rank of the matrix of (17) must equal 2 which yields the following possibilities

(a) $a^4=d^4=1,\quad b=c=0$,

(b) $a=d=0,\quad c=-1/b$,

(c) $a=-d,\quad abcd\neq 0$,

(d) $a=d,\quad abcd\neq 0$.

In cases (a)–(c), Δ_N^4 equals the identity and the NVE have periodic solution. Case (d) must be excluded since the independent equations of (17) yield in this case

$$-\frac{b}{c}=\frac{3I_{22}}{I_{40}}=\frac{I_{04}}{3I_{22}} \tag{18}$$

and this last equation is not true for the particular integrals. The quartic oscillator admits also a SLS along the x axis. On this SLS the same results are obtained by virtue of the symmetry of the potential (1).

The remaining six cases of the Hénon-Heiles system are presumably nonintegrable but the monodromy matrix of the NVE in real time happens to be resonant[4,5] so that these equations admit a periodic solution. Hioe mentions that cases 7 and 8 ($A/B=1$ and 4, $C/D=-5/2$ and $-5/2$, respectively) have been proved to be nonintegrable by Yoshida. This is not correct. Yoshida[4] actually proved nonintegrability for the homogeneous Hénon-Heiles system

$$V_h(x,y)=Cx^2y-Dy^3/3 \tag{19}$$

for $C/D=-5/2$ by showing the existence of two nonresonant, noncommuting monodromy matrices of the NVE along the SLS,

$$x^2/y^2=2+D/C \ . \tag{20}$$

In order to apply Yoshida's[7] theorem for the nonhomogeneous potential (2), the above SLS must persist when the quadratic terms are added in the potential (19) which is not the case. The only SLS admitted by the full potential (2) is (4) and one cannot show the existence of two nonresonant monodromy matrices on this SLS for the above value of C/D. It may however be shown that no polynomial invariant exists in these cases, by a corollary of Hietarinta (p. 96).[8]

Cases 2 and 3 ($A/B=1/16$ and $9/16$ and $C/D=-5/16$ and $-5/16$, respectively), and cases 7 and 8 seem to be nonintegrable since narrow chaotic zones are present in the surface of section. In Fig. 1 we show a surface of section for case 3. In case 4 ($A/B=0$, $C/D=-1/2$), all orbits are unbounded while no chaos appeared in case 5 ($A/B=1$, $C/D=-1/2$). This case seems to be a good candidate for integrability, although it does not possess the Painlevé property since a logarithmic term enters in the expansion.[9]

We may note in conclusion that the existence of a second integral of motion I results in a periodic solution of the VE around a SLS if ∇I is independent of ∇H on the SLS. Otherwise, the existence of such a periodic solution can be shown only if I satisfies certain conditions. In any case, Hioe's results provide a good demonstration of the fact that one cannot disprove integrability by the real-time monodromy matrix of the NVE only, since this matrix may or may not be resonant, independently of the existence of a second integral of motion. Nonintegrability may be proved by Ziglin's theorem[4,5,7,10] if at least two independent monodromy matrices in the complex domain are known.

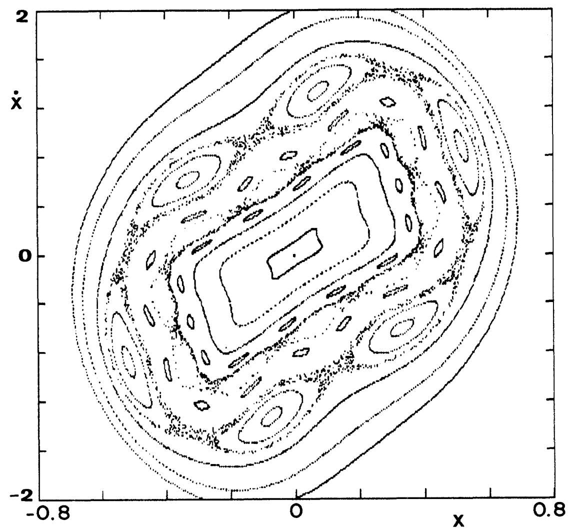

FIG. 1. Poincaré section for the generalized Hénon-Heiles system for $A=9$, $B=16$, $C=-5$, $D=16$, and $h=2.58$.

[1]F. T. Hioe, Phys. Rev. A **39**, 2628 (1989).

[2]See, e.g., H. Poincaré, NASA Report No. TT F-450, 1967 (unpublished); J. D. Hadjidemetriou, European Program Erasmus, Monograph No. ICP-88-016-GR (Mathematics and Fundamental Applications) (Erasmus, Thessaloniki, Greece, 1988).

[3]V. V. Kozlov, Russ. Math. Survey **38**, 1 (1983).

[4]H. Yoshida, Physica D **29**, 128 (1987).

[5]S. L. Ziglin, Funct. Anal. Appl. **16**, 181 (1983); **17**, 6 (1983).

[6]A. Ramani, B. Dorizzi, and B. Grammatikos, Phys. Rev. Lett. **49**, 1539 (1982).

[7]H. Yoshida, Commun. Math. Phys. **116**, 529 (1988).

[8]J. Hietarinta, Phys. Rep. **147**, 87 (1987).

[9]T. Bountis, H. Segur, and F. Vivaldi, Phys. Rev. A **25**, 1257 (1982).

[10]R. C. Churchill and D. L. Rod, J. Diff. Equations **76**, 91 (1988).